ON AUTOFRETTAGE OF CYLINDERS BY LIMITING CIRCUMFERENTIAL RESIDUAL STRESS BASED ON MISES YIELD CRITERION

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The autofrettage technique is an effective and important measure to improve load-bearing capacity and safety and to even distributions of stresses for pressure vessels. Based on the classical fundamental theory on autofrettage and by theoretical analysis of residual stresses, the total stresses, the overstrain and the load-bearing capacity, etc., in view of the cylindrical pressure vessels with outside-to-inside radius ratio larger than the critical ratio, the laws contained in the autofrettage theory are revealed, the essential cause and reason for the obtained laws are analyzed, the inherent and meaning relations between various parameters in the autofrettage theory are brought to light, and the safe depth of the plastic zone (overstrain) as well as the conditions of loading or optimum operation conditions are found out. It is shown that under the optimum operation conditions, pressure vessels are not only safe but also economic, and the equations of the autofrettage theory are simplified greatly and are quite terse, as a result, the essential relations between various parameters in the autofrettage theory are distinct, and these equations are convenient for application in engineering practice.

Key words: pressure vessel, autofrettage, load-bearing capacity, strength theory

Nomenclature

\[ r_i, r_j, r_o \] – Inside radius, radius of elastic-plastic juncture, outside radius, respectively
\[ k \] – Outside to inside radius ratio \((k = r_o/r_i)\)
\[ k_j \] – Depth of plastic zone \((k_j = r_j/r_i)\)
\[ k_c \] – Critical radius ratio
\[ p, p_a, p_y \] – Internal, autofrettage and entire yield pressure, respectively
\[ p_e \] – Maximum elastic load-bearing capability of unaufactrettaged pressure vessel or initial yield pressure
\[ \sigma_r, \sigma_\theta, \sigma_z, \sigma_e \] – Radial, circumferential, axial and equivalent stress, respectively
\[ \sigma_y \] – Yield strength

Superscripts: ‘, T, p – Residual, total and stress caused by \(p\), respectively.

1. Introduction

Cylindrical pressure vessels are important mechanical equipment and are widely applied in the industry. To raise load-bearing capacity of a pressure vessel and even distribution of stresses in the wall of a pressure vessel, the “autofrettage” technique for thick pressure vessels was created, which is an interesting and absorbing subject. Parker (2001a,b) extended a procedure to model the addition of pressure or material to a tube, providing an associated residual stress

The autofrettage damage mechanics model was setup from the ultra-high pressure vessel autofrettage damage mechanism (Lin et al., 2009) and a brief introduction of the autofrettage method, such as mechanical extrusion, direct pressure, expansion of the explosion pressure and solid autofrettage was given by Gao et al. (2008). In terms of the point of view of avoiding compressive yield for cylinders experiencing autofrettage handling and raising load-bearing capacity as far as possible simultaneously, Zhu (2008a,b) studied autofrettage of cylinders based on the 3rd and 4th strength theory, respectively. By studying the results published by Zhu (2008a,b), it is found that there are still some problems needed to be studied further, especially when the strength theory (Mises yield criterion) is applied to study the autofrettage for pressure vessels. It is also necessary to pay attention to some problems, such as overstrain to avoid compressive yield, load-bearing capacity, safe conditions for operation, and so on. Based on the work by Zhu (2008a,b), this paper discusses the stresses and depth of the plastic zone (overstrain), which is done in Section 2 and followed by conclusions in Section 3.

2. Discussion about stresses and safe conditions for operation

Figure 1 shows a cross section of a cylinder wall with partially plastic and partially elasticity.

![Cross section of a cylinder wall with marked regions of partial plasticity and partial elasticity](image)

Fig. 1. Cross section of a cylinder wall with marked regions of partial plasticity and partial elasticity

The residual stress at a general radial location within the plastic zone (Zhu, 2008a,b)

\[
\sigma'_r = \frac{1}{\sqrt{3}} \frac{k_j^2}{k^2} + \ln \left( \frac{r/r_i}{k_j^2} \right) \frac{1}{k^2 - 1} - \left( 1 - \frac{k_j^2}{k^2} + \ln k_j^2 \right) \frac{1}{k^2 - 1}
\]

\[
\sigma'_\theta = \frac{1}{\sqrt{3}} \frac{k_j^2}{k^2} + 1 + \ln \left( \frac{r/r_i}{k_j^2} \right) \frac{1}{k^2 - 1} - \left( 1 - \frac{k_j^2}{k^2} + \ln k_j^2 \right) \frac{1}{k^2 - 1} \left( 1 + \frac{k_j^2}{k^2 - 1} \right)
\]

(2.1)

Therefore, the equivalent residual stress at a general radial location within the plastic zone is (Zhu, 2008a,b)
\[
\frac{\sigma'_r}{\sigma_y} = \frac{\sqrt{3}}{2} \left( \frac{\sigma'_\theta}{\sigma_y} - \frac{\sigma'_r}{\sigma_y} \right) = 1 - \frac{k^2 - k_j^2 + k^2 \ln k_j^2}{(k^2 - 1)(r/r_i)^2}
\]  \tag{2.2}
\]

which is the same as that based on the 3rd strength theory (Tresca yield criterion). The reason is as follows.

For a cylinder with closed ends \( \sigma_z = (\sigma_r + \sigma_\theta)/2 \) (Yu, 1990), then, under the identical stress state and according to the 3rd and 4th strength theories, there is: \( \sigma'_c = \sqrt{3} \sigma^\text{III}_c / 2 \), while each component of the residual stresses based on the 4th strength theory is \( 2/\sqrt{3} \) times of that based on the 3rd strength theory (Zhu, 2008a,b) for the same \( k \) and \( k_j \). This is the reason why the equivalent residual stress based on the 3rd strength theory equals that based on the 4th strength theory.

