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FACE WRINKLING OF SANDWICH BEAMS UNDER PURE BENDING

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The paper is devoted to sandwich beams under pure bending. The local buckling problem is analysed. The analytical description of the upper face wrinkling is proposed. From the principle of stationary total potential energy, formulae describing critical stresses in the faces of the beam are derived. The algorithm for determining the critical stresses is shown. Two particular cases of the solution following the core properties are mentioned. The finite element model of the sandwich beam is formulated. The comparison of the results obtained from the proposed analytical model and from FEM analysis is shown for a family of sandwich beams with different thicknesses and core properties.

Key words: sandwich beam, local buckling, wrinkling, elastic foundation

1. Introduction

A typical sandwich structure consist of thin stiff faces, upper and lower, and a light flexible core. The monographs concerning theoretical investigation of such structures are these, for example, by Plantema (1966), Allen (1969), Libove and Butdorf (1948) and Reissner (1948). The core of a sandwich element can be made of polyurethane foam, metal foam or thin shaped metal sheet – corrugated core. In the case of a soft core, for which Young's modulus is at a level of 10 MPa, the stiff face may be treated as a beam on an elastic foundation. This phenomenon has been described in many monographs, like those by Życzkowski (1988), Woźniak (2001) or Bažant and Cedolin (1991). Stability problems of beams on an elastic foundation were presented by Vlasov and Leontev (1960). Discrete model of elastic-plastic problem of the beam on foundation is analysed by Chen and Yu (2000).

The behaviour of sandwich members as well as their failure modes may be determined in simple tests like axial compression or bending. As to the first load, the sandwich column may buckle both globally and locally. An example of an analytical model describing global and local buckling of sandwich columns can be found in Léotoing *et al.* (2002). Numerical investigation of this problem was presented by Hadi (2001).

The second load case, the bending, may be realised in two ways. When the beam supported at both ends is loaded with one force, the load scheme is called the three point bending. The collapse mechanism of sandwich beams under this kind of load was analytically described by Steeves and Fleck (2004). The numerical analysis and experimental results of three point bending can be found in Bart-Smith *et al.* (2001).

The three point bending leads usually to the global mode of failure. If local phenomena are under consideration the four point bending is a more adequate load. This kind of load induces pure bending conditions between the applied forces. In sandwich structures it causes local wrinkling of the upper compressed face. Since the local buckling-wrinkling is a small scale phenomenon, the properties of the core are of high importance. When the core is a metallic foam, the cell size and its homogeneity is important since it influences the shear stiffens of the core. Moreover some defects in the foam structure, like large holes, may induce the buckling process (see Kesler and Gibson, 2002; Rakow and Waas, 2005).

In the present paper, the sandwich beam under pure bending is considered. The analytical model of the face wrinkling is proposed. The work is a continuation of two papers by Jasion and Magnucki (2011) and Jasion *et al.* (2011). In the latter, the experimental results on sandwich beams with a metal foam core under axial compression and pure bending are presented. Other papers devoted to this subject are by Koissin *et al.* (2010), in which the influence of physical nonlinearities on the wrinkling of the upper compressed face is analysed and by Stifinger and Rammerstorfer (1997) who presented analytical and FEM analysis of the face wrinkling in shell structures including post-buckling analysis.

2. Analytical analysis of local buckling-wrinkling of the face

2.1. Pre-buckling state

The sandwich beam considered in the paper is simply supported at both ends and loaded with two equal transverse forces F placed symmetrically. The load and support scheme as well as the dimensions of the beam are shown in Fig. 1a. Two forces applied to the beam induce a pure bending state in the area between them.

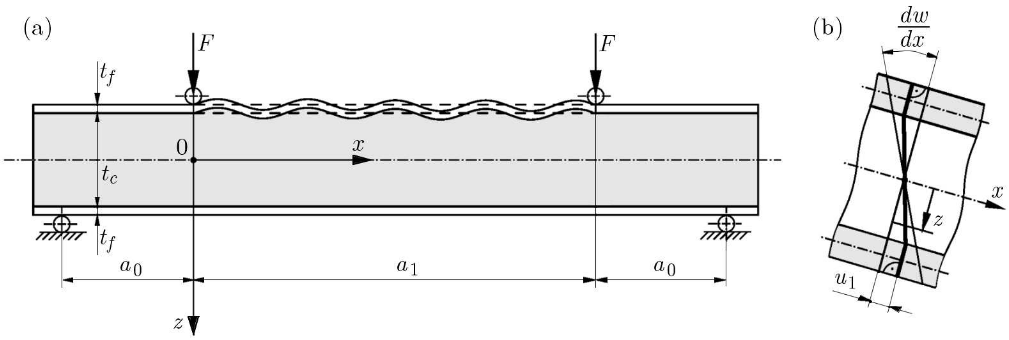

Fig. 1. The scheme of the load and dimensions of the sandwich beam (a); displacements of the particular faces (b)

