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THERMOELASTIC DISTURBANCES IN A TRANSVERSELY ISOTROPIC HALF-SPACE DUE TO THERMAL POINT LOAD

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The objective of this paper is to study disturbances due to thermal point load in a homogeneous transversely isotropic half-space in generalized thermoelasticity. A combination of the Fourier and Hankel transform technique is applied to obtain the solutions to governing equations. Cagniared's technique is used to invert the transformed solutions for small times. Theoretically obtained results, for temperature, stresses are computed numerically for a zinc material. It is found that variations in stresses and temperature are more prominent at small times and decrease with passage of time. Theg results obtained theoretically are represented graphically at different values of thermal relaxation times.

Key words: transversely isotropic, generalized thermoelasticity, Cagniard technique, thermal point load

Nomenclature

T_0	–	uniform temperature
C_{ij}	–	elastic parameters
λ, μ	–	thermal conductivity
ρ	–	density of medium
C_e	–	specific heat at constant strain
τ_0	–	thermal relaxation time
K_3, K_1	–	coefficients of thermal conductivities
α_3, α_1	–	coefficients of linear thermal expansions
ε_1	–	thermoelastic coupling constant
V_R	–	Rayleigh waves velocity
T	–	temperature
v	–	velocity of compressional waves

L^{-1}	–	inverse Laplace transform
$\delta(x)$	–	Dirac delta function

1. Introduction

Thermoelasticity theory, Chadwick (1960, 1979) and Nowacki (1962, 1975), of thermal disturbances has aroused considerable interest in the last century, but systematic research started only after thermal waves – called second sound – were first measured in materials like solid helium, bismuth and sodium fluoride. Thus, the thermoelasticity theories, which admit a finite speed for thermal signals, have been receiving a lot of attention for the past thirty years. In contrast to the conventional coupled thermoelasticity theory based on a parabolic heat equation, Biot (1956), which predicts an infinite speed for the propagation of heat, these theories involve a hyperbolic heat equation and are referred to as generalized thermoelasticity theories.

The Lord and Shulman (1967) theory introduces a single time constant to dictate the relaxation of thermal propagation as well as the rate of change of strain rate and the rate of change of heat generation, and obtained a wave-type heat equation by postulating a new law of heat conduction to replace the classical Fourier law for isotropic bodies. Later, the theory was developed and extended to anisotropic solids by Dhaliwal and Sherief (1980).

These thermoelastic models are based on hyperbolic-type equations for temperature, and are closely connected with the theories of second sound, which view heat propagation as a wave-like phenomenon. The majority of the work by Chandrasekharaiah (1986, 1998) in this field has been devoted to various aspects of linear thermoelastic models considering isotropic materials, very little work has been done considering materials which are anisotropic in nature. Hence, the study of thermo-mechanical interactions and thermoelastic disturbances in anisotropic materials is justified and is of great importance and practical use in engineering applications especially in the context of generalized theory of thermoelasticity.

Verma (1999) and Verma and Hasabe (2002) studied thermoelastic problems by considering equations for transversely isotropic heat conducting plates with thermal relaxations times. Harinath (1975, 1980) considered the problems of surface point and line source over a homogeneous isotropic thermoelastic halfspace in thermoelasticity. De Hoop (1959) modified and used a method originally presented by Cagniard (1962) to solve the disturbances that are generated by an impulsive, concentrated load applied along a line on the free surface of a homogeneous isotropic elastic half-space. Nayfeh and Nasser (1972) developed the displacements and temperature fields in a homogeneous isotropic generalized thermoelastic halfspace subjected on the free surface to an instantaneously applied heat source using the Cagniard-De Hoop method (Cagniard, 1962).

In this paper, using a combination of the Laplace and Hankel transforms, the governing equations of transversely isotropic thermoelastic solid half-space, which are subjected to thermal point load on its free surface are solved. The resulting equations are then inverted using the Cagniard-De Hoop method for small times. The results obtained theoretically have been verified numerically and illustrated graphically for a single crystal of zinc.

2. Formulation of the problem

We consider thermal and elastic wave motion of small amplitude in homogeneous heat conducting transversely isotropic elastic solids with thermal relaxation, at a uniform temperature T_0, and considering the plane of isotropy is perpendicular to z-axis. We take z-axis pointing normally into the half space, which is thus represented by $z \geqslant 0$. The disturbance is caused by a suddenly applied thermal point source on the free surface of the initially undisturbed elastic solid. This source is acting in the direction of z-axis at the origin of the cylindrical coordinate system (r, θ, z) which is any point of the plane boundary $z = 0$. The problem is axi-symmetric with respect to the $z = 0$. The governing equations of motion and heat conduction for the displacement vector $\boldsymbol{u}(r, z, t) = (u, 0, w)$ and temperature $T(r, z, t)$ for such a medium in the absence of the heat source and the body forces in the context of generalized theory of linear thermoelasticity are given by