The residual stress at a general radial location within the elastic zone is (Zhu, 2008a,b)

\[
\frac{\sigma'_r}{\sigma_y} = \frac{1}{\sqrt{3}} \left[ \frac{k_j^2}{k^2} - \left( 1 - \frac{k_j^2}{k^2} + \ln k_j^2 \right) \frac{1}{k^2 - 1} \right]
\]

\[
\frac{\sigma'_\theta}{\sigma_y} = \frac{1}{\sqrt{3}} \left( 1 - \frac{k^2}{(r/r_i)^2} \right) \left[ \frac{k_j^2}{k^2} - \left( 1 - \frac{k_j^2}{k^2} + \ln k_j^2 \right) \frac{1}{k^2 - 1} \right] = \left( 1 - \frac{k^2}{(r/r_i)^2} \right) \frac{\sigma'_e}{\sigma_y}
\]  \tag{2.3}

\[
\frac{\sigma'_e}{\sigma_y} = \frac{1}{\sqrt{3}} \left( 1 + \frac{k^2}{(r/r_i)^2} \right) \left[ \frac{k_j^2}{k^2} - \left( 1 - \frac{k_j^2}{k^2} + \ln k_j^2 \right) \frac{1}{k^2 - 1} \right] = \left( 1 + \frac{k^2}{(r/r_i)^2} \right) \frac{\sigma'_e}{\sigma_y}
\]

Therefore, the equivalent residual stress at a general radial location within the elastic zone is

\[
\frac{\sigma'_r}{\sigma_y} = \frac{\sqrt{3}}{2} \left( \frac{\sigma'_\theta}{\sigma_y} - \frac{\sigma'_r}{\sigma_y} \right) = \frac{k^2(k_j^2 - 1) - \ln k_j^2}{(k^2 - 1)(r/r_i)^2}
\]  \tag{2.4}

which is also the same as that based on the 3rd strength theory. The reason is as above mentioned.

In the whole elastic zone \( \sigma'_e/\sigma_y > 0 \) for \( k_j^2 - 1 > \ln k_j^2 \) \cite{14}, but in the plastic zone \( \sigma'_e/\sigma_y \) may be positive or negative. For example, on the inside surface of the cylinder where \( r/r_i = 1 \), from Eq. (2.2), the equivalent residual stress is

\[
\frac{\sigma'_e}{\sigma_y} = \frac{k^2 - 1 - k^2 \ln k_j^2}{k^2 - 1}
\]  \tag{2.5}

Since \( \ln k_j^2 = 2(k_j^2 - 1)/(k_j^2 + 1) + \ldots \) (Zhu, 2008a,b), \( \ln k_j^2 > 2(k_j^2 - 1)/(k_j^2 + 1) \), hence, \( k^2 \ln k_j^2 > 2k^2(k_j^2 - 1)/(k_j^2 + 1) > k_j^2 - 1 \).

This means that on the inside surface of the cylinder \( \sigma'_e/\sigma_y \) is always negative. It is not difficult to know that \( d\sigma'_r/dr > 0 \) within the plastic zone and \( d\sigma'_r/dr < 0 \) within the elastic zone, and, at the elastic-plastic juncture where \( r/r_i = k_j \), both Eq. (2.2) and (2.4) become

\[
0 < \frac{\sigma'_r}{\sigma_y} = \frac{\sigma'_e}{\sigma_y} = \frac{k^2(k_j^2 - 1) - \ln k_j^2}{(k^2 - 1)k_j^2} \leq 1
\]  \tag{2.6}

Therefore, within the plastic zone \( \sigma'_e/\sigma_y \) goes up from the negative value on the inside surface till a positive value at the elastic-plastic juncture, and goes down within the elastic zone from the positive value at the elastic-plastic juncture to a smaller positive value on the outside surface.

In Eq. (2.2), letting

\[
1 - \frac{k^2 - k_j^2 + k^2 \ln k_j^2}{(k^2 - 1)(r/r_i)^2} = 0
\]
one obtains
\[
\frac{r}{r_i} = \sqrt{\frac{k^2 - k_i^2 + k^2 \ln k_j^2}{k^2 - 1}} = \sqrt{\frac{\sigma_{el}'^2}{\sigma_y} + 1} < k_j
\]  
(2.7)

On the other hand, the solution for \( \sigma_z' = \sigma_r', \sigma_r' = \sigma_y' \) and \( \sigma_y' = \sigma_z' \) within the plastic zone (see Eqs. (2.1)) is also Eq. (2.7). This means that the three curves for the residual stress at a general radial location collect at one point within the plastic zone, and the abscissa of the intersection is just Eq. (2.7), where \( \sigma_y' / \sigma_y = 0 \).

Since the depth of the plastic zone \( (k_j) \) affects the residual stresses and the greater the \( k_j \), the greater the equivalent residual stress, so it is inadvisable to raise \( k_j \), otherwise, compressive yield may occur after removing the autofrettage pressure \( p_a \). However, raising \( k_j \) can raise load-bearing capacity of the cylinder thereupon, Zhu (2008a,b) put forward an equation for \( \sigma_y' \), the maximum and optimum plastic depth \( k_j \) for a certain \( k \) to avoid the compressive yield

\[
k^2 \ln k_{j*}^2 - k^2 - k_{j*}^2 + 2 = 0 \quad \text{or} \quad k = \sqrt{\frac{k_{j*}^2 - 2}{\ln k_{j*}^2 - 1}} \quad (k_{j*} \geq \sqrt{\epsilon})
\]  
(2.8)

where \( \sqrt{\epsilon} \leq k_{j*} \leq k_c = 2.2184574899167 \ldots \) and \( k \geq 2.2184574899167 \ldots \), when \( k \leq 2.2184574899167 \ldots = k_c \), \( |\sigma_{el}/\sigma_y| > 1 \) never occurs irrespective of \( k_j \).

When \( k_j \) is determined by Eq. (2.8) \( - k_j = k_{j*} \), Eqs. (2.1)-(2.4) can be rearranged as follows within the plastic zone

\[
\frac{\sigma_z'}{\sigma_y} = \frac{\ln(r/r_i)^2}{\sqrt{3}} - \frac{1}{\sqrt{3}} = \ln \frac{x^2}{\sqrt{3}} - \frac{1}{\sqrt{3}} \quad \frac{\sigma_r'}{\sigma_y} = \frac{\ln x^2}{\sqrt{3}} + \frac{2}{\sqrt{3}x^2} - \frac{2}{\sqrt{3}}
\]

\[
\frac{\sigma_r'}{\sigma_y} = 1 - \frac{2}{(r/r_i)^2}
\]

within the elastic zone

\[
\frac{\sigma_z'}{\sigma_y} = \frac{k^2 - 2}{\sqrt{3}k^2} \quad \frac{\sigma_r'}{\sigma_y} = \left(1 - \frac{k^2}{(r/r_i)^2}\right) \frac{1 - \frac{k^2}{x^2}}{\sigma_y}
\]

\[
\frac{\sigma_r'}{\sigma_y} = \frac{k^2 - 2}{(r/r_i)^2} \quad \frac{\sigma_y'}{\sigma_y} = \left(1 + \frac{k^2}{(r/r_i)^2}\right) \frac{1 + \frac{k^2}{x^2}}{\sigma_y}
\]  
(2.10)

where \( x \) represents \( r/r_i \), the same below.