Since the wrinkling of the upper face is analysed, the internal longitudinal force N_f acting in that face will be determined first. The field of displacements is assumed in accordance with the broken line hypothesis presented *e.g.* by Volmir (1967) (see Fig. 1b). According to that, individual displacement components for three layers have the form

— lower face: $-\Big(\frac{1}{2}+x_1\Big) \leqslant \zeta \leqslant -\frac{1}{2}$

$$u(x,\zeta) = -t_c\Big[\zeta\frac{dw}{dx} + \psi_0(x)\Big] \qquad \gamma_{xz} = 0 \tag{2.1}$$

— core: $-\frac{1}{2} \leqslant \zeta \leqslant \frac{1}{2}$

$$u(x,\zeta) = -t_c\zeta\Big[\frac{dw}{dx} - 2\psi_0(x)\Big] \qquad \gamma_{xz} = 2\psi_0(x) \tag{2.2}$$

— upper face: $\frac{1}{2} \leqslant \zeta \leqslant \frac{1}{2} + x_1$

$$u(x,\zeta) = -t_c\Big[\zeta\frac{dw}{dx} - \psi_0(x)\Big] \qquad \gamma_{xz} = 0 \tag{2.3}$$

where $\zeta = z/t_c$, $x_1 = t_f/t_c$ and $\psi_0(x) = u_1(x)/t_c$ are dimensionless parameters and $-\Big(\frac{1}{2}+x_1\Big) \leqslant \leqslant \zeta \leqslant \frac{1}{2} + x_1$.

The internal bending moment $M_b(x)$ and longitudinal N_f and shear Q forces acting in the beam are defined as follows:
— bending moment

$$\begin{aligned} M_b(x) = \int\limits_A \sigma z\, dA = -\frac{1}{12}E_c b t_c^3 \Big\{ [1 + 2(3 + 6x_1 + 4x_1^2)x_1 e_1]\frac{d^2 w}{dx^2} \\ - 2[1 + 6(1 + x_1)x_1 e_1]\frac{d\psi_0(x)}{dx}\Big\} \end{aligned} \tag{2.4}$$

— longitudinal force

$$N_f = \int\limits_{-\left(\frac{1}{2}+x_1\right)}^{-\frac{1}{2}} b\sigma\, dz = E_f b t_c^2 \Big[\frac{1}{2}(1 + x_1)x_1 \frac{d^2 w}{dx^2} + x_1 \frac{d\psi_0(x)}{dx}\Big] \tag{2.5}$$

— shear force

$$Q = \int\limits_A \tau\, dA = 2G_c b t_c \psi_0(x) \tag{2.6}$$

Since the pure bending of the beam is considered, the shear force Q equals zero. In such a case, from equation (2.6) there is $\psi_0 = 0$. Then the equations (2.4) and (2.5) take the form

$$M_b(x) = -\frac{1}{12}E_c b t_c^3 [1 + 2(3 + 6x_1 + 4x_1^2)x_1 e_1]\frac{d^2 w}{dx^2} \qquad N_f = E_f b t_c^2 \frac{1}{2}(1 + x_1)x_1 \frac{d^2 w}{dx^2} \tag{2.7}$$

The bending moment induced by the couple of forces N_f acting in the upper and lower face can be written

$$M_b^{(N_f)} = -N_f(t_c + t_f) = -\frac{1}{12}E_c b t_c^3 6(1 + x_1)^2 x_1 e_1 \frac{d^2 w}{dx^2} \tag{2.8}$$

From Fig. 1a there is $M_0 = Fa_0$. Assuming that $M_b(x) = M_b^{N_f} = M_0$, the normal force N_f can be finally written

$$N_f = \frac{6(1 + x_1)x_1 e_1}{1 + 2(3 + 6x_1 + 4x_1^2)x_1 e_1}\frac{M_0}{t_c} \tag{2.9}$$

The longitudinal force acting in the lower face is expressed by the same equation, but its value is opposite.

