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DETERMINATION OF MATERIAL PARAMETERS OF QUASI-LINEAR VISCOELASTIC RHEOLOGICAL MODEL FOR THERMOPLASTICS AND RESINS

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In this study, an algorithm for identification of elastic and viscoelastic constants of a recently developed constitutive equation for thermoplastics and resins is presented. The equation has been applied to model the viscoelastic response of ultra high molecular weigth polyethylene. In order to determine the material parameters, a series of rheological tests has been performed. A set of equations describing one-dimensional processes has been derived and is utilized to determine the material constants. The material parameter identification leads to a set of 9 constants, i.e. 3 constants of elasticity and 6 constants of viscoelasticity. For the determined values of parameters, several validation tests have been performed.

Key words: rheology, constitutive equation, quasi-linear viscoelasticity

1. Introduction

Although the determination of constants of viscoelasticity is a problem widely discussed in numerous papers, e.g. Wilczyński (1968), Bradshaw and Brinson (1997), no universal identification algorithm has been proposed so far. There are many opinions regarding the sort of experimental data and the technique that should be used for material parameter identification.

Both in the case of linear and nonlinear viscoelasticity, the data utilized for determining material constants are usually obtained from creep or relaxation experiments, cf Christensen (1971). It is often assumed that the loading (force-induced or kinematic) is given by the Heaviside function, thus the loading ramp is simply not taken into account during the identification of the material parameters. In linear theory, this assumption is admissible if the ramp time is short compared to the time of the examined rheological process. For nonlinear viscoelastic models, the described simplification may lead to incorrect values of the parameters describing the instantaneous material response. Thus, in the case of nonlinear viscoelastic models, it is usually recommended that a complete deformation history should be used for the parameter identification, e.g. Goh *et al.* (2004).

An increasingly large number of researchers recommend determination of material constants from experiments involving complex loading histories. For instance, Garbarski (1988) proposed a rheological test which includes loading alternation from force-induced to kinematic. The complete deformation paths were used for the parameter identification. Usually, complex load histories are limited to the cases when a closed form solution of the hereditary integral can be found. However, Goh *et al.* (2004) demonstrated that for the class of quasi-linear viscoelastic (QLV) constitutive equations, replacing a hereditary integral with Taylor's recurrence-update numerical formula (Taylor *et al.*, 1970) and using it in the least squares optimization procedure, does not influence the determined values of material constants seriously. Thus, for this particular class of viscoelastic materials, even the complex loading histories for which the hereditary integral has no analytical solution are admissible for the parameter identification.

In the case of linear viscoelasticity, a mixed analytical-numerical algorithm for the parameter identification is often postulated, e.g. Wilczyński (1968). In this method, some of the parameter values are calculated analytically utilizing the experimental data, whereas the remaining constants, usually the characteristic times, are determined numerically using the least squares optimization procedure. The identification of material constants is usually a strongly nonlinear optimization problem. This is the case even for the simplest generic function, i.e. a series of exponentials, also called the Prony series. Thus, in order to simplify the optimization problem, it is often postulated that the values of characteristic times should be predetermined, e.g. Bradshaw and Brinson (1997), Laksari *et al.* (2012), Pawlikowski (2012), and should not participate in the optimization. Usually, it is assumed that the material characteristic times are distributed uniformly in a logarithmic scale of time of the experiment. The major drawback of this approach is that the final set of material parameters does not include the optimal values of the characteristic times.

Ciambella *et al.* (2010) proposed an algorithm for the material parameter identification where the optimization process is launched iteratively with a predetermined set of characteristic times being updated for each successive iteration according to the rule proposed by the authors. This algorithm allows for a better estimation of the material characteristic times without involving them in the least squares optimization.

In this paper, the concept presented by Ciambella *et al.* (2010) is used to develop a computer-aided material parameter identification algorithm utilizing Taylor's recurrence-update formula for discretization of the constitutive equation. This combination enables the algorithm to be used for arbitrary deformation histories. A Matlab program has been written based on the developed algorithm. It is applied to the identification of material parameters of the recently developed QLV rheological model (see Suchocki, 2013) corresponding to the mechanical properties of ultra high molecular weigth polyethylene (UHMWPE). An appropriate set of equations describing one-dimensional processes has been derived from the QLV constitutive equation. The developed algorithm is used to determine two separate sets of material constants. The first set of constants is determined by utilizing test data as recommended in the literature (e.g. Goh *et al.*, 2004). The second set of parameters is determined by using a combination of relaxation and cyclic test data. The obtained results are compared.

2. Quasi-linear viscoelastic constitutive model

This point summarizes the basic notions of the QLV model for polymeric materials developed by Suchocki (2013).

The following assumptions are adopted:

- The polymer is a homogeneous and isotropic, nonlinear viscoelastic body.
- The considered deformation processes are isothermal.
- The deformation is restricted to the processes where no plastic strains occur or the plastic strains are negligible.
- The nonlinear viscoelastic material properties are described by a modification of the Boltzmann superposition principle, cf Fung (1981).
- The instantaneous material elastic response follows from the modified Knowles stored-energy potential, cf Knowles (1977), Suchocki (2011).
- The mentioned Prony series is used as the relaxation function.

The formulation of the constitutive equation involves uncoupling of the volumetric and deviatoric stress tensor components.