$$\begin{gathered}
\Big[C_{11}\Big(\frac{\partial^2}{\partial r^2} + r^{-1}\frac{\partial}{\partial r} - r^{-2}\Big) + C_{44}\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2}\Big]u + (C_{13} + C_{44})\frac{\partial^2 w}{\partial r \partial z} = \beta_1\frac{\partial T}{\partial r} \\
\Big[C_{44}\Big(\frac{\partial^2}{\partial r^2} + r^{-1}\frac{\partial}{\partial r}\Big) + C_{33}\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2}\Big]w + (C_{13} + C_{44})\frac{\partial^2 u}{\partial r \partial z} = \beta_3\frac{\partial T}{\partial z} \\
K_1\Big(\frac{\partial^2}{\partial r^2} + r^{-1}\frac{\partial}{\partial r}\Big)T + K_3\frac{\partial^2 T}{\partial z^2} - \rho C_e\Big(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\Big)T \\
= \beta_1 T_0\Big(\frac{\partial^2 u}{\partial r \partial t} + r^{-1}\frac{\partial u}{\partial t} + \overline{\beta}\frac{\partial^2 w}{\partial z \partial t} + \tau_0\frac{\partial^3 u}{\partial r \partial t^2} + r^{-1}\frac{\partial u}{\partial t} + \overline{\beta}\frac{\partial^3 w}{\partial z \partial t^2}\Big)
\end{gathered} \tag{2.1}$$

where

$$\beta_1 = (C_{11} + C_{12})\alpha_1 + C_{13}\alpha_3 \qquad \beta_3 = 2C_{13}\alpha_1 + C_{33}\alpha_3 \qquad \overline{\beta} = \frac{\beta_3}{\beta_1} \tag{2.2}$$

C_{ij} are being the isothermal parameters, C_e and τ_0 are the specific heat at constant strain and thermal relaxation time, respectively. K_3, K_1 and α_3, α_1 are the coefficients of thermal conductivities and linear thermal expansions respectively, along and perpendicular to the axis of symmetry. If we take

$$\begin{aligned} &C_{11} = C_{33} = \lambda + 2\mu && C_{44} = 2\mu && C_{13} = \lambda \\ &K_3 = K_1 = K && \alpha_1 = \alpha_3 = \alpha_t && \beta_1 = \beta_3 = (3\lambda + 2\mu)\alpha_t \end{aligned} \tag{2.3}$$

then equations (2.1) reduce to the corresponding form for an isotropic body, with Lamé's parameters λ, μ, thermal conductivity K and the coefficients of linear thermal expansion α_t. We define the dimensionless quantities

$$\begin{aligned} &r' = \frac{w^*}{v}r && z' = \frac{w^*}{v}z && t' = w^*t \\ &\tau_0' = w^*\tau_0 && w' = \frac{\rho w^* v}{\beta_1 T_0}w && T' = \frac{T}{T_0} \\ &\overline{k} = \frac{K_3}{K_1} && c_1 = \frac{C_{33}}{C_{11}} && c_2 = \frac{C_{44}}{C_{11}} \\ &c_3 = \frac{C_{13} + C_{44}}{C_{11}} && \varepsilon_1 = \frac{\beta_1^2 T_0}{\rho C_e C_{11}} \end{aligned} \tag{2.4}$$

where $k_1 = K_1/(\rho C_e)$ and $v = \sqrt{C_{11}/\rho}$ are the thermal diffusivity and the velocity of compressional waves in the x-direction, respectively. Here ε_1 is the thermoelastic coupling constant.

Introducing above quantities (2.4) into equations (2.1), we obtain (on suppressing the primes throughout)

$$\begin{aligned} &\Big(\frac{\partial^2}{\partial r^2} + r^{-1}\frac{\partial}{\partial r} - r^{-2} + c_2\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}\Big)u + c_3\frac{\partial^2 w}{\partial r \partial z} = \frac{\partial T}{\partial r} \\ &\Big[c_2\Big(\frac{\partial^2}{\partial r^2} + r^{-1}\frac{\partial}{\partial r}\Big) + c_1\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}\Big]w + c_3\frac{\partial^2 u}{\partial r \partial z} = \overline{\beta}\frac{\partial T}{\partial z} \\ &\Big(\frac{\partial^2}{\partial r^2} + r^{-1}\frac{\partial}{\partial r}\Big)T + \overline{k}\frac{\partial^2 T}{\partial z^2} - \Big(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\Big)T \\ &\qquad = \varepsilon_1\Big(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\Big)\Big(\frac{\partial u}{\partial r} + r^{-1}u + \overline{\beta}\frac{\partial w}{\partial z}\Big) \end{aligned} \tag{2.5}$$

The boundary conditions at the surface z = 0 are

$$\sigma_{rz} = \sigma_{zz} = 0 \qquad hT + \frac{\partial T}{\partial z} = \frac{Q_0\delta(r)f(t)}{2\pi r} \tag{2.6}$$

where σ_{rz}, σ_{zz} are thermal stresses, Q_0 is a constant and $\delta(r)$ is the Dirac delta function, h is Biot's heat transfer coefficient and $f(t)$ is an arbitrary single-valued finite and continuous function of time and must have only one numerical value.

Equations (2.6) may also be written as

$$\begin{aligned} &(c_3 - c_2)\frac{\partial u}{\partial x} + c_1\frac{\partial w}{\partial z} - \overline{\beta}T = 0 \\ &\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \qquad\qquad hT + \frac{\partial T}{\partial z} = Q_0^*\delta(x)f(t) \end{aligned} \tag{2.7}$$

where $Q_0^* = vQ_0/T_0$. The condition at infinity requires that the solutions be bounded as z becomes large. Finally, the initial conditions are such that the medium is at rest for $t < 0$.

3. Solution of the problem

The condition at infinity requires that the solutions be bounded as z becomes large. Finally, the initially conditions are such that the medium is at rest for $t < 0$.