2.2. Local elastic buckling

The sandwich beam under pure bending may buckle locally. The buckling shape has the form of short wrinkles appearing on the upper face. Here it is assumed that the deformation of the upper compressed face has the shape of longitudinal waves of a constant amplitude. The deformation of the core follows the upper face but its magnitude diminishes to zero near the lower face. Additional assumption is that there is no longitudinal displacements. Then the field of displacements can be defined as follows (see Fig. 2a):

$$u(x, z) \equiv 0 \qquad\qquad w(x, z) = w_1 w(z) \sin\frac{m\pi x}{a_1} \tag{2.10}$$

The unknown displacements function $w(z)$, which describes the core deformation has the following boundary conditions (see Fig. 2b)

$$w\Big(-\frac{t_c}{2}\Big) = 1 \qquad\qquad w\Big(\frac{t_c}{2}\Big) = 0 \tag{2.11}$$

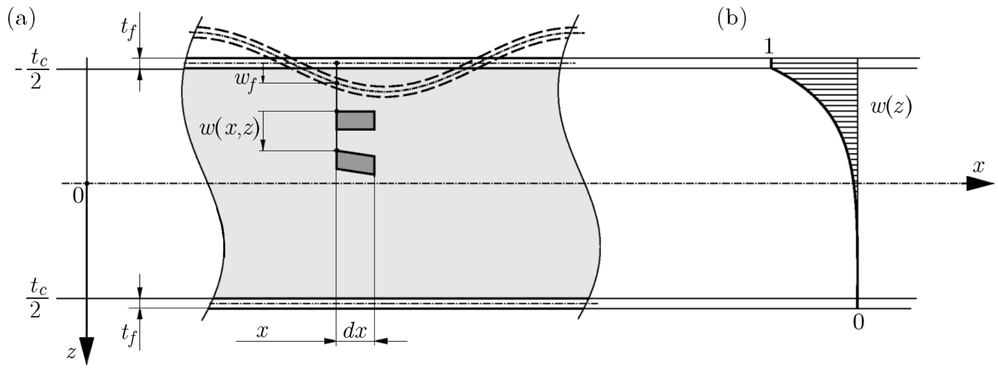

Fig. 2. Deformation of the upper face and core (a); the unknown function $w(z)$ (b)

The deflection of the upper face is then

$$w(x) = w_1 \sin\frac{m\pi x}{a_1} \tag{2.12}$$

Limiting considerations to the elastic range, the strains in the core have the form

$$\varepsilon_x = \frac{\partial u}{\partial x} \equiv 0 \qquad\qquad \varepsilon_z = \frac{\partial w}{\partial z} \qquad\qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial w}{\partial x} \tag{2.13}$$

Since the critical load will be determined with the use of the principle of stationary total potential energy, the elastic strain energy has to be formulated for the buckled beam. The strain energy of the core $U_\varepsilon^{(c)}$ can be written as follows

$$U_\varepsilon^{(c)} = \frac{E_c b}{2(1-\nu_c^2)} \int\limits_{-\frac{t_c}{2}}^{\frac{t_c}{2}} \int\limits_0^{a_1} \Big(\varepsilon_x^2 + 2\nu_c\varepsilon_x\varepsilon_z + \varepsilon_z^2 + \frac{1-\nu_c}{2}\gamma_{xz}^2\Big)\, dx\, dz \tag{2.14}$$

After substitution function (2.10) into Eq. (2.14) and integrating through the length, Eq. (2.14) takes the form

$$U_\varepsilon^{(c)} = \frac{E_c a_1 b}{4(1-\nu_c^2)} w_1^2 \int\limits_{-\frac{t_c}{2}}^{\frac{t_c}{2}} \Big[\Big(\frac{dw}{dz}\Big)^2 + \frac{1-\nu_c}{2}\Big(\frac{m\pi}{a_1}\Big)^2 w^2\Big]\, dz \tag{2.15}$$

The strain energy of the compressed face $U_\varepsilon^{(f)}$ for which $z = -t_c/2$ is

$$U_\varepsilon^{(f)} = \frac{1}{2} E_f J_z^{(f)} \int\limits_0^{a_1} \Big(\frac{d^2 w}{dx^2}\Big)\, dx = \frac{E_f a_1 b t_f^3}{48}\Big(\frac{m\pi}{a_1}\Big)^4 w_1^2 \tag{2.16}$$