By taking \( k = 3 \), \( k_j = 1.5, 1.748442 \) and 2, respectively, the residual stresses are illustrated as shown in Fig. 2.

In Fig. 2b, \( k_j = 1.748442 = k_{j*} \), as is just determined by Eq. (2.8). \( k_j = 1.5 \neq k_{j*} \) in Fig. 2a, and \( k_j = 2 \neq k_{j*} \) in Fig. 2c or none of them is determined by Eq. (2.8). The former residual stresses are less than those when \( k_j = 1.748442 = k_{j*} \), and the latter residual stresses are greater than those when \( k_j = 1.748442 = k_{j*} \).

Substituting Eq. (2.7) into Eq. (2.9)\(_{1-3}\) one obtains

\[
\frac{\sigma_z'}{\sigma_y} = \frac{\sigma_r'}{\sigma_y} = \frac{\sigma_y'}{\sigma_y} = \frac{\ln 2 - 1}{\sqrt{3}}
\]

When \( k_j = k_{j*} \), Eq. (2.7) becomes \( r/r_i = \sqrt{2} \). This means that when \( k_j = k_{j*} \), for any \( k \) the three curves for the residual stress at a general radial location collect at the same point within the plastic zone, and the intersection is

\[
\left(\sqrt{2}, \frac{\ln 2 - 1}{\sqrt{3}}\right) = (1.414214 \ldots, -0.17716 \ldots)
\]
The residual stresses at the inside surface \((r = r_i)\) can be obtained by letting \(r = r_i\) in Eqs. (2.1) or (2.3)

\[
\frac{\sigma'_{zi}}{\sigma_y} = -\frac{1}{\sqrt{3}(k^2 - 1)} \left( 1 - k_i^2 + k^2 \ln k_j^2 \right) < 0 \quad \quad \sigma'_{ri} = 0
\]

\[
\frac{\sigma'_{bi}}{\sigma_y} = -\frac{2}{\sqrt{3}(k^2 - 1)} \left( 1 - k_i^2 + k^2 \ln k_j^2 \right) = 2 \frac{\sigma'_{zi}}{\sigma_y} = \sigma'_{ei} = \frac{\sigma'_{ji}}{\sigma_y} < -1 \quad \quad (2.11)
\]

The equivalent residual stress at \(r_i\) is

\[
\frac{\sigma'_{zi}}{\sigma_y} = \frac{\sqrt{3}}{2} \left( \frac{\sigma'_{bi}}{\sigma_y} - \frac{\sigma'_{ri}}{\sigma_y} \right) = \sqrt{3} \frac{\sigma'_{zi}}{\sigma_y} = -\frac{1}{k^2 - 1} \left( 1 - k_i^2 + 2k^2 \ln k_j^2 \right) \quad \quad (2.12)
\]

If \(k_j = k_j^*\), then, Eqs. (2.11) and (2.12) become

\[
\frac{\sigma'_{zi}}{\sigma_y} = -\frac{1}{\sqrt{3}} \quad \quad \frac{\sigma'_{zi}}{\sigma_y} = 0 \quad \quad \frac{\sigma'_{bi}}{\sigma_y} = -\frac{2}{\sqrt{3}} < -1 \quad \quad \frac{\sigma'_{zi}}{\sigma_y} = \frac{\sigma'_{ei}}{\sigma_y} = -1 \quad \quad (2.13)
\]

Therefore, if \(k_j = k_j^*\), the residual stresses and their equivalent stress at the inside surface are always compressive and constant irrespective of \(k\) and \(k_j\) or the thickness and overstrain of the cylinder. The equivalent residual stress at \(r_i\) just reaches \(-\sigma_y\). However, \(\sigma'_{bi}\) exceeds the compressive yield strength \(\sigma_y\). This is an inexorable law, for \(\sigma'_{ei} = \sqrt{3}(\sigma'_{bi} - \sigma'_{ri})/2\), at the inside surface \(\sigma'_{ri} = 0\), then \(\sigma'_{ei} = \sqrt{3}\sigma'_{bi}/2\), when \(\sigma'_{ei}/\sigma_y = -1\), there must be \(\sigma'_{bi} = -2/\sqrt{3}\sigma_y < -\sigma_y\).

The residual stresses at elastic-plastic juncture \((r = r_j)\) can be obtained by letting \(r = r_j\) in Eqs. (2.1) or (2.3)
\[
\frac{\sigma'_{ij}}{\sigma_y} = \frac{1}{\sqrt{3}} \left[ \frac{r_i^2}{r_0^2} - \left( 1 - \frac{r_i^2}{r_0^2} + 2 \ln \frac{r_i}{r_0} \right) \frac{1}{k^2} \left( 1 - \frac{k^2}{k_i^2} + 2 \ln k_j \right) \frac{1}{k^2} \right] = \frac{k^2 - 1 - \ln k_j^2}{\sqrt{3}(k^2 - 1)}
\]

\[
\frac{\sigma'_{ij}}{\sigma_y} = \left( 1 - \frac{k^2}{k_j^2} \right) \frac{\sigma'_{ij}}{\sigma_y} = \frac{1}{1 + \frac{k^2}{k_j^2}} \frac{\sigma'_{ij}}{\sigma_y}
\]

The solution to Eq. (2.18) is

\[
\sigma_y \left( 1 - \frac{k^2}{k_j^2} \right) \sigma_y = \left( 1 - \frac{k^2}{k_j^2} \right) \sigma_y
\]

In Eq. (2.14)\textsubscript{1}, when \( k_j = 1 \), the numerator = 0, and \( d(k_j^2 - 1 - \ln k_j^2)/dk_j = 2(k_j^2 - 1)/k_j > 0 \), so, \( \sigma'_{ij} > 0 \) or tension. Then, \( \sigma'_{ij} < 0 \), \( \sigma'_{ij} > 0 \).