The total stress is given by the following equation (see Suchocki, 2013)

$$\mathbf{S} = g_\infty \mathbf{S}^e + \sum_{j=1}^{N} \mathbf{H}_j \tag{2.1}$$

where

$$\begin{aligned}
&\mathbf{S}^e = J\frac{\partial U}{\partial J}\mathbf{C}^{-1} + 2J^{-\frac{2}{3}}\,\mathrm{DEV}\Big[\frac{\partial \overline{W}}{\partial \overline{\mathbf{C}}}\Big] \\
&\dot{\mathbf{H}}_j + \frac{1}{\tau_j}\mathbf{H}_j = g_j \dot{\mathbf{S}}^e \qquad j = 1,2,\ldots,N \\
&\mathrm{DEV}\,[\bullet] = [\bullet] - \frac{1}{3}([\bullet]\cdot\overline{\mathbf{C}})\overline{\mathbf{C}}^{-1}
\end{aligned} \tag{2.2}$$

The utilized symbols have the following meaning: $\mathbf{S}$ – second Piola-Kirchhoff (P-K) total stress tensor, $\mathbf{S}^e$ – instantaneous second P-K elastic stress tensor, $\mathbf{H}_j$ – viscoelastic overstress tensor ($j = 1,2,\ldots,N$), U – volumetric stored-energy potential, $\overline{W}$ – isochoric stored-energy potential, $\mathbf{C}$ – right Cauchy-Green (C-G) deformation tensor, $\overline{\mathbf{C}}$ – isochoric C-G deformation tensor, J – Jacobian determinant, τ_j – relaxation time ($j = 1,2,\ldots,N$), g_∞ and g_j – relaxation coefficients ($j = 1,2,\ldots,N$), N – assumed number of overstress tensors. The position and time arguments are omitted for convenience.

After time-integrating, Eq. $(2.2)_2$ takes the form of a functional

$$\mathbf{H}_j(t) = \int_0^t g_j e^{-\frac{t-\tau}{\tau_j}} \dot{\mathbf{S}}^e \, d\tau \qquad j = 1,2,\ldots,N \tag{2.3}$$

Thus, by substituting Eq. (2.3) into Eq. (2.1), one can obtain

$$\mathbf{S}(t) = \int_0^t G(t-\tau)\frac{\partial \mathbf{S}^e[\mathbf{E}(\tau)]}{\partial \tau}\, d\tau \tag{2.4}$$

where

$$G(t) = g_\infty + \sum_{j=1}^{N} g_j e^{-\frac{t}{\tau_j}} \tag{2.5}$$

is the reduced relaxation function, cf Fung (1981).

Following the postulate mentioned above, the isochoric stored-energy potential $\overline{W}(\overline{\mathbf{C}})$ is taken in the form of the Knowles model, whereas the volumetric stored-energy $U(J)$ is assumed in the simplest possible form, i.e.

$$U(J) = \frac{1}{D_1}(J-1)^2 \qquad \overline{W}(\overline{\mathbf{C}}) = \frac{\mu}{2b}\Big\{\Big[1 + \frac{b}{\kappa}(\bar{I}_1 - 3)\Big]^{\kappa} - 1\Big\} \tag{2.6}$$

Inserting Eqs (2.6) into Eq. $(2.2)_1$, yields

$$\mathbf{S}^e = \frac{2}{D_1}J(J-1)\mathbf{C}^{-1} + \mu\Big[1 + \frac{b}{\kappa}(\bar{I}_1 - 3)\Big]^{\kappa-1}\Big(J^{-\frac{2}{3}}\mathbf{1} - \frac{1}{3}\bar{I}_1\mathbf{C}^{-1}\Big) \tag{2.7}$$

The symbols used in Eqs (2.6) and (2.7) denote: μ – shear modulus, κ – "hardening" parameter, b – auxiliary constant, D_1 – inverse of the bulk modulus, $\bar{I}_1$ – first invariant of the isochoric right C-G tensor, $\mathbf{1}$ – identity tensor.

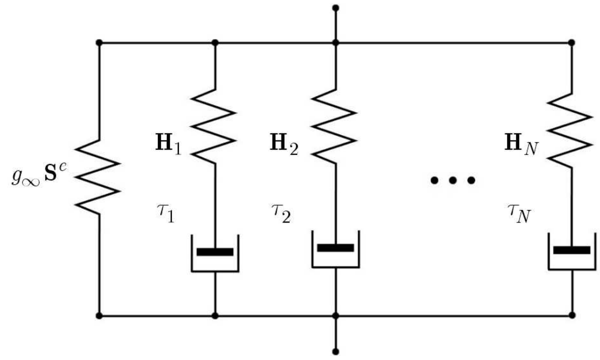

Fig. 1. Mechanical scheme of the rheological model

Summing up, the developed model is described by four elasticity constants and $N+1$ viscoelasticity constants, of which N are independent.

It follows from Eqs (2.1) and $(2.2)_2$ that the presented constitutive equation has an interpretation of the generalized Maxwell model (Fig. 1).