Apply the Laplace transform with respect to time and the Hankel transform with respect to r to the system of equations (2.5) to (2.7). The appropriate solution of the resulting equation is then constructed and subsequently inverted. The Laplace and the exponential Fourier transforms are defined respectively as

$$\overline{\phi}(r,z,p) = \int\limits_0^\infty \phi(r,z,p)e^{-pt}\,dt \qquad\qquad \hat{\phi}(q,z,p) = \int\limits_0^\infty r\overline{\phi}(q,z,p)J_n(qr)\,dr \tag{3.1}$$

where $n = 1$ in the case of $\overline{u}(r,z,p)$ and $n = 0$ for $\overline{w}(r,z,p)$ and $\overline{T}(r,z,p)$ to equation $(3.1)_2$, we obtain

$$\begin{aligned} \hat{u}'' &= \frac{1}{c_2}[(q^2 + p^2)\hat{u} - q\hat{T} + c_3q\hat{w}'] \\ \hat{w}'' &= \frac{1}{c_1}[(c_2q^2 + p^2)\hat{u} - c_3q\hat{u}' + \overline{\beta}\hat{T}] \\ \hat{T}'' &= \frac{1}{\overline{k}}[(q^2 + \tau p^2)T + \varepsilon_1\tau p^2(q\hat{u} + \overline{\beta}\hat{w})] \end{aligned} \tag{3.2}$$

where $\tau = \tau_0 + p^{-1}$.

The system of equations (3.2) can be written as

$$\frac{d}{dz}\boldsymbol{W}(q,z,p) = \mathbf{A}(q,p)\boldsymbol{W}(q,z,p) \tag{3.3}$$

where

$$\boldsymbol{W} = \begin{bmatrix} U \\ U' \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{A}_2 & \mathbf{A}_1 \end{bmatrix} \qquad \mathbf{A}_1 = \begin{bmatrix} 0 & \dfrac{c_3 q}{c_2} & 0 \\ -\dfrac{c_3 q}{c_1} & 0 & \dfrac{\overline{\beta}}{c_1} \\ 0 & \dfrac{\varepsilon_1 \tau p^2 \overline{\beta}}{\overline{k}} & \end{bmatrix}$$

$$\boldsymbol{U} = \begin{bmatrix} \hat{u} \\ \hat{w} \\ \hat{T} \end{bmatrix} \qquad \mathbf{A}_2 = \begin{bmatrix} \dfrac{q^2+p^2}{c_1} & 0 & -\dfrac{q}{c_2} \\ 0 & \dfrac{c_2 q^2 + p^2}{c_1} & 0 \\ \dfrac{\varepsilon_1 \tau p^2 q}{\overline{k}} & 0 & \dfrac{q^2+\tau p^2}{\overline{k}} \end{bmatrix} \tag{3.4}$$

$$\mathbf{O} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To solve equation (3.3), we have

$$\boldsymbol{W}(q,z,p) = \mathbf{X}(q,p)\exp(mz)$$

So that $\mathbf{A}(q,p)\boldsymbol{W}(q,z,p) = m\boldsymbol{W}(q,z,p)$, which leads to the eigenvalue problem. The characteristic equation corresponding to the matrix $\mathbf{A}$ is given by

$$\det(\mathbf{A} - m\mathbf{I}) = 0 \tag{3.5}$$

on expansion we have

$$m^6 - \lambda_1 m^4 + \lambda_2 m^2 - \lambda_3 = 0 \tag{3.6}$$

where

$$\begin{aligned} \lambda_1 &= \frac{Pq^2 + Jp^2}{c_1 c_2} + \frac{q^2+\tau p^2}{\overline{k}} + \frac{\varepsilon_1 \tau p^2 \overline{\beta}^2}{\overline{k} c_1} \\ \lambda_2 &= \Big\{\overline{k}(q^2+p^2)(c_2 q^2 + p^2) + (Pq^2 + Jp^2)(q^2 + \tau p^2) \\ &\qquad + \varepsilon_1 \tau p^2 q^2 [p^2 \overline{\beta}^2 + (c_1 - 2c_3\overline{\beta} + \overline{\beta}^2)]\Big\}\frac{1}{\overline{k} c_1 c_2} \\ \lambda_3 &= (c_2 q^2 + p^2)[(q^2+p^2)(q^2+\tau p^2) + \varepsilon_1 \tau p^2 q^2]\frac{1}{\overline{k} c_1 c_2} \\ P &= c_1 + c_2^2 + c_3^2 \qquad J = c_1 + c_2 \end{aligned} \tag{3.7}$$

The eigenvalues of the matrix $\mathbf{A}$ are the characteristic roots $\pm m_i$ $(i = 1, 2, 3)$ of equation (3.6). We assume that real parts of are positive. The eigenvector $\boldsymbol{X}(q, p)$ corresponding to the eigenvalue m can be determined by solving the homogeneous equation

$$(\mathbf{A} - m\mathbf{I})\boldsymbol{X}(q, p) = \mathbf{0} \tag{3.8}$$

The set of eigen-vectors $\boldsymbol{X}_i(q, p)$ $(i = 1, 2, \dots, 6)$ may be obtained as

$$\boldsymbol{X}_i(q, p) = \begin{bmatrix} \boldsymbol{X}_{i1}(q, p) \\ \boldsymbol{X}_{i2}(q, p) \end{bmatrix} \tag{3.9}$$