The work of load related to the deformation of the upper face caused by the force N_f is

$$W = \frac{1}{2} N_f \int\limits_0^{a_1} \Big(\frac{dw}{dx}\Big)^2\, dx = \frac{1}{4} N_f a_1 \Big(\frac{m\pi}{a_1}\Big)^2 w_1^2 \tag{2.17}$$

The equation of stationary total potential energy has the form

$$\delta(U_{\varepsilon}^{(f)} + U_{\varepsilon}^{(c)} - W) = 0 \tag{2.18}$$

Solving Eq. (2.18), the equation of equilibrium is obtained as follows

$$\frac{d^2 w(z)}{dz^2} - k^2 w(z) = 0 \qquad \text{where} \qquad k^2 = \frac{1-\nu_c}{2}\Big(\frac{m\pi}{a_1}\Big)^2 \tag{2.19}$$

Taking into account boundary conditions (2.11), the solution to the above equation is

$$w(z) = \frac{1}{\sinh C_c}\sinh\Big[C_c\Big(\frac{1}{2} - \frac{z}{t_c}\Big)\Big] \qquad \text{where} \qquad C_c = kt_c = m\pi\sqrt{\frac{1-\nu_c}{2}}\frac{t_c}{a_1} \tag{2.20}$$

Knowing the function of displacements $w(z)$, the formulae for critical stresses in the compressed face can be obtained from equation (2.18)

$$\tilde{\sigma}_{cr}^{(f)} = \frac{\sigma_{cr}^{(f)}}{E_c} = \min_{C_c}\Big(\frac{\alpha_1}{C_c \tanh C_c} + \alpha_2 C_c^2\Big) \tag{2.21}$$

where

$$\alpha_1 = \frac{1}{2(1+\nu_c)x_1} \qquad \alpha_2 = \frac{e_1 x_1^2}{6(1-\nu_c)} \qquad x_1 = \frac{t_f}{t_c} \qquad e_1 = \frac{E_f}{E_c} \tag{2.22}$$

Equation (2.21) is a general formula for dimensionless critical stresses in the compressed face of the sandwich beam. If the character of hyperbolic tangent is taken into account (see Fig. 3) two particular cases of Eq. (2.21) can be distinguished:
— if $C_c \geqslant 2$ then $\tanh C_c = 1$; the critical stress can be written as follows

$$\tilde{\sigma}_{cr,I}^{(f)} = \frac{\sigma_{cr}^{(f)}}{E_c} = \Big\{\frac{1}{2(1+\nu_c)}\Big[\frac{2(1+\nu_c)}{3(1-\nu_c)}\Big]^{\frac{1}{3}} + \frac{1}{6(1-\nu_c)}\Big[\frac{3(1-\nu_c)}{2(1+\nu_c)}\Big]^{\frac{2}{3}}\Big\}e_1^{\frac{1}{3}} \tag{2.23}$$

— if $C_c \ll 1$ then $\tanh C_c = C_c$; the formula for the critical stress reduces to the form

$$\tilde{\sigma}_{cr,II}^{(f)} = \sqrt{\frac{e_1 x_1}{3(1-\nu_c^2)}} \tag{2.24}$$

It can be shown that Eq. (2.24) is consistent with the classical solution of the beam on the Winkler foundation.

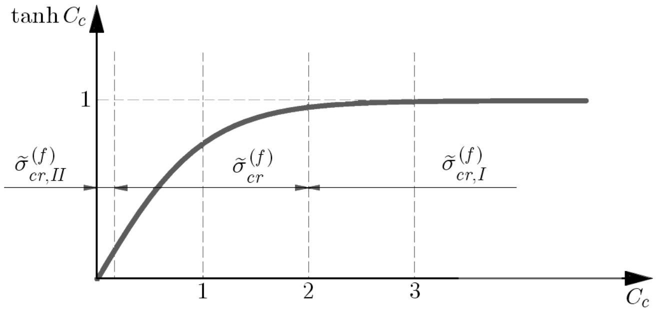

Fig. 3. Hyperbolic tangent

2.3. Numerical example

An example, in which the buckling analysis of one beam is presented, is shown below. The results obtained from the proposed model are compared with those given by FEM analysis. Broader investigation on a family of beams as well as details concerning the FEM model are shown in the next section.