If \( k_j = k_{j^r} \), then Eqs. (2.14) and (2.15) become

\[
\frac{\sigma'_{ij}}{\sigma_y} = \frac{1}{\sqrt{3}} \frac{k^2}{k^2(k^2 - 1)} = \frac{k^2}{1 + 2 \ln k_j}
\]

\[
\sigma_y = \left( 1 - \frac{k^2}{k_j^2} \right) \sigma_y = \frac{k^2}{k^2} \left( 1 - \frac{2k^2}{k_j^2} \right) \sigma_y
\]

It is clear from Eq. (2.15) that \( \sigma'_{ij}/\sigma_y < 1 \), as a result \( \sigma'_{ij}/\sigma_y < 1 \) in Eq. (2.15) or no matter whether \( k_j = k_{j^r} \), the equivalent residual stress at the elastic-plastic juncture is certainly tension. So, it is quite necessary to pay close attention to the residual stresses at the inside surface instead of those at the elastic-plastic juncture.

When \( k = 3, k_{j^r} = 1.748442 \), at the inside surface \( |\sigma'_{ij}| \) and \( |\sigma'_{ij}| \) just reach \( \sigma_y \) according to the 3rd strength theory, but \( |\sigma'_{ij}| \) exceeds \( \sigma_y \) according to the 4th strength theory. This may lower the safety of a pressure vessel. So, it is necessary to limit \( |\sigma'_{ij}| \) to find the maximum \( k_j \) for \( |\sigma'_{ij}| \) to be below \( \sigma_y \), written as \( k_{j^\theta} \). Then, letting

\[
\frac{\sigma'_{ij}}{\sigma_y} = - \frac{2}{\sqrt{3}(k^2 - 1)}(1 - k_j^2 + k^2 \ln k_j^2) = -1
\]

obtains

\[
2k^2 \ln k_j^2 - 2k_j^2 - \sqrt{3}k^2 + 2 + \sqrt{3} = 0 \quad \text{or} \quad k = \sqrt{\frac{2(k_j^2 - 1) - \sqrt{3}}{2 \ln k_j^2 - \sqrt{3}}}
\]

From Eq. (2.17), \( k \to \infty, k_{j^\theta} = e^{\sqrt{3}/4} = 1.541896 \ldots \). Letting \( k_{j^\theta} = k \) in Eq. (2.17), one obtains

\[
\frac{k^2 \ln k^2}{k^2 - 1} = \frac{2 + \sqrt{3}}{2}
\]

The solution to Eq. (2.18) is \( k = 2.024678965 \ldots = k_{\theta^r} \), which means that when \( k \leq k_{\theta^r} \), no matter how deep is the depth of the plastic zone, even when the whole cylinder is yielded \( k_j = k \), \( \sigma'_{ij} \) can not exceed \( \sigma_y \) after removing \( p_a \) from the cylinder. When \( k > k_{\theta^r} \), if \( k_j \) is greater than what is determined by Eq. (2.17) \( (k_j > k_{j^\theta}) \), \( \sigma'_{ij} \) will exceed \( \sigma_y \) after removing \( p_a \). This paper is in the light of the cylindrical pressure vessels with \( k \geq k_{\theta^r} \). The relation between \( k_j \) and \( k \) is shown in Fig. 3, the solution to Eq. (2.8) is shown in this figure too.
The pressure which a cylinder with \( k_j \) can contain is (Yu, 1990)

\[
\frac{p}{\sigma_y} = \frac{2}{\sqrt{3}} \ln k_j + \frac{k^2 - k_j^2}{\sqrt{3} k^2} \tag{2.19}
\]

Combining Eq. (2.17) with Eq. (2.19), one obtains the ultimate allowable loading of the cylinder with \( k_j \) determined by Eq. (2.17) \((k_j = k_j^{\theta})\) or by Fig. 3 under the condition \( \sigma_{\theta i} = -\sigma_y \) \((|\sigma_{ei}| < \sigma_y)\) based on the 4th strength theory

\[
\frac{p}{\sigma_y} = \frac{\sqrt{3} + 2 k^2 - 1}{2 k^2} = \frac{\sqrt{3} + 2 p_e}{2 \sigma_y} \tag{2.20}
\]

Therefore, when \( k > k_{\theta i} \) and under the condition \( \sigma_{\theta i} = -\sigma_y \) \((|\sigma_{ei}| < \sigma_y)\), the load-bearing capacity of the autofrettaged cylinder should be determined by Eq. (2.20). The value determined by Eq. (2.20) is less than that determined by another equation proposed by Zhu (2008b) based on the 4th strength theory and when \( k_j = k_j^{\ast} \): \( p_4/\sigma_y = (2/\sqrt{3})(k^2 - 1)/k^2 \) but greater than that proposed by Zhu (2008a) based on the 3rd strength theory and when \( k_j = k_j^{\ast} \): \( p_3/\sigma_y = (k^2 - 1)/k^2 \), and greater than the maximum elastic load-bearing capability of the autofrettaged cylinder: \( p_e/\sigma_y = (k^2 - 1)/(\sqrt{3}k^2) \) or \( 1 + \sqrt{3}/2(= 1.866025) \) times the initial yield pressure of the autofrettaged cylinder. It is seen that \( p = (p_3 + p_4)/2 \), and this is too big to be coincidental – the authors think. Another interesting thing is that letting Eq. (2.20) equal the entire yield pressure \( p_y/\sigma_y \) or \( p/\sigma_y = (2\sqrt{3} + 3)/(6(k^2 - 1)/k^2) = p_y/\sigma_y = (2/\sqrt{3}) \ln k \) one obtains Eq. (2.18). \( p, p_3, p_4 \) and \( p_e \) are plotted in Fig. 4.