where

$$\begin{aligned} \boldsymbol{X}_{i1}(q, p) &= \begin{bmatrix} -q \\ a_i m_i \\ b_i \end{bmatrix} \qquad & \boldsymbol{X}_{i2}(q, p) &= \begin{bmatrix} -q m_i \\ a_i m_i^2 \\ b_i m_i \end{bmatrix} \\ \boldsymbol{X}_{j1}(q, p) &= \begin{bmatrix} -q \\ -a_i m_i \\ b_i \end{bmatrix} \qquad & \boldsymbol{X}_{j2}(q, p) &= \begin{bmatrix} q m_i \\ a_i m_i^2 \\ -b_i m_i \end{bmatrix} \qquad j = i + 3 \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} a_i &= \frac{1}{\Delta_i}[c_2\overline{\beta} m_i^2 + (c_3 - \overline{\beta})q^2 - p^2\overline{\beta}] \\ b_i &= \frac{1}{\Delta_i}[(q^2 + p^2 - c_2 m_i^2)(c_2 q^2 + p^2 - c_1 m_i^2) + c_3^2 q^2 m_i^2] \\ \Delta_i &= (c_1 - c_3\overline{\beta})m_i^2 - c_2 q^2 - p^2 \end{aligned} \tag{3.11}$$

Thus the solution to (3.3) is given by

$$\boldsymbol{W}(q, z, p) = \sum_{i=1}^{3}[B_i \boldsymbol{X}_i(q, p)\exp(m_i z) + B_{i+3}\boldsymbol{X}_{i+3}(q, p)\exp(-m_i z)] \tag{3.12}$$

where B_i, $(i = 1, 2, \dots, 6)$ are arbitrary constants. Equation (3.12) represents the general problem in the axi-symmetric case of generalized homogeneous transversely isotropic thermoelasticity by employing the eigenvalue approach.

The displacements, temperature, stresses and temperature gradient in the transformed domain which satisfy the radiation conditions can be written from equations (3.1) and (3.2) as

$$\begin{aligned} \hat{u} &= -q[B_4\exp(-m_1 z) + B_5\exp(-m_2 z) + B_6\exp(-m_3 z)] \\ \hat{w} &= -[B_4 a_1 m_1\exp(-m_1 z) + B_5 a_2 m_2\exp(-m_2 z) + B_6 a_3 m_3\exp(-m_3 z)] \end{aligned}$$

$$
\begin{aligned}
\hat{T} &= B_4 b_1 \exp(-m_1 z) + B_5 b_2 \exp(-m_2 z) + B_6 b_3 \exp(-m_3 z)\\
\hat{\sigma}_{zz} &= \sum_{i=1}^{3} [(c_3 - c_2)q^2 + a_i c_1 m_i^2 - \overline{\beta} b_i] B_{i+3} \exp(-m_3 z)\\
\hat{\sigma}_{rz} &= \frac{c_2}{2} \sum_{i=1}^{3} B_{i+3} m_i (1 + a_i) \exp(-m_3 z)\\
\hat{T}' &= -\sum_{i=1}^{3} B_{i+3} m_i b_i \exp(-m_3 z)
\end{aligned}
\tag{3.13}
$$

where $\hat{T}' = d\hat{T}/dz$.

Applying transforms $(3.1)_{1,2}$ to the boundary conditions, and above relations

$$
\begin{aligned}
&\hat{\sigma}_{zz} = 0 \qquad\qquad \hat{\sigma}_{rz} = 0\\
&h\hat{T} + \hat{T}' = \frac{-Q^* q \hat{f}}{2\pi} \qquad \text{at} \quad z = 0
\end{aligned}
\tag{3.14}
$$

we obtain

$$
\begin{aligned}
&\sum_{i=1}^{3} [(c_3 - c_2)q^2 + a_i c_1 m_i^2 - \overline{\beta} b_i] B_{i+3} = 0\\
&\sum_{i=1}^{3} B_{i+3} m_i (1 + a_i) = 0 \qquad\qquad \sum_{i=1}^{3} (h - m_i) b_i B_{i+3} = \frac{-Q^* q \hat{f}}{2\pi}
\end{aligned}
\tag{3.15}
$$

For a stress free thermally insulated boundary (heat transfer coefficient $h \to 0$), and for a stress free isothermal boundary $(h \to \infty)$.

Solving equations (3.15) for B_4, B_5 and B_6, we get

$$
\begin{aligned}
B_4 &= -\frac{Q^* q}{2\pi \Delta^* p} \Big\{ m_3(1 + a_3)[(c_3 - c_2)q^2 - \overline{\beta} b_2 + a_2 c_1 m_2^2]\\
&\quad - m_2(1 + a_2)[(c_3 - c_2)q^2 - \overline{\beta} b_3 + a_3 c_1 m_3^2] \Big\}\\
B_5 &= \frac{Q^* q}{2\pi \Delta^* p} \Big\{ m_3(1 + a_3)[(c_3 - c_2)q^2 - \overline{\beta} b_1 + a_1 c_1 m_1^2]\\
&\quad - m_1(1 + a_1)[(c_3 - c_2)q^2 - \overline{\beta} b_3 + a_3 c_1 m_3^2] \Big\}\\
B_6 &= -\frac{Q^* q}{2\pi \Delta^* p} \{ m_2(1 + a_2)[(c_3 - c_2)q^2 - \overline{\beta} b_1 + a_1 c_1 m_1^2]\\
&\quad - m_1(1 + a_1)[(c_3 - c_2)q^2 - \overline{\beta} b_2 + a_2 c_1 m_2^2] \Big\}
\end{aligned}
\tag{3.16}
$$