In this example, the sandwich beam has the following parameters:

- material properties of the faces: $E_f = 65600\,\mathrm{MPa}$, $\nu_f = 0.33$
- material properties of the core: $E_c = 100\,\mathrm{MPa}$, $\nu_c = 0.3$
- dimensions according to Fig. 1: $a_0 = 350\,\mathrm{mm}$, $a = 300\,\mathrm{mm}$, $b = 100\,\mathrm{mm}$, $t_f = 1\,\mathrm{mm}$, $t_f = 48\,\mathrm{mm}$.

Following Eq. (2.22), the dimensionless parameters equal: $e_1 = 656$, $x_1 = 1/48$, $\alpha_1 = 18.46$, $\alpha_2 = 0.0678$. According to Eq. (2.21), the dimensionless critical stresses are

$$\tilde{\sigma}_{cr}^{(f)} = \min_{C_c}\Big(\frac{18.46}{C_c \tanh C_c} + 0.0678C_c^2\Big)$$

The above function is presented in Fig. 4a. Its minimum equals 5.383 at $C_c = 5.146$. The critical stress are then

$$\sigma_{cr}^{(f)} = \tilde{\sigma}_{cr}^{(f)} E_c = 538.3\,\mathrm{MPa}$$

Knowing that $N_{cr}^{f} = \sigma_{cr}^{f} t_c b$, the critical bending moment can be determined from Eq. (2.9)

$$M_{0,cr} = 2669\,\mathrm{Nm}$$

A good agreement can be seen between the shape of the displacement function $w(z)$ obtained analytically and these given by FEM analysis (Fig. 4b).

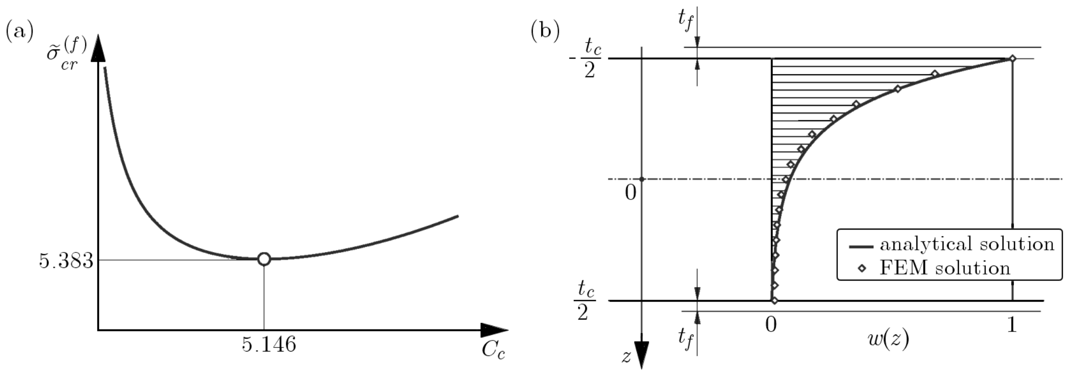

Fig. 4. Minimisation of $\tilde{\sigma}_{cr}^{(f)}$ (a); shape of the displacement function $w(z)$ (b)

3. FEM analysis of local buckling-wrinkling of the face

The finite element analysis has been performed with the use of ABAQUS software. The finite elements and procedures available in this package have been used. The FE model consists of 3D brick elements used for modelling of the core and 2D shell elements used to model the faces. The tie constrains have been applied between the faces and the core. The boundary conditions correspond to those shown in Fig. 1a. The size of finite elements has been determined in analysis of the mesh convergence. A family of sandwich beams has been analysed for which dimensions and material properties were as follows: face thickness $t_f = 1\,\text{mm}$, core thickness $t_c = 18$, 28, 38, 48 mm, width of the beam $b = 100\,\text{mm}$, distance between the force and support $a_0 = 350\,\text{mm}$, distance between forces $a_1 = 300\,\text{mm}$, Poisson's ratios and Young's moduli of the faces and the core $\nu_f = 0.3$, $\nu_c = 0.33$, $E_f = 65600\,\text{MPa}$, $E_c = 10$, 50, 100, 400, 800, 1200 MPa. Buckling analysis has been performed to determine the critical bending moment. The critical stresses were derived from static analysis by applying the critical bending moments determined in the buckling analysis.