Equations (2.1) and (2.3) can be discretized by taking advantage of the algorithm developed by Taylor *et al.* (1970), i.e.

$$\begin{aligned} \mathbf{S}_{n+1} &= g_\infty \mathbf{S}^e_{n+1} + \sum_{j=1}^{N} \mathbf{H}_{j\,n+1} \\ \mathbf{H}_{j\,n+1} &= e^{-\frac{\Delta t}{\tau_j}} \mathbf{H}_{j\ n} + g_j \frac{1 - e^{-\frac{\Delta t}{\tau_j}}}{\frac{\Delta t}{\tau_j}} \Big(\mathbf{S}^e_{n+1} - \mathbf{S}^e_n \Big) \end{aligned} \tag{2.8}$$

The recurrence-update formula requires stresses at the time increment n to be stored in memory for computation of stresses in the successive time increment $n+1$.

3. Algorithm for identification of elasticity and viscoelasticity constants

In order to develop an algorithm for determination of material parameters, the neccessary equations describing the selected deformation process have to be derived. The uniaxial compression experiments are the easiest to perform by means of the standard material testing toolset. Thus, an appropriate set of scalar equations has been derived, which describes a one-dimensional stress state and a three-dimensional strain state in a QLV body subjected to a compressive loading. The obtained equations are valid for arbitrary uniaxial stress loading histories.

By using Ogden's (1997) approach, the most general equation describing a homogeneous deformation of a continuum takes the form

$$\mathbf{x} = \mathbf{F}\mathbf{X} + \mathbf{c} \tag{3.1}$$

where the following symbols are used: $\mathbf{x}$ – position vector of a particle in the current configuration, $\mathbf{X}$ – position vector of a particle in the reference configuration, $\mathbf{F}$ – deformation gradient, $\mathbf{c}$ – vector representing rigid translation of the whole body.

In the case of a cylindrical specimen subjected to axial compression it is convenient to use one Cartesian basis for analyzing both reference and current configurations. It is assumed that one axis is aligned with the axis of the specimen and the other pair lying in a plane perpendicular to the specimen axis, so as to form a rectangular coordinate system.

Since the specimen does not undergo rigid translational motion, the vector $\mathbf{c}$ is equal to zero. Thus, Eq. (3.1) has the following component form

$$x_1 = \lambda_1 X_1 \qquad x_2 = \lambda_2 X_2 \qquad x_3 = \lambda_3 X_3 \tag{3.2}$$

where λ_k $(k = 1, 2, 3)$ are the principal stretches.

The component matrix of the deformation gradient tensor takes the form

$$\mathbf{F}_{3\times 3} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{3.3}$$

where the stretch ratio λ_1 is chosen to correspond to the axial direction. Thus $\lambda_1 = l/l_0$, where l_0 and l denote initial height of the specimen and height at a certain time instant, accordingly.

Equation (2.4), related to the axial compression of a cylindrical specimen, gives the following relations for the second P-K stress components

$$\begin{aligned} S_{11}(t) &= \int_0^t G(t-\tau)\frac{\partial S^e_{11}(\tau)}{\partial \tau}\,d\tau \qquad & S_{22}(t) &= \int_0^t G(t-\tau)\frac{\partial S^e_{22}(\tau)}{\partial \tau}\,d\tau \\ S_{33}(t) &= \int_0^t G(t-\tau)\frac{\partial S^e_{33}(\tau)}{\partial \tau}\,d\tau \qquad & S_{12}(t) &= S_{23}(t) = S_{31}(t) = 0 \end{aligned} \tag{3.4}$$

where the relations for instantaneous elastic stress components follow from Eq. (2.7) and the diagonal form of the deformation gradient, i.e.

$$\begin{aligned} S^e_{11} &= \frac{2}{D_1}J(J-1)C^{-1}_{11} + \mu\Big[1 + \frac{b}{\kappa}(\bar{I}_1 - 3)\Big]^{\kappa-1}\Big(J^{-\frac{2}{3}} - \frac{1}{3}\bar{I}_1 C^{-1}_{11}\Big) \\ S^e_{22} &= \frac{2}{D_1}J(J-1)C^{-1}_{22} + \mu\Big[1 + \frac{b}{\kappa}(\bar{I}_1 - 3)\Big]^{\kappa-1}\Big(J^{-\frac{2}{3}} - \frac{1}{3}\bar{I}_1 C^{-1}_{22}\Big) \\ S^e_{33} &= \frac{2}{D_1}J(J-1)C^{-1}_{33} + \mu\Big[1 + \frac{b}{\kappa}(\bar{I}_1 - 3)\Big]^{\kappa-1}\Big(J^{-\frac{2}{3}} - \frac{1}{3}\bar{I}_1 C^{-1}_{33}\Big) \\ S^e_{12} &= S^e_{23} = S^e_{31} = 0 \end{aligned} \tag{3.5}$$

with

$$\bar{I}_1 = J^{-\frac{2}{3}}\Big(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\Big) \qquad J = \lambda_1\lambda_2\lambda_3 \tag{3.6}$$

For convenience, the time and position arguments are omitted.