where we have taken $\hat{f}(p) = 1/p$, and

$$
\begin{aligned}
\Delta^* &= (h_1 - h_2 m_1) b_1 \{m_3(1+a_3)[(c_3 - c_2)q^2 - \overline{\beta} b_2 + a_2 c_1 m_2^2] \\
&\quad - m_2(1+a_2)[(c_3 - c_2)q^2 - \overline{\beta} b_3 + a_3 c_1 m_3^2]\} \\
&\quad - (h_1 - h_2 m_2) b_2 \{m_3(1+a_3)[(c_3 - c_2)q^2 - \overline{\beta} b_1 + a_1 c_1 m_1^2] \\
&\quad - m_1(1+a_1)[(c_3 - c_2)q^2 - \overline{\beta} b_3 + a_3 c_1 m_3^2]\} \\
&\quad - (h_1 - h_2 m_3) b_3 \{m_2(1+a_2)[(c_3 - c_2)q^2 - \overline{\beta} b_1 + a_1 c_1 m_1^2] \\
&\quad - m_1(1+a_1)[(c_3 - c_2)q^2 - \overline{\beta} b_2 + a_2 c_1 m_2^2]\}
\end{aligned}
\tag{3.17}
$$

Thus a formal solution to equations (2.5) is given by

$$
(u, w, T) = L^{-1} \left\{ \int\limits_{-\infty}^{\infty} \sum_{k=1}^{3} (a_{1k}, a_{2k}, a_{3k}) J_n(qr) \mathrm{e}^{-m_k z} \, dq \right\} \tag{3.18}
$$

where L^{-1} designate the inverse Laplace transform and where we have set $(i = 1, 2, 3)$

$$
a_{1i} = -qB_{i+3} \qquad a_{2i} = -qm_i B_{i+3} \qquad a_{3i} = b_i B_{i+3} \tag{3.19}
$$

4. Inversion of transforms

To obtain the solution, we use the Cagniard (1962) method to. This method consists of recasting each integral in (3.18) into the Lapalace transform of a known function, thus allowing one to write down the inverse transform by inspection. Mathematically, this procedure is based on De-Hoop (1959), Cagniard (1962) and Fung (1965) a rather elementary observation that

$$
\begin{aligned}
&L^{-1} \left\{ \frac{p^n}{2\pi} \int\limits_{t_0}^{\infty} f(t) \mathrm{e}^{-pt} \, dt - p^{n-1} f(0) - p^{n-2} f'(0) - \ldots - f^{(n-1)}(0) \right\} \\
&\quad = \frac{d^n f(t)}{dt^n} H(t - t_0)
\end{aligned}
\tag{4.1}
$$

and

$$
L^{-1} \left\{ \frac{1}{2\pi p^n} \int\limits_{t_0}^{\infty} f(t) \mathrm{e}^{-pt} \, dt \right\} = \int\limits_{1} \int\limits_{2} \cdots \int\limits_{n} f(\overline{t}) H(\overline{t} - t_0) \, d\overline{t} \qquad n = 0, 1, 2, \ldots \tag{4.2}
$$

For this technique to apply, it is therefore essential that we obtain an explicit expression for m_k and that we isolate the Laplace transform parameter p as shown in (4.1) and (4.2). To this end, we observe that equation (3.6) pertain to the coupled, dilatational, distortional and thermal waves. To find the explicit expression, we seek for solution to (3.6) for small values of the thermoelastic coupling constant ε_1. Assuming that ε_1 is sufficiently small, we find that

$$m_j^2 = m_{j0}^2 + \varepsilon_1 m_{j1}^2 + \dots \qquad j = 1,2,3 \tag{4.3}$$

where m_{j0}^2 are given

$$\begin{aligned}
m_{10}^2, m_{20}^2 &= \frac{Pq^2 + Jp^2 \pm \sqrt{Pq^2 + Jp^2 - 4c_1c_2(c_2q^2 + p^2)(p^2 + q^2)}}{\overline{k}c_1c_2} \\
m\mathbb{N}_{30}^2 &= \frac{q^2 + \tau p^2}{\overline{k}} \\
m_{j1}^2 &= \frac{\tau p^2\{(c_2q^2 + p^2)q^2 - m_{j0}^2[(c_1 - 2c_3\overline{\beta} + \overline{\beta}^2)q^2 + \overline{\beta}^2 p^2 - c_2\overline{\beta}^2 m_{j0}^2]\}}{\overline{k}c_1c_2(m_{j0}^2 - m_{i0}^2)(m_{j0}^2 - m\mathbb{N}_{k0}^2)} \\
&\quad i \neq j \neq k = 1,2,3
\end{aligned} \tag{4.4}$$

In view

$$J_n(\xi) = \sqrt{\frac{2}{\pi\xi}}\cos\Big[\xi - \Big(n + \frac{1}{2}\Big)\frac{\pi}{2}\Big] = \mathrm{Re}\Big\{\sqrt{\frac{2}{\pi}}\exp\Big[-\mathrm{i}\Big(\xi - \Big(n + \frac{1}{2}\Big)\Big)\frac{\pi}{2}\Big]\Big\} \tag{4.5}$$

of Watson (1945), formal solution (3.18) can be we written as

$$(u,w,T) = L^{-1}\left\{\mathrm{Re}\Big[\int\limits_0^\infty \Big(\sum_{i=1}^3 a_{1i}^*, \sum_{i=1}^3 a_{2i}^*, \sum_{i=1}^3 a_{3i}^*\Big)\exp(-\mathrm{i}qr - m_k z)\, dq\Big]\right\} \tag{4.6}$$

and

$$a_{1i}^* = a_{1i}\sqrt{\frac{2q}{\pi r}}\exp\Big(\mathrm{i}\frac{\pi}{4}\Big) \qquad (a_{2i}^*, a_{3i}^*) = \sqrt{\frac{2q}{\pi r}}\exp\Big(\mathrm{i}\frac{3\pi}{4}\Big)(a_{2i}, a_{3i}) \tag{4.7}$$