![Fig. 3. The depth of plastic zone](image)

![Fig. 4. Comparison between load-bearing capacity](image)
When \( k_j = k_j^* \), uniting Eq. (2.17) and Eq. (2.1)-(2.4), one obtains
\[
\frac{\sigma'_i}{\sigma_y} = \frac{1}{\sqrt[3]{3}} \ln (\frac{r}{r_i})^2 - \frac{1}{2} = \frac{1}{\sqrt[3]{3}} \ln x^2 - \frac{1}{2} \quad \frac{\sigma'_j}{\sigma_y} = \frac{1}{\sqrt[3]{3}} \left( \ln x^2 + \frac{2 + \sqrt{3}}{2x^2} - \frac{2 + \sqrt{3}}{2} \right)
\]
\[
\frac{\sigma'_y}{\sigma_y} = \frac{1}{\sqrt[3]{3}} \left( 2 - \frac{2 + \sqrt{3}}{2} - \frac{2 - \sqrt{3}}{2} \right) \quad \frac{\sigma'_e}{\sigma_y} = 1 - \frac{\sqrt{3} + 2}{2(r/r_i)^2} = 1 - \frac{\sqrt{3} + 2}{2x^2}
\]
(2.21)

It is seen that Eqs. (2.21) are unconcerned with \( k_j \) and \( k \), meaning that when \( \sigma'^i_{\theta i} \) is controlled, or \( k_j = k_{j\theta} \), the residual stresses and their equivalent stress are determined only by the radial relative location \( (r/r_i) \) within the plastic wall of the cylinder and independent of \( k_j \) and \( k \) or at a certain \( r/r_i \), the residual stresses and their equivalent stress within the plastic zone are identical for various \( k_j \) and \( k \). From Eq. (2.21), there are:

when \( x \leq (\sqrt{3} + 2)/2, \sigma'_e \leq 0 \), at the inside surface where \( x = 1, \sigma'_e = -\sqrt{3}/2 \) reaching the minimum for \( d\sigma'_e/dr > 0 \). When \( x > (\sqrt{3} + 2)/2, \sigma'_e > 0 \). Within the whole wall of a cylinder, \( |\sigma'_e| < \sigma_y \). At the elastic-plastic juncture where \( x = k_j, 0 < \sigma'_e/\sigma_y = 1 - (\sqrt{3} + 2)/(2k_j^2) < 1 \), for \( k_j^2 > 1 + \sqrt{3}/2 \), meaning that \( \sigma'_e \) at the elastic-plastic juncture is tension and can not reach \( \sigma_y \).

At \( r/r_i = (\sqrt{3} + 2)/2 = 1.366025 \ldots, \sigma'_e = 0 \), and this is just the location where \( \sigma'^i_{\theta i} = \sigma'^i_z \), that is to say, when \( k_j = k_{j\theta} \), no matter how great \( k \) is, the three curves for the residual stress at a general radial location collect at the same point within the plastic zone, and the intersection is

\[
\left( \frac{\sqrt{3} + 2}{2}, \frac{1}{\sqrt{3}} \ln \frac{\sqrt{3} + 2}{2} - \frac{1}{2} \right) = (1.366025 \ldots, -0.13984 \ldots)
\]

where \( r/r_i = (\sqrt{3} + 2)/2 \) is just Eq. (2.7) when \( k \) and \( k_j \) meet the relation expressed by Eq. (2.17)

\[
\frac{\sigma'_i}{\sigma_y} = \frac{k_j^2 - (\sqrt{3} + 2)/2}{\sqrt{3}k_j^2} \quad \frac{\sigma'_j}{\sigma_y} = \left( 1 - \frac{k_j^2}{(r/r_i)^2} \right) \frac{\sigma'_y}{\sigma_y} = \left( 1 - \frac{k_j^2}{x^2} \right) \frac{\sigma'_y}{\sigma_y}
\]
\[
\frac{\sigma'_z}{\sigma_y} = \frac{k_j^2 - (\sqrt{3} + 2)/2}{x^2}
\]
(2.22)

The distribution of equivalent residual stress within the whole wall is demonstrated in Fig. 5, the related parameters are shown in the figure as well.

\( k_j = 1.748442 \) for \( k = 3 \) in Fig. 2b is determined by Eq. (2.8) or \( k = 1.748442 \ldots \). It is seen from the figure that \( |\sigma'^i_{\theta i}/\sigma_y| = 1 \), but \( |\sigma'^i_{\theta i}/\sigma_y| > 1 \). For \( |\sigma'^i_{\theta i}/\sigma_y| = 1 \), letting \( k = 3 \) in Eq. (2.17), one obtains \( k_j = k_{j\theta} = 1.603502 \ldots \). Similarly, letting \( k = 4 \) in Eq. (2.17) one finds \( k_{j\theta} = 1.571211 \ldots \), letting \( k = 4 \) in Eq. (2.8) \( k_{j\theta} = 1.694172 \ldots \). For \( k = 3, k_{j\theta} = 1.748442 \ldots \) and \( k = 3, k_{j\theta} = 1.603502 \ldots \), the residual stresses in three directions and their equivalent stress are plotted in Fig. 6a. For \( k = 4, k_{j\theta} = 1.694172 \ldots \) and \( k = 4, k_{j\theta} = 1.571211 \ldots \), the residual stresses in the three directions and their equivalent stress are plotted in Fig. 6b.

The stresses caused by the internal pressure \( p \) at a general radius location are
\[
\frac{\sigma'^i_p}{\sigma_y} = \frac{1}{k^2 - 1} \frac{p}{\sigma_y} \quad \frac{\sigma'^i_p}{\sigma_y} = \left( 1 - \frac{k^2}{(r/r_i)^2} \right) \frac{\sigma'^i_p}{\sigma_y} = \left( 1 + \frac{k^2}{(r/r_i)^2} \right) \frac{\sigma'^i_p}{\sigma_y}
\]
(2.23)

The equivalent stress of the stresses caused by the internal pressure \( p \) is
\[
\frac{\sigma'^i_p}{\sigma_y} = \frac{\sqrt{3}}{2} \left( \frac{\sigma'^i_p}{\sigma_y} - \frac{\sigma'^i_{\theta i}}{\sigma_y} \right) = \frac{\sqrt{3}k^2}{k^2 - 1} \frac{p}{\sigma_y} \left( \frac{r}{r_i} \right)^{-2}
\]
(2.24)
On autofrettage of cylinders by limiting circumferential residual stress ...