By inserting into Eqs (3.5), the right C-G deformation tensor components expressed by means of the principal stretches, i.e.

$$C^{-1}_{11} = \frac{1}{\lambda_1^2} \qquad C^{-1}_{22} = \frac{1}{\lambda_2^2} \qquad C^{-1}_{33} = \frac{1}{\lambda_3^2} \tag{3.7}$$

and substituting $p = 2(J-1)/D_1$, one obtains

$$\begin{aligned} S^e_{11} &= Jp\frac{1}{\lambda_1^2} + \mu\Big[1 + \frac{b}{\kappa}(\bar{I}_1 - 3)\Big]^{\kappa-1}\Big(J^{-\frac{2}{3}} - \frac{1}{3}\bar{I}_1\frac{1}{\lambda_1^2}\Big) \\ S^e_{22} &= Jp\frac{1}{\lambda_2^2} + \mu\Big[1 + \frac{b}{\kappa}(\bar{I}_1 - 3)\Big]^{\kappa-1}\Big(J^{-\frac{2}{3}} - \frac{1}{3}\bar{I}_1\frac{1}{\lambda_2^2}\Big) \\ S^e_{33} &= Jp\frac{1}{\lambda_3^2} + \mu\Big[1 + \frac{b}{\kappa}(\bar{I}_1 - 3)\Big]^{\kappa-1}\Big(J^{-\frac{2}{3}} - \frac{1}{3}\bar{I}_1\frac{1}{\lambda_3^2}\Big) \end{aligned} \tag{3.8}$$

Due to the symmetry of the specimen and material isotropy, the principal stretches in the directions perpendicular to the axis are equal, i.e., $\lambda_2 = \lambda_3 = \lambda$. The boundary condition of the stress-free lateral surface requires $S_{22} = S_{33} = 0$. In addition, the reference configuration is assumed to be stress-free. It follows from Eqs $(3.4)_2$ and $(3.4)_3$ that these conditions are satisfied only if the instantenous stresses $S^e_{22} = S^e_{33} = 0$. Hence, Eqs $(3.8)_2$ and $(3.8)_3$ are equivalent and allow for determining the term Jp, i.e.

$$Jp = -\mu\Big[1 + \frac{b}{\kappa}(\bar{I}_1 - 3)\Big]^{\kappa-1}\Big(J^{-\frac{2}{3}}\lambda^2 - \frac{1}{3}\bar{I}_1\Big) \tag{3.9}$$

where

$$\bar{I}_1 = J^{-\frac{2}{3}}(\lambda_1^2 + 2\lambda^2) \qquad\qquad J = \lambda_1\lambda^2 \tag{3.10}$$

Substituting Eq. (3.9) into Eq. $(3.8)_1$ yields

$$S^e_{11} = J^{-\frac{2}{3}}\mu\Big[1 + \frac{b}{\kappa}(\bar{I}_1 - 3)\Big]^{\kappa-1}\Big[1 - \Big(\frac{\lambda}{\lambda_1}\Big)^2\Big] \tag{3.11}$$

which is valid for both compressible and incompressible materials.

In the case of material incompressibility, it follows from the constraint $J = 1$, cf Holzapfel (2010), that

$$\lambda^2 = \frac{1}{\lambda_1} \qquad\qquad \bar{I}_1 = I_1 = \Big(\lambda_1^2 + 2\frac{1}{\lambda_1}\Big) \tag{3.12}$$

Thus, by inserting Eqs (3.12) into Eq. (3.11) one finally obtains

$$S^e_{11} = \mu\Big[1 + \frac{b}{\kappa}(I_1 - 3)\Big]^{\kappa-1}\Big(1 - \frac{1}{\lambda_1^3}\Big) \tag{3.13}$$

The Taylor numerical integration algorithm (see Taylor *et al.*, 1970) is utilized to discretize Eqs $(3.4)_1$ and (3.13), thus

$$\begin{aligned}
S_{11}(t_{n+1}) &= g_\infty S^e_{11}(t_{n+1}) + \sum_{j=1}^{N} H_j(t_{n+1})\\
H_j(t_{n+1}) &= e^{-\frac{\Delta t}{\tau_j}} H_j(t_n) + g_j \frac{1 - e^{-\frac{\Delta t}{\tau_j}}}{\frac{\Delta t}{\tau_j}} [S^e_{11}(t_{n+1}) - S^e_{11}(t_n)]\\
S^e_{11}(t_{n+1}) &= \mu\Big[1 + \frac{b}{\kappa}(I_1(t_{n+1}) - 3)\Big]^{\kappa-1}\Big(1 - \frac{1}{\lambda_1^3(t_{n+1})}\Big)\\
T_{11}(t_{n+1}) &= \lambda_1(t_{n+1}) S_{11}(t_{n+1})
\end{aligned} \tag{3.14}$$

where $I_1(t_{n+1}) = \lambda_1^2(t_{n+1}) + 2/\lambda_1(t_{n+1})$, T_{11} is the first P-K stress, and the axial components of the overstress tensors are denoted as H_j $(j = 1, 2, \ldots, N)$.

A discretized time history of λ_1 and corresponding Δt are the input arguments of Eqs (3.14) allowing one to determine the material response. In each time increment, Eq. $(3.14)_3$ is used to calculate the instantenous elastic stress, subsequently Eq. $(3.14)_2$ for $j = 1, 2, \ldots, N$ is used to update the viscoelastic overstresses. The total stress is updated according to Eq. $(3.14)_1$.