The buckling shapes for two beams are shown in Fig. 5. The longitudinal waves appear in the area between forces applied to the upper face. It can be seen that the amplitude of the waves is the highest in the mid-length of the beam and it diminishes moving to the point where the force is applied.

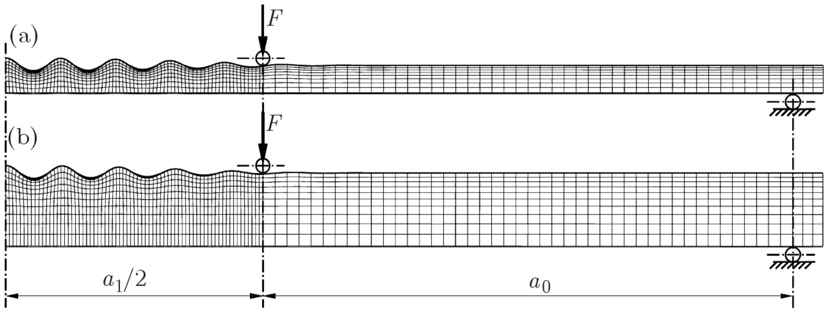

Fig. 5. Buckling shapes of the sandwich beams ($E_c = 100\,\text{MPa}$): $t_c = 18\,\text{mm}$ (a); $t_c = 48\,\text{mm}$ (b)

The critical stresses and the critical bending moments for the family of beams described above are shown in Fig. 6. In the plots the analytical and numerical (FEM) results are compared. A good agreement can be seen between the results obtained with both approaches. The biggest discrepancy appear for beams with the stiffest core, but it does not exceed 7%.

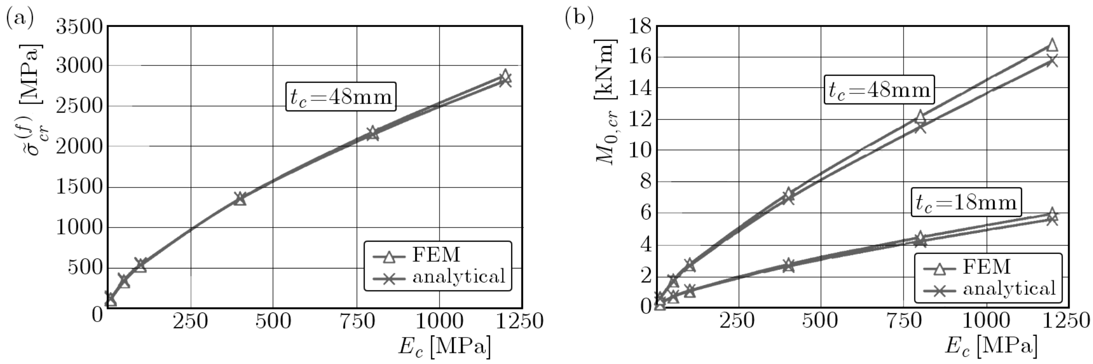

Fig. 6. Critical stresses and bending moments for a family of sandwich beams

4. Conclusions

In the paper, a mathematical model of the face wrinkling of the sandwich beam has been presented. It allows one to estimate the buckling load for the beam subjected to pure bending. The results of calculations made on a family of beams show that the elastic local buckling may appear only for beams with a soft core for which Young's modulus is not higher than about 100 MPa.

The results obtained from the analytical model have been compared with those given by the finite element method. The discrepancy as to the critical bending moment is less than 7%. The reason for this difference may be the buckling shape assumed in the analytical considerations. It has the form of a sine function, whereas in the FEM results the amplitude of the waves diminishes near the applied force.

An algorithm has been proposed, which gave the possibility to determine critical stresses in the compressed face. Three cases are possible to occur depending on the parameter C_c describing properties of the core.