The equivalent stress of the total stress $\sigma^T/\sigma_y$ (ETS) is

$$\frac{\sigma^T}{\sigma_y} = \frac{\sqrt{3} \sigma_y}{2} \left( \frac{\sigma_y + \sigma_y'^*}{\sigma_y} \right) = \frac{\sqrt{3}}{2} \left( \frac{\sigma_y' - \sigma_r'}{\sigma_y} \right) = \frac{\sigma_e}{\sigma_y} + \frac{\sigma_e'}{\sigma_y} \quad (2.25)$$

Then, within the plastic zone

$$\frac{\sigma^T}{\sigma_y} = 1 - \frac{k^2 - k_j^2 + k^2 \ln k_j^2}{(k^2 - 1)(r/r_i)^2} + \frac{\sqrt{3}k^2 p}{(r/r_i)^2} \left( \frac{k^2}{k^2 - 1} \right)$$

within the elastic zone

$$\frac{\sigma^T}{\sigma_y} = \frac{k^2(k^2 - 1 - \ln k_j^2)}{(k^2 - 1)(r/r_i)^2} + \frac{\sqrt{3}k^2 p}{k^2 - 1} \left( \frac{r}{r_i} \right)^{-2} \quad (2.27)$$

At the elastic-plastic juncture ($r/r_i = k_j$), Eqs. (2.26) and (2.27) both become

$$\frac{\sigma^T}{\sigma_y} = \frac{k^2(k^2 - 1 - \ln k_j^2)}{(k^2 - 1)k_j^2} + \frac{\sqrt{3}k^2 p}{k^2 - 1} \frac{1}{k_j^2} \quad (2.28)$$

If $k_j = k_j^\theta$, Eqs. (2.26) and (2.27) become respectively

$$\frac{\sigma^T}{\sigma_y} = 1 - \frac{\sqrt{3} + 2}{2x^2} + \frac{\sqrt{3}k^2 p}{k^2 - 1} \frac{1}{x^2} \quad (2.29)$$
From Eq. (2.29)1, it is known that:

1. provided \( p/\sigma_y > -|4x^2 - (\sqrt{3} + 2)|/(2\sqrt{3})(k^2 - 1)/k^2 \) (negative), \( \sigma_e^T/\sigma_y > -1 \), this is definitely feasible for \( p/\sigma_y > 0 \) in engineering;
2. as long as \( p/\sigma_y > \left[ (\sqrt{3} + 2) - 2x^2 \right]/(2\sqrt{3})(k^2 - 1)/k^2 \), \( \sigma_e^T > 0 \), while \( \left[ (\sqrt{3} + 2) - 2x^2 \right]/(2\sqrt{3})(k^2 - 1)/k^2 < p_c/\sigma_y \), so when \( p > p_c \), \( \sigma_e^T > 0 \);
3. so long as \( p/\sigma_y \leq (\sqrt{3} + 2)/(2\sqrt{3})(k^2 - 1)/k^2 = (\sqrt{3} + 2)/2p_c/\sigma_y, \sigma_e^T \leq \sigma_y \). When \( x = 1 \), if \( p/\sigma_y = (k^2 - 1)/2k^2 = (\sqrt{3}/2)p_c/\sigma_y \), then \( \sigma_e^T/\sigma_y = 0 \).

From Eq. (2.29)2:

1. provided \( p/\sigma_y > -[k_j^2 - (\sqrt{3} + 2)/2]/\sqrt{3}(k^2 - 1)/k^2 \) (negative), \( \sigma_e^T/\sigma_y > 0 \), this is definitely feasible for \( p/\sigma_y > 0 \) in engineering, so the equivalent residual stress within the elastic zone is always tension;
2. as long as \( p/\sigma_y < [x^2 - k_j^2 + (\sqrt{3} + 2)/2]/\sqrt{3}(k^2 - 1)/k^2 \), \( \sigma_e^T < \sigma_y \), so when \( p/\sigma_y \leq (\sqrt{3} + 2)/(2\sqrt{3})(k^2 - 1)/k^2 = (\sqrt{3} + 2)/2p_c/\sigma_y, \sigma_e^T \leq \sigma_y \).

At the inside surface, \( x = r/r_i = 1 \), then, from Eq. (2.29)1

\[
\frac{\sigma_e^T}{\sigma_y} = \frac{\sqrt{3}k^2}{k^2-1}\cdot \frac{p}{\sigma_y} - \frac{\sqrt{3}}{2}
\]

(2.30)

Unless \( p/\sigma_y < -(2 - \sqrt{3})/(2\sqrt{3})(k^2 - 1)/k^2 \) (negative), \( \sigma_e^T \) can not be lower than \( -\sigma_y \). Unless \( p/\sigma_y > 2p_c/\sigma_y \), \( \sigma_e^T/\sigma_y \) can not be higher than \( \sigma_y \). So, when \( 0 \leq p/\sigma_y \leq (\sqrt{3} + 2)/2p_c/\sigma_y, \sigma_e^T/\sigma_y \leq 1 \). Especially, when \( p/\sigma_y = (\sqrt{3} + 2)/2p_c/\sigma_y, \sigma_e^T/\sigma_y \equiv 1 \) within the whole plastic zone.

At the elastic-plastic juncture, \( x = r/r_i = k_j \), from (2.29)1 or (2.29)2

\[
\frac{\sigma_e^T}{\sigma_y} = \frac{k_j^2 - (\sqrt{3} + 2)/2}{k_j^2} + \frac{\sqrt{3}k^2}{k^2-1}\cdot \frac{p}{\sigma_y}
\]

(2.31)

Clearly, \( \sigma_e^T > 0 \) within the whole elastic zone. If \( p/\sigma_y \leq (\sqrt{3} + 2)/2p_c/\sigma_y \), \( \sigma_e^T \) can not be higher than \( \sigma_y \). So, when \( 0 \leq p/\sigma_y \leq (\sqrt{3} + 2)/2p_c/\sigma_y, 0 < \sigma_e^T/\sigma_y < 1 \). Especially, when \( p/\sigma_y = (\sqrt{3} + 2)/2p_c/\sigma_y, \sigma_e^T/\sigma_y = 1 \) at the elastic-plastic juncture and \( \sigma_e^T/\sigma_y = k_j^2/x^2 \) at a general radial location within the elastic zone.