Equations (3.14) are utilized within the least squares optimization procedure to calculate the theoretical values of stresses for a given deformation process. The material parameters are found by minimization of the following stress error function

$$\mathcal{F}(\mathbf{p}) = \sum_{k=1}^{M} \Big[\big(T_{11}(\mathbf{p})\big)_k - (\widetilde{T}_{11})_k\Big]^2 \tag{3.15}$$

where $T_{11}(\mathbf{p})$ is the theoretical and $\widetilde{T}_{11}$ is the experimental Lagrange stress at the time instant t_k $(k = 1, 2, \ldots, M)$, M being the number of collocation points, and

$$\mathbf{p} = [\mu, b, \kappa, g_1, g_2, \ldots, g_N, g_\infty]^{\mathrm{T}} \tag{3.16}$$

is a vector of the material parameters involved in the least squares optimization. The subsequent constraints must be satisfied

$$\begin{array}{lll} \mu \geqslant 0 & b \geqslant 0 & \kappa \geqslant 0 \\ 0 \leqslant g_j \leqslant 1 & 0 \leqslant g_\infty \leqslant 1 & j = 1, 2, \ldots, N \end{array} \tag{3.17}$$

with

$$g_\infty + \sum_{j=1}^{N} g_j = 1 \tag{3.18}$$

The reduced relaxation function given by Eq. (2.5) must be monotonically decreasing, thus

$$\tau_j \geqslant 0 \qquad j = 1, 2, \ldots, N \tag{3.19}$$

The concept presented in Ciambella *et al.* (2010) is utilized to define a recurrence-update scheme in which the error function given by Eq. (3.15) is iteratively minimized for a predetermined set of relaxation times τ_j $(j = 1, 2, \ldots, N)$. The relaxation times are updated for each successive iteration according to a rule specified below.

The initial values of the relaxation times, used in the iteration $i = 1$, are distributed uniformly in a logarithmic scale within the range $\langle \tau_{min}, \tau_{max} \rangle$, i.e.

$$\tau_1^{(i=1)} = \tau_{min}, \ldots, \tau_j^{(i=1)} = \tau_{min} 10^{(j-1)\Delta}, \ldots, \tau_N^{(i=1)} = \tau_{max} \tag{3.20}$$

where τ_{min} is set equal to the sampling interval, τ_{max} is equal to the total time of the experiment and Δ is the logarithimic time step.

The logarithmic step and the initial values of the viscoelastic coefficients are defined as follows

$$\Delta = \frac{1}{N-1} \log \frac{\tau_{max}}{\tau_{min}} \qquad g_j^{(i=1)} = g_\infty^{(i=1)} = \frac{1}{N+1} \qquad j = 1, 2, \ldots, N \tag{3.21}$$

After minimization of $\mathcal{F}(\mathbf{p})$ at a least squares optimization step, all the viscoelastic coefficients g_j $(j = 1, 2, \ldots, N)$ are checked upon overcoming a predefined threshold. If a coefficient is below the threshold, then both the given coefficient and the corresponding relaxation time are discarded. If the viscoelastic coefficient overcomes the threshold, then the corresponding relaxation time is split according to the rule

$$\tau_j^{(i)} \Rightarrow \tau_j^{(i)} \Big\{ 10^{-\left(\frac{\Delta}{3}\right)^i}, 1, 10^{\left(\frac{\Delta}{3}\right)^i} \Big\} \qquad j = 1, 2, \ldots, N \tag{3.22}$$

where i denotes the number of the optimization step.

The initial values of the viscoelastic coefficients associated with the new relaxation times are set to zero. Subsequently, the least squares optimization procedure is launched again. This process is repeated iteratively in a loop until the stop condition is satisfied.

4. Experimental set-up and rheological tests

The material chosen for the rheological tests was CHIRULEN 1050 medical ultra high molecular weight polyethylene (UHMWPE). UHMWPE is a popular biomaterial with numerous applications as hip joint implants, elbow implants or artificial intervertebral discs for instance. The usual role of the elements made of UHMWPE is dissipating the mechanical energy, thus making an implant less vulnerable to a failure. The nonlinear viscoelastic properties of UHMWPE are representative for a large number of polymers, aspecially thermoplastics and resins.

Cylindrical specimens of dimensions $\emptyset 17$ and $l_0 = 21$ mm were machined from a rod made of CHIRULEN 1050. The tests were performed on MTS Bionix 2500 testing machine at the constant room temperature of 20°C. Two types of tests were performed, i.e. loading/unloading compression experiments and relaxation tests. The lowest deformation rate used was $0.00023\,\mathrm{s}^{-1}$ whereas the maximum was $0.03\,\mathrm{s}^{-1}$. During the relaxation tests the specimens were allowed to relax for 30 minutes.

The strain measurement method utilizes a video extensometer. The elements of the measuring system are a CCD camera (Charge Coupled Device) with a resolution of 1600×1200 pixels, a camera lens with focal length of 35 mm. The axial deformation was additionally measured with a strain gauge extensometer in order to verify the measured values. A sketch of the experimental setup can be seen in Fig. 2.