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References

1. Allen H.G., 1969, *Analysis and Design of Structural Sandwich Panels*, Pergamon Press, London
2. Bart-Smith H., Hutchinson J.W., Evans A.G., 2001, Measurement and analysis of the structural performance of cellular metal sandwich construction, *Mechanical Science*, **43**, 1945-1963
3. Bažant Z.P., Cedolin L., 1991, *Stability of structures. Elastic, Inelastic, Fracture, and Damage Theories*, Oxford University Press, New York, Oxford
4. Chen X W, Yu T.X., 2000, Elastic-plastic beam-on-foundation under quasi-static loading, *Int. Journal of Mechanical Sciences*, **42**, 2261-2281
5. Hadi B.K., 2001, Wrinkling of sandwich column: comparison between finite element analysis and analytical solutions, *Composite Structures*, **53**, 477-482
6. Jasion P., Magnucka-Blandzi E., Szyc W., Wasilewicz P., Magnucki K., 2011, Global and local buckling of a sandwich beam-rectangular plate with metal foam core, *Proc. 6th Int. Conference on Thin-walled Structures*, Vol. 2, Dubina D., Ungureanu V. (Eds.), ECCS Publication, Printed in Multicomp Lda, Mem Martins, Portugal, ISBN(ECCS) 978-92-9147-102-7, 707-714
7. Jasion P., Magnucki K., 2011, Buckling-wrinkling of a face of sandwich beam under pure bending, *Modelowanie Inżynierskie*, **41**, 151-156, ISSN 1896-771X [in Polish]
8. Kesler O., Gibson L.J., 2002, Size effects in metallic foam core sandwich beams, *Materials Science and Engineering A*, **326**, 228-234
9. Koissin V., Shipsha A., Skvortsov V., 2010, Effect of physical nonlinearity on local buckling in sandwich beams, *Journal of Sandwich Structures and Materials*, **12**, 7, 477-494
10. Léotoing L., Drapier S., Vautrin A., 2002, First applications of a novel unified model for global and local buckling of sandwich columns, *European J. of Mech. A/Solids*, **21**, 683-701
11. Libove C., Butdorf S.B., 1948, *A General Small-Deflection Theory for Flat Sandwich Plates*, NACA TN 1526
12. Plantema F.J., 1966, *Sandwich Construction: The Bending and Buckling of Sandwich Beams, Plates and Shells*, John Wiley and Sons, New York
13. Rakow J.F., Waas A.M., 2005, Size effects and the shear response of aluminium foam, *Mechanics of Materials*, **37**, 69-82
14. Reissner E., 1948, Finite deflections of sandwich plates, *Journal of the Aeronautical Science*, **15**, 7, 435-440
15. Steeves C.A., Fleck N.A., 2004, Collapse mechanisms of sandwich beams with composite faces and a foam core, loaded in three-point bending. Part I: analytical models and minimum weight design, *Int. Journal Mechanical Sciences*, **46**, 561-583
16. Stiftinger M.A., Rammerstorfer F.G., 1997, Face layer wrinkling in sandwich shells – Theoretical and experimental investigations, *Thin-Walled Structures*, **29**, 1/4, 113-127

17. Vlasov V.Z., Leontev N.N., 1960, *Beams, Plates and Shells on Elastic Foundation*, Gosud. Izd. Fiz-Mat-Lit., Moscow [in Russian]

18. Volmir A.S., 1967, *Stability of Deformable Systems*, Izd. Nauka, Fiz-Mat-Lit, Moscow [in Russian]

19. Woźniak M., 2001, Interaction of a plate with elastic foundation, [In:] *Mechanika Techniczna, Mechanics of Elastic Plates and Shells*, **VIII**, C. Woźniak (Edit.), Wyd. Naukowe PWN, Warszawa, 510-541 [in Polish]

20. Życzkowski M., 1988, *Stability of bars and system bars*, [In:] *Mechanika Techniczna, Strength of Structures Elements*, **IX**, M. Życzkowski (Edit.), PWN, Warszawa, 241-380 [in Polish]

Marszczenie ściskanej okładziny belki trójwarstwowej poddanej czystemu zginaniu

Streszczenie

W pracy omówiono zagadnienie lokalnej stateczności belki trójwarstwowej poddanej czystemu zginaniu. Zaproponowano analityczny model marszczenia górnej, ściskanej okładziny. Z zasady stacjonarności energii potencjalnej wyprowadzono równanie opisujące naprężenia krytyczne. Zaproponowano algorytm pozwalający określić wartość naprężeń krytycznych w zależności od własności rdzenia. Opracowano model numeryczny MES belki trójwarstwowej. Dla rodziny belek przeprowadzono analizę numeryczną na wartości własne, a wyniki porównano z otrzymanymi z zaproponowanego modelu analitycznego.

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