Due to existence of the damping term in temperature field equation $(2.5)_3$, isolation of p is impossible. However, this isolation of p may be achieved for small time, i.e. if we assume p to be large. Hence, an expansion in the inverse power of p followed by the change of variable $q = p\eta$, reduces $m\mathbb{N}_{k0}$ and m_{k1}^2 to

$$\begin{aligned}
&m_{10} = p\alpha_{10} \qquad m_{20} = p\alpha_{20} \qquad m\mathbb{N}_{30} = p\alpha_{30} + \frac{1}{2}\overline{k}\alpha_{30} \\
&m_{j1}^2 = p^2\Big(\alpha_{j1}^2 + \frac{\alpha_{j1}^{*2}}{p}\Big) \qquad j = 1,2,3
\end{aligned} \tag{4.8}$$

and

$$
\begin{aligned}
&\alpha_{10}^2, \alpha_{20}^2 = \Big[P\eta^2 + J \pm \sqrt{(P\eta^2+J)^2 - 4c_1c_2(\eta^2+1)(c_2\eta^2+1)}\Big]\frac{1}{2c_1c_2}\\
&\alpha_{30}^2 = \frac{\eta^2+\tau_0}{\overline{k}}\\
&\alpha_{ji}^2 = \tau_0\eta^2(c_2\eta^2+1) - \frac{\alpha_{j0}^2}{\alpha_{jik}}[(c_1 - 2c_3\overline{\beta} + \overline{\beta}^2)\eta^2 + \overline{\beta}^2 - c_2\overline{\beta}^2\alpha_{j0}^2]\\
&\alpha_{jik} = \overline{k}c_1c_2(\alpha_{j0}^2 - \alpha_{i0}^2)(\alpha_{j0}^2 - \alpha_{k0}^2) \qquad i \neq j \neq k = 1,2,3\\
&\alpha_{11}^{*2} = \alpha_{11}^2\Big(\tau_0 + \frac{1}{\overline{k}}\Big(\alpha_{10}^2 - \alpha_{30}^2\Big)\Big)\\
&\alpha_{21}^{*2} = \alpha_{21}^2\Big[\tau_0 + \frac{1}{\overline{k}}(\alpha_{20}^2 - \alpha_{30}^2)\Big]\\
&\alpha_{31}^{*2} = \alpha_{31}^2\Big\{\tau_0 + [(\alpha_{10}^2 + \alpha_{20}^2) - 2\alpha_{30}^2]\frac{c_1c_2}{\alpha_{312}}\Big\}\\
&\qquad - \frac{\tau_0}{\overline{k}\alpha_{312}}[(c_1 - 2c_3\overline{\beta} + \overline{\beta}^2)\eta^2 + \overline{\beta}^2 - 2c_2\overline{\beta}^2\alpha_{30}^2]
\end{aligned}
\tag{4.9}
$$

Special Case: We take $f(t) = H(t)$, the unit step function so that the surface of the half-space is subjected to a thermal source of magnitude Q_0^* and $\overline{f}(p) = 1/p$. Substitution of equations (4.1) to $(4.9)_2$ in equations (3.13) and then into equation (3.3) yields

$$
(u, w, T) = L^{-1}\left\{\sum_{i=1}^{3}\overline{u}_i, \sum_{i=1}^{3}\overline{w}_i, \sum_{i=1}^{3}\overline{T}_i\right\} \tag{4.10}
$$

and

$$
\begin{aligned}
\overline{u}_k &= \mathrm{Re}\int_{-\infty}^{\infty}\Big(\sqrt{p}A_{1k} + \sqrt{\frac{1}{p}}B_{1k}\Big)\exp[-p(z\alpha_{k0} + \mathrm{i}\eta r)]\,d\eta\\
\overline{w}_k &= \mathrm{Re}\int_{-\infty}^{\infty}\Big(\sqrt{p}A_{2k} + \sqrt{\frac{1}{p}}B_{2k}\Big)\exp[-p(z\alpha_{k0} + \mathrm{i}\eta r)]\,d\eta\\
\overline{T}_k &= \mathrm{Re}\int_{-\infty}^{\infty}\Big(\sqrt{p}A_{3k} + \sqrt{\frac{1}{p}}B_{3k}\Big)\exp[-p(z\alpha_{k0} + \mathrm{i}\eta r)]\,d\eta
\end{aligned}
\tag{4.11}
$$

A_{ij} and B_{ij} are given in the Appendix A and B.