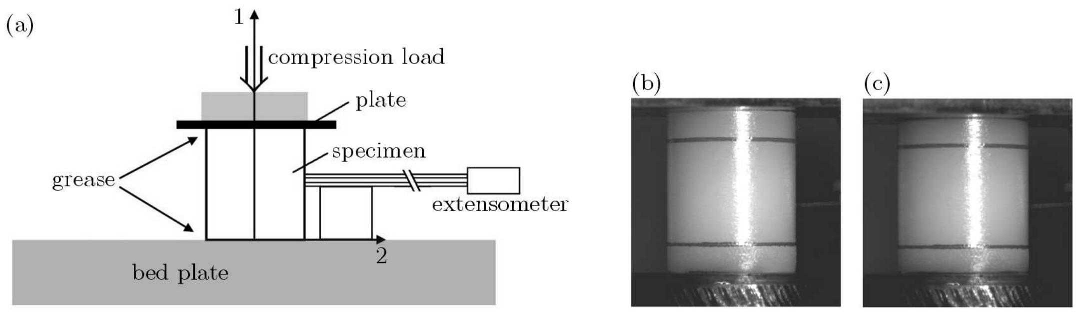

Fig. 2. (a) Experimental set-up, (b) view of undeformed specimen, (c) view of deformed specimen

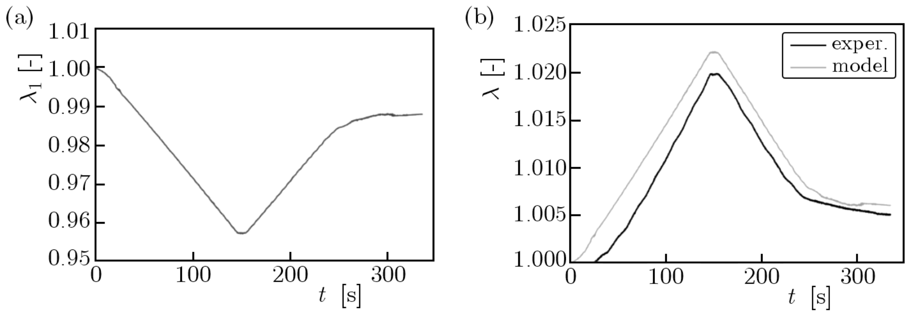

Fig. 3. Exemplary deformation histories recorded by video extensometer; (a) axial stretch ratio versus time, (b) transverse stretch ratio versus time

In Fig. 3, the principal stretches recorded by the video extensometer during a compression loading/unloading experiment are presented. The experimental values of the principal stretch in the transverse direction were compared to the theoretical values calculated from Eq. $(3.12)_1$. The comparison shows good agreement, thus the assumption of incompressibility is justified in the case of UHMWPE.

5. Identification of material parameters for uniaxial compression and relaxation tests data

The procedure proposed by Goh *et al.* (2004) was applied to determine the parameters of incompressible QLV model for UHMWPE. The initial values of the constants were evaluated using a Matlab program based on the algorithm discussed in Section 3. The relaxation test data were utilized. Subsequently, the elasticity constants were recalculated to fit the test data from compression experiments performed at different deformation rates. The results of curve fitting can be seen in Fig. 4.

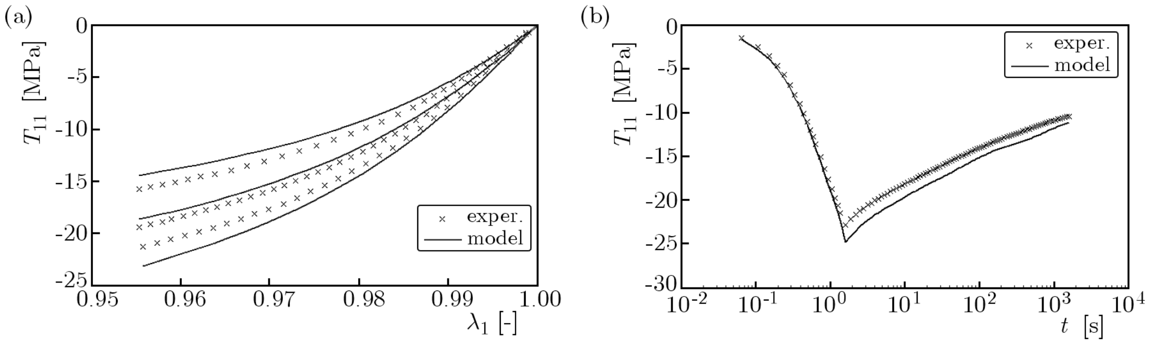

Fig. 4. Fitting of quasi-linear viscoelastic material model to experimental data; (a) compression ramp tests with deformation rate of: 0.0003, 0.003 and $0.03\,\mathrm{s}^{-1}$, (b) stress relaxation test

The identification led to a set of 15 independent parameters, i.e., 3 elasticity constants and 12 viscoelasticity constants. The parameters have been gathered in Table 1.

Table 1. Incompressible QLV material parameters of UHMWPE

Elastic constants	Viscoelastic coefficients	Relaxation time [s]
$\mu = 296.13\,\mathrm{MPa}$	$g_1 = 0.096$	$\tau_1 = 0.49$
$b = 314.85$	$g_2 = 0.12$	$\tau_2 = 0.97$
$\kappa = 0.67$	$g_3 = 0.034$	$\tau_3 = 3.73$
	$g_4 = 0.10$	$\tau_4 = 7.34$
	$g_5 = 0.16$	$\tau_5 = 55.81$
	$g_6 = 0.15$	$\tau_6 = 834.49$

The constant D_1 did not participate in the optimization. In FE implementation, it was assumed that $D_1 = 33\text{E-}9\,\mathrm{MPa}^{-1}$, which corresponds to the bulk modulus high enough to account for material near incompressibility.

In order to verify the performance of the model, a complex multistep rheological test was performed. The comparison of the experimental results and model predictions can be seen in Fig. 5.

6. Identification of material parameters for uniaxial compressive loading/unloading and relaxation tests data

In the second approach, the material parameters were evaluated from compressive loading/unloading and relaxation test data.