When \( p/\sigma_y = (\sqrt{3} + 2)/2p_c/\sigma_y \) or the load-bearing capacity is determined by Eq. (2.20), Eqs. (2.29) become respectively

\[
\frac{\sigma_e^T}{\sigma_y} = 0 \quad \frac{\sigma_e^T}{\sigma_y} = \frac{k_j^2}{x^2}
\]

(2.32)

Figure 7 shows the distribution of the equivalent stress of the total stress.

The supplemental explanation for Fig. 7 is as follows.

1. Horizontal line \( bao \): \( k = 2.024678965 \ldots = k_{e\theta}, k_j = k = k_{j\theta} = 2.024678965 \ldots \).

   Within the plastic zone, \( \sigma^T/\sigma_y \) is a horizontal line: \( \sigma^T/\sigma_y = 1 \), \( x \) varies from 1 to \( k_{j\theta}(= k_{e\theta} = 2.024678965 \ldots ) \) (from point \( b \) to \( a \)), and then from 2.024678965 \ldots \( (k_{j\theta}) \) to 2.024678965 \ldots \( (k) \) (from point \( a \) to \( a \)) within the elastic zone (no elastic zone, the “curve” of the equivalent stress of the total stress is actually a point within the elastic zone).

2. Curve \( bcd \): \( k = 2.1 \ldots, k_{j\theta} = 1.806908 \ldots \). Within the plastic zone, \( \sigma^T/\sigma_y \) is a horizontal line: \( \sigma^T/\sigma_y = 1 \), \( x \) varies from 1 to \( k_{j\theta}(1.806908 \ldots ) \) (from point \( b \) to \( c \)), and then from 1.806908 \ldots \( \) to 2.1 \ldots \( (k) \) (from point \( c \) to \( d \)) within the elastic zone.
Fig. 7. Distribution of the equivalent stress of the total stress for $p/\sigma_y = (\sqrt{3} + 2)/2p_e/\sigma_y$ and $k_j = k_{j\theta}$

(3) Curve bef: $k = 2.5 \ldots, k_{j\theta} = 1.6522121 \ldots$. Within the plastic zone, $\sigma^T/\sigma_y$ is a horizontal line: $\sigma^T/\sigma_y = 1, x$ varies from 1 to $k_{j\theta}(1.6522121 \ldots)$ (from point b to e), and then from 1.6522121 \ldots to $2.5 \ldots(k)$ (from point e to f) within the elastic zone.

(4) Curve bgf: $k = 3, k_{j\theta} = 1.60350225 \ldots$. Within the plastic zone, $\sigma^T/\sigma_y$ is a horizontal line: $\sigma^T/\sigma_y = 1, x$ varies from 1 to $k_{j\theta}(1.644363 \ldots)$ (from point b to g), and then from 1.644363 \ldots to 3(k) (from point g to h) within the elastic zone.

(5) Curve bkl: $k = 4, k_{j\theta} = 1.57121054 \ldots$. Within the plastic zone, $\sigma^T/\sigma_y$ is a horizontal line: $\sigma^T/\sigma_y = 1, x$ varies from 1 to $k_{j\theta}(1.57121054 \ldots)$ (from point b to k), and then from 1.57121054 \ldots to 4(k) (from point k to l) within the elastic zone.

(6) Curve bmn: $k \to \infty, k_{j\theta} = e^{\sqrt{3}/4}$. Within the plastic zone, $\sigma^T/\sigma_y$ is a horizontal line: $\sigma^T/\sigma_y = 1, x$ varies from 1 to $k_{j\theta} = e^{\sqrt{3}/4} = 1.541896 \ldots$ (from point b to m), and then from $k_{j\theta}$ to $\infty(k)$ (from point m to n) within the elastic zone.

The prerequisite to the above arguments is that $k$ and $k_j$ meet Eq. (2.17) or $k_j = k_{j\theta}$ and $p/\sigma_y = (\sqrt{3} + 2)/(2\sqrt{3})(k^2 - 1)/(k^2) = (\sqrt{3} + 2)/2p_e/\sigma_y$. Grasping these laws is helpful to the design of high and ultrahigh pressure vessels. If $k_j \neq k_{j\theta}$ or $p/\sigma_y \neq (\sqrt{3} + 2)/(2\sqrt{3})(k^2 - 1)/k^2$, the above facts are untenable, and $|\sigma^T|_T$ may exceed $\sigma_y$.

When $p/\sigma_y = (\sqrt{3} + 2)/2p_e/\sigma_y$, Eqs. (2.23) and (2.24) become

$$\frac{\sigma^p}{\sigma_y} = \frac{\sqrt{3} + 2}{2\sqrt{3}k^2}, \quad \frac{\sigma^p}{\sigma_y} = \frac{\sqrt{3} + 2}{2\sqrt{3}k^2} - \frac{\sqrt{3} + 2}{2\sqrt{3}k^2} \quad (2.33)$$

When $p/\sigma_y = (\sqrt{3} + 2)/2p_e/\sigma_y$ and $k_j = k_{j\theta}$, the components of the total stresses are:

— within the plastic zone

$$\frac{\sigma^T}{\sigma_y} = \frac{\ln x^2 - 1}{2\sqrt{3}k^2} + \frac{\sqrt{3} + 2}{2\sqrt{3}k^2}, \quad \frac{\sigma^T}{\sigma_y} = \frac{\ln x^2}{\sqrt{3}k^2} + \frac{\sqrt{3} + 2}{2\sqrt{3}k^2} - \frac{\sqrt{3} + 2}{2\sqrt{3}} \quad (2.34)$$

From Eqs. (2.34) 2 and (2.34) 3, $\sigma^T_e/\sigma_y = (\sqrt{3}/2)[\sigma^T_e/\sigma_y - \sigma^T_e/\sigma_y] \equiv 1$, the same as from Eq. (2.32) 1. This proves the above arguments are correct and reasonable.
Within the elastic zone
\[
\frac{\sigma_z^T}{\sigma_y} = \frac{\sigma_z}{\sigma_y} + \frac{\sigma_y^p}{\sigma_y} = \frac{k_j^2}{\sqrt{3}k^2}, \quad \frac{\sigma_x^T}{\sigma_y} = \frac{\sigma_x}{\sigma_y} + \frac{\sigma_y^p}{\sigma_y} = -\frac{k_j^2(x^2 - k^2)}{\sqrt{3}}
\]
\[
\frac{\sigma_y^T}{\sigma_y} = \frac{\sigma_y}{\sigma_y} + \frac{\sigma_y^p}{\sigma_y} = \frac{k_j^2(x^2 + k^2)}{\sqrt{3}}.
\]

From Eqs. (2.35)_2 and (2.35)_3, \(\sigma_y^T/\sigma_y = k_j^2/x^2\), the same as from Eq. (2.32)_2. This as well proves that the above arguments are correct and reasonable.