5. Singularities of the integrals

In order to evaluate integrals (4.10) and (4.11), we consider a complex variable and distort the path of integration in the η-plane. The integrals are hexa-valued functions of η, when the choice of signs in α_{10}, α_{20} and α_{30} is unrestricted, and these representations require a six-leaved Riemann surface. However, at all points of the path of integration, we have confined to the leaf of the Riemann sheet defined by $\mathrm{Re}(\alpha_{j0}) \geqslant 0$, $(j = 1,2,3)$ everywhere due to our choice that $\mathrm{Re}(m_k) \geqslant 0$, and these are called the upper leaf. The possible singular points of the integrals are.

a) Branch points. The branch points are given by (discriminant of Eq. $(4.9)_1$)

$$\sqrt{(P\eta^2 + J)^2 - 4c_1c_2(\eta^2+1)(c_2\eta^2+1)} = 0 \qquad \alpha_{k0} = 0 \qquad k = 1,2,3 \tag{5.1}$$

and

$$\alpha_{k0} = 0 \quad \text{for} \quad k = 1,2,3 \quad \text{provide} \quad \eta = \pm\mathrm{i},\ \eta = \pm\frac{\mathrm{i}}{\sqrt{c_2}},\ \eta = \pm\mathrm{i}\sqrt{\tau_0} \tag{5.2}$$

For an isotropic medium, it reduces to $\eta = \pm\mathrm{i}$, $\eta = \pm\mathrm{i}v_1/v_2$, $\eta = \pm\mathrm{i}\sqrt{\tau_0}$ which are same as obtained by Nayfeh and Nasser (1972) and Sharma (1986) , where v_1 and v_2 are the velocities of dilatational and distortional waves. Again first equation of (5.1) is quadratic equation in η^2 and has real roots if the discriminant of this equation is positive.

Further, if

$$PJ > 2c_1c_2(c_2+1) \qquad P^2 > 4c_1c_2^2 \tag{5.3}$$

then equation (5.1) cannot have positive roots in η^2. Therefore, assume that equation (5.1) is hold and its discriminant is positive, thus the quartic equation has only pure imaginary pure roots. Physically, it is justified since we do not want the solution assumed to break down for points of the real η-axis. Otherwise, the waves of some wavelengths which correspond to these singular points of the real η-axis are propagated with amplitudes which are linear functions of depth in the medium. The corresponding branch points are of the type $\eta = \pm\mathrm{i}\eta_0$.

b) Poles. Other singular points of the integrands are its poles, which are given by

$$\begin{aligned} &(\alpha_{10}^2 - \alpha_{20}^2)(\alpha_{20}^2 - \alpha_{30}^2)(\alpha_{30}^2 - \alpha_{10}^2) = 0 \\ &\alpha_{k0} = 0 \qquad \varDelta'(\eta) = 0 \end{aligned} \tag{5.4}$$

Equation $(5.4)_1$ provides $\alpha_{10}^2 = \alpha_{20}^2 = \alpha_{30}^2$. This does not hold true as $\mathrm{Re}(\alpha_{k0}) \geqslant 0$ and $\alpha_{10} \neq \alpha_{20} \neq \alpha_{30}$. Therefore, this yields no singularities. The poles of $(5.4)_2$ coincide with branch points (5.2). Now to find poles given by $(5.4)_3$, on taking $\eta = \mathrm{i}/V$, rationalizing and simplifying it, reduces to Eq. (45) of Verma (2001), giving phase velocity for isothermal Rayleigh waves in a transversely isotropic half-space in thermoelasticity. It can be easily verified (see Abubakar, 1961) that on the assumption $P > Jc_2$, only one root of the resulting equation (see Eq. (45), Verma (2001)), satisfies $(5.4)_3$ on the upper leaf of the Riemann surface, and that is the root which lies in the range $0 < V^2 < c_2$. Let it be V_R^2, where V_R is the Rayleigh waves velocity in uncoupled theory of thermoelasticity, which is the same as obtained by Verma (2001). Thus, on the assumption made, the singularities of integrands $(4.9)_{4,5,6}$, which lie on the upper leaf of the Riemann surface are

$$\begin{array}{lll} \eta = \pm \mathrm{i} & \eta = \pm \dfrac{\mathrm{i}}{\sqrt{c_2}} & \eta = \pm \mathrm{i}\sqrt{\tau_0} \\ \eta = \pm \mathrm{i}\eta_0 & \eta = \pm \dfrac{\mathrm{i}}{V_r} & \end{array} \tag{5.5}$$

In the special case of $\tau_0 < 1$ and $V_R^2 = 0.1834$ for a zinc crystal, the path of integration is along the real axis. To make the functions of η single-valued in the complex plane of integration, we make a cut joining the singularities $\mathrm{i}/\sqrt{c_2}$ and $-\mathrm{i}/\sqrt{c_2}$ in the η-plane.

First, we consider one of integrals $(4.9)_{3-6}$, say

$$\overline{u}_1(x,z,p) = \frac{1}{\pi}\mathrm{Im}\int\limits_{z/\sqrt{c_2}}^{\infty} \Big(\frac{A_{11}}{p} + \frac{B_{11}}{p^2}\Big)\frac{d\eta}{dt}\mathrm{e}^{-pt}\, dt \tag{5.6}$$

Using the equation, we get

$$u_1(x,z,t) = \mathrm{Re}\Big[\int\limits_0^t A_{11}H\Big(\overline{t} - \frac{z}{\sqrt{c_2}}\Big)\frac{\partial\eta}{\partial\overline{t}}\, d\overline{t} + \int\limits_0^t dt\int\limits_0^{\overline{t}} B_{11}H\Big(t_1 - \frac{z}{\sqrt{c_2}}\Big)\frac{\partial\eta}{\partial t}\, dt\Big] \tag{5.7}$$