Firstly, the algorithm discussed in Section 3 was utilized in curve-fitting of the relaxation data. Subsequently, both elasticity constants and viscoelastic coefficients were recalculated by fitting the loading/unloading data. The results of the optimization are shown in Fig. 6.

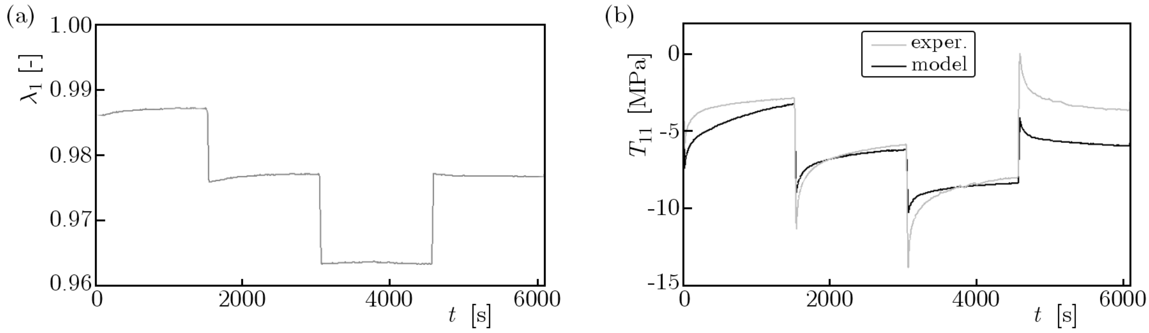

Fig. 5. Model validation; (a) experimental deformation history, (b) material response, comparison of experimental data and model predictions

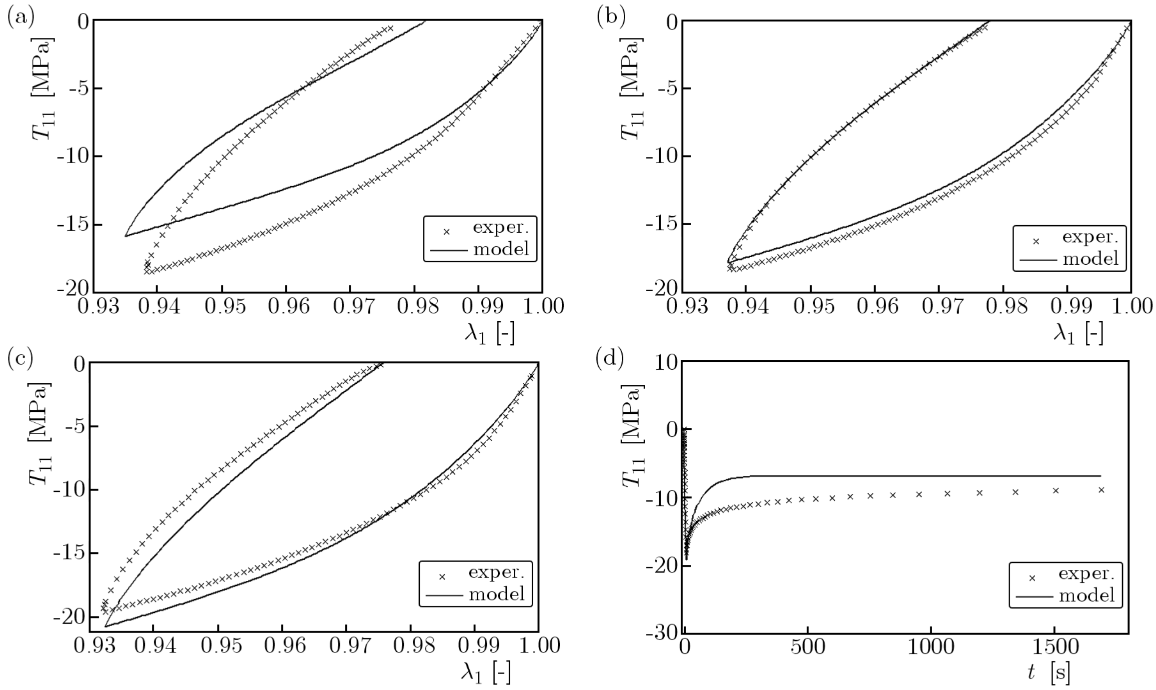

Fig. 6. Fitting of quasi-linear viscoelastic material model to experimental data. Cyclic compression tests with deformation rate of: (a) 0.00023, (b) 0.00038 and (c) 0.0005 s^{-1}, (d) stress relaxation test

The identification led to a set of 9 independent parameters, i.e., 3 elasticity constants and 6 viscoelasticity constants. The parameters have been gathered in Table 2. Again, they are supplemented by $D_1 = 33\text{E-}9\,\text{MPa}^{-1}$.

Table 2. Incompressible QLV material parameters of UHMWPE

Elastic constants	Viscoelastic coefficients	Relaxation time [s]
$\mu = 309.85\,\text{MPa}$	$g_1 = 0.14$	$\tau_1 = 3.73$
$b = 247.93$	$g_2 = 0.19$	$\tau_2 = 7.34$
$\kappa = 0.79$	$g_3 = 0.41$	$\tau_3 = 55.81$

In order to verify the performance of the model, several validation tests were performed including a complex, multistep rheological test and compression loading/unloading experiments with varied deformation rates. The comparison of the experimental results and model predictions can be seen in Fig. 7 and Fig. 8.