Figure 8 shows a comparison between the equivalent stresses of the total stresses under different internal pressures and \(k_j = k_j^0\), from which it is known that only when \(p/\sigma_y = (\sqrt{3} + 2)/(2\sqrt{3})(k^2 - 1)/k^2 = (\sqrt{3} + 2)/2p_e/\sigma_y\) and \(k_j = k_j^0\), is the operation state optimum, otherwise, or \(p/\sigma_y \neq (\sqrt{3} + 2)/2p_e/\sigma_y\) and/or \(k_j \neq k_j^0\), either \(\sigma_y^T > \sigma_y\) or the load-bearing capacity is lowered or compressive yield occurs. In Fig. 8, curve 1 is just curve bgh in Fig. 7, in this case, \(p/\sigma_y = (\sqrt{3} + 2)/2p_e/\sigma_y\), \(\sigma_y^T \equiv \sigma_y\) within the whole plastic zone. Curve 2: \(p/\sigma_y = (\sqrt{3} + 2.1)/2p_e/\sigma_y\), \(\sigma_y^T > \sigma_y\). Curve 3: \(p/\sigma_y = (\sqrt{3} + 1.9)/2p_e/\sigma_y\), \(0 < \sigma_y^T < \sigma_y\). Curve 4: \(p = p_e\), \(0 < \sigma_y^T < \sigma_y\). Curve 5: \(p/\sigma_y = (k^2 - 1)/2k^2 = (\sqrt{3} + 2)p_e/\sigma_y\), \(\sigma_y^T/\sigma_y = 0\). Curve 6: \(p = 0.8p_e\), \(\sigma_y^T/\sigma_y < 0\).

![Fig. 8. Comparison between the equivalent stresses of the total stresses under different internal pressures and \(k_j = k_j^0\).](image)

Besides, for certain \(k\), when \(k_j < k_j^0\), though the residual stresses are less than when \(k_j = k_j^0\), the load-bearing capacity is dropped. For example, for \(k = 3\), if \(k_j = k_j^0\) (= 1.60350225...), from Eq. (2.19) or (2.20), \(p/\sigma_y = 0.9576...\); while if \(k_j = 1.5\), from Eq. (2.19), \(p/\sigma_y = 0.9012... < 0.9576...\)

### 3. Conclusions

- For \(|\sigma_{xi}'| \leq \sigma_y\), the depth of the plastic zone is \(k^2 \ln k_j^0 + k^2 - k_j^0 + 2 = 0\), where \(\sqrt{e} \leq k_j \leq k_c = 2.2184574899167...\), when \(k \geq 2.2184574899167...\), the load-bearing capacity is
  \[
  \frac{p_4}{\sigma_y} = \frac{2k^2 - 1}{\sqrt{3}k^3} = 2p_e/\sigma_y
  \]

- For \(|\sigma_{bi}'| \leq \sigma_y\), the depth of the plastic zone is \(2k^2 \ln k_j^0 - 2k_j^0 - \sqrt{3}k^2 + (2 + \sqrt{3}) = 0\), where \(e\sqrt{3}/4 \leq k_j \leq k_c = 2.024678965...\), when \(k \geq 2.024678965...\), the load-bearing capacity is
  \[
  \frac{p}{\sigma_y} = \frac{\sqrt{3} + 2k^2 - 1}{2\sqrt{3}k^2} = \frac{\sqrt{3} + 2p_e}{2\sigma_y}
  \]
The data \(2.024678965\ldots\) is the solution to \((k^2 \ln k^2)/(k^2 - 1) = (2 + \sqrt{3})/2\).

- For an autofrettaged cylinder, \(k_j = k_{j\theta}\) with
  \[
  \frac{p}{\sigma_y} = \frac{\sqrt{3} + 2 k^2 - 1}{2 \sqrt{3}} = \frac{\sqrt{3} + 2 p_r}{2 \sigma_y}
  \]
is the optimum operation state — not only safe but also highly capable, under the state \(\sigma^T_e \equiv \sigma_y\) within the plastic zone and \(\sigma^T_e/\sigma_y = k^2_j/x^2 < 1\) within the elastic zone.

- For the same \(k\), \(k_{j\theta} < k^{*}\), as means that the depth of plastic zone necessary to ensure \(|\sigma'_{\theta i}| \leq \sigma_y\) is less than that to ensure \(|\sigma'_{ei}| \leq \sigma_y\).

- Irrespective of \(k\), the three curves for the residual stress at a general radial location collect at the same point within the plastic zone, and the intersection is
  \[
  \left(\sqrt{\frac{3}{2}} + \ln \frac{3}{2} \frac{1}{\sqrt{3}} - \frac{1}{2}\right) = (1.366025\ldots, -0.13984\ldots) \quad \text{when} \quad k_j = k_{j\theta}
  \]
  or
  \[
  \left(\sqrt{2}, \frac{1}{\sqrt{3}} - \ln 2 - \frac{1}{2}\right) = (1.414214\ldots, -0.17716\ldots) \quad \text{when} \quad k_j = k^{*}.
  \]

- When \(\sigma'_{\theta i}\) is controlled, or \(k_j = k_{j\theta}\), the residual stresses and their equivalent stress are determined only by the radial relative location \((r/r_i)\) within the plastic zone and independent of \(k_j\) and \(k\), or once the location \(r/r_i\) is determined, the residual stresses and their equivalent stress within the plastic zone are identical for any \(k_j\) and \(k\).

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**References**


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