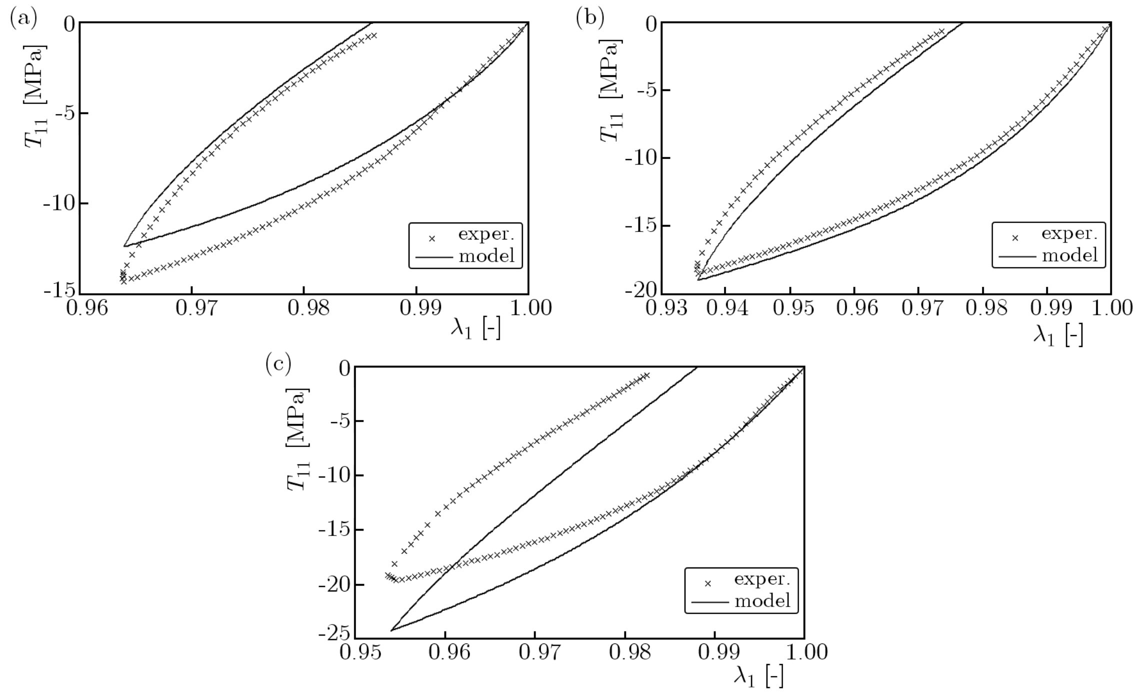

Fig. 7. Validation of quasi-linear viscoelastic material model. Experimental data and model predictions for cyclic compression tests with deformation rate of: (a) 0.00028, (b) 0.00046 and (c) $0.0022\,\mathrm{s}^{-1}$

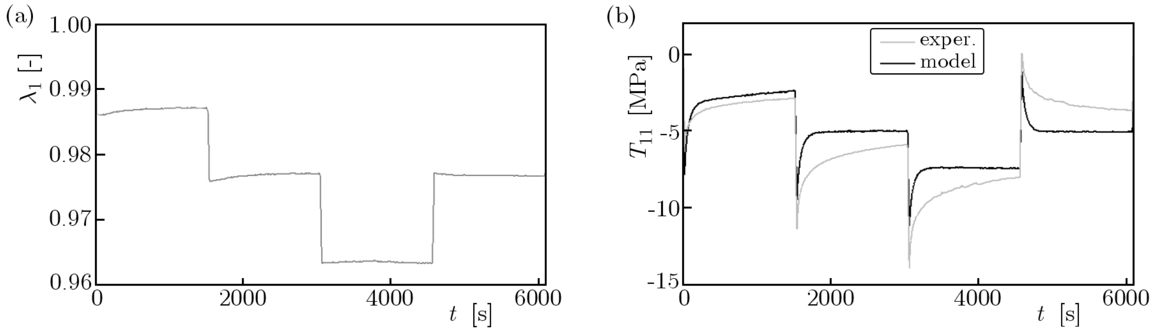

Fig. 8. Model validation; (a) experimental deformation history, (b) material response, comparison of experimental data and model predictions

7. Conclusions

In the paper, an algorithm allowing one to identify the material constants of the QLV constitutive equation is presented. The algorithm combines the concept of iterative determination of relaxation times with discretization of QLV model by Taylor's recurrence-update formula (Taylor *et al.*, 1970). This enables evaluating the constants from arbitrary deformation histories.

Two different experimental data sets were used for the curve-fitting. In the first approach, the data from compression tests performed at different deformation rates were used along with the relaxation test data, as proposed in Goh *et al.* (2004). For the determined set of constants, the model predictions are in a good agreement with the experimental data, which was confirmed by the complex validation test (see Figs 4 and 5). However the number of material parameters is large (15 independent constants). What is more, it was found that for loading/unloading simulations the error of predicted energy dissipation is significant. Thus in the second approach the loading/unloading data were used for parameter evaluation. The spectrum of deformation rates was narrowed. This resulted in the number of constants decreasing to 9 and obtaining much better prediction of the energy dissipation, which was confirmed by the validation tests (see Figs 6, 7 and 8).

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