Similarly

$$\begin{aligned} u_2(x,z,t) = \mathrm{Re}\Big\{ &\int\limits_0^t A_{12}H\Big(\overline{t} - \frac{z}{\sqrt{c_1}}\Big)\frac{\partial\eta_2}{\partial\overline{t}}\, d\overline{t} \\ &+ \int\limits_0^t \Big[\int\limits_0^{\overline{t}} B_{12}H\Big(t_1 - \frac{z}{\sqrt{c_1}}\Big)\frac{\partial\eta_2}{\partial t_1}\, dt_1\Big]\, dt\Big\} \end{aligned} \tag{5.8}$$

$$u_3(x,z,t) = \mathrm{Re}\Big\{ \int\limits_0^t A_{13} H\Big(\overline{t} - z\sqrt{\frac{\tau_0}{\overline{k}}}\Big) \frac{\partial \eta_3}{\partial \overline{t}}\, d\overline{t}$$
$$+ \int\limits_0^t \Big[\int\limits_0^{\overline{t}} B_{13} H\Big(t_1 - z\sqrt{\frac{\tau_0}{\overline{k}}}\Big) \frac{\partial \eta_3}{\partial t_1}\, dt_1 \Big]\Big\}$$

Thus, we have

$$u(x,y,t) = \sum_{k=1}^{3} \mathrm{Re}\Big\{ \int\limits_0^t (\overline{t} + s_k z) \frac{\partial \eta_k}{\partial \overline{t}}\, d\overline{t} + \int\limits_0^t \Big[\int\limits_0^{\overline{t}} B_{1k} H(t_1 - s_k z) \frac{\partial \eta_k}{\partial t_1}\, dt_1 \Big]\, dt \Big\} \tag{5.9}$$

where $s_1 = 1/\sqrt{c_2}$, $s_2 = 1/\sqrt{c_1}$, $s_3 = \sqrt{\tau_0/\overline{k}}$ are the slowness of the transverse dilatational and the thermal waves, respectively.

Similarly

$$w(x,z,t) = \sum_{k=1}^{3} \mathrm{Re}\Big\{ \int\limits_0^t A_{2k} H(\overline{t} - s_k z) \frac{\partial \eta_k}{\partial \overline{t}}\, d\overline{t}$$
$$+ \int\limits_0^t \Big[\int\limits_0^{\overline{t}} B_{2k} H(t_1 - s_k z) \frac{\partial \eta_k}{\partial t_1}\, dt_1 \Big]\, dt \Big\}$$
$$T(x,z,t) = \sum_{k=1}^{3} \mathrm{Re}\Big[A_{3k} H(\overline{t} - s_k z) \frac{\partial \eta_k}{\partial t} + \int\limits_0^t B_{3k} H(\overline{t} - s_k z) \frac{\partial \eta_k}{\partial \overline{t}}\, dt \Big] \tag{5.10}$$

where η_k, $k = 1,2,3$ can be determined from $t = \alpha_{k0} z + \mathrm{i}\eta_k x$. Also when the thermoelastic coupling constant ε_1 vanishes, then the temperature field vanishes as well.

6. Numerical results and discussions

The results obtained theoretically for temperature and stresses are computed numerically for a single crystal of zinc for which the physical data is given as

$\varepsilon_1 = 0.0221$	$c_1 = 0.385$	$c_2 = 0.2385$
$c_3 = 0.549$	$\overline{k} = 1.0$	$\overline{\beta} = 0.9$
$\tau_0 = 0.02$	$T_0 = 296\,\mathrm{K}$	$c_{11} = 1.628 \cdot 10^{11}\,\mathrm{Nm}^{-2}$
$\rho = 7.14 \cdot 10^3\,\mathrm{kmg}^{-3}$	$w^* = 5.01 \cdot 10^{11}\,\mathrm{s}^{-1}$	

The computations were carried out for four values of time, namely $\tau = 0.05$, 0.1, 0.2, 0.5 at the surface $z = 0$ and for a stress free heat transfer coefficient $h \to 0$. The results for thermal stresses with respect to distance are shown in Figs. 1-4, respectively. From the figures, it is observed that the negative values of stresses are due to compression by a point load at the surface, and they increase in magnitude with the passage of time. The temperature also decreases from a positive value with the passage of time. Also the variations of all these quantities are more prominent at small times and decrease with the passage of time. This established the fact that the second sound effect is short lived. All these quantities vanish when we move away from the heat source at a certain distance at all times, which shows the existence of the wave front and ascertain the fact that the generalized theory of thermoelasticity admits a finite velocity of heat.

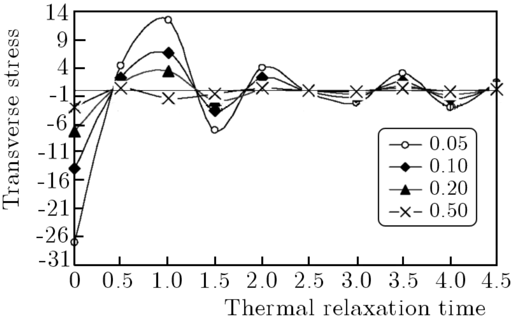

Fig. 1. Variation of the transverse stress near the surface with distance and time

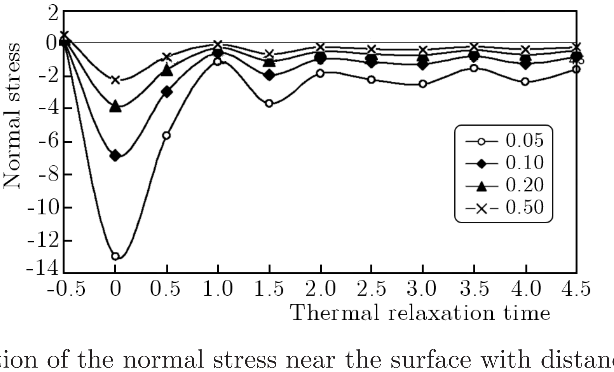

Fig. 2. Variation of the normal stress near the surface with distance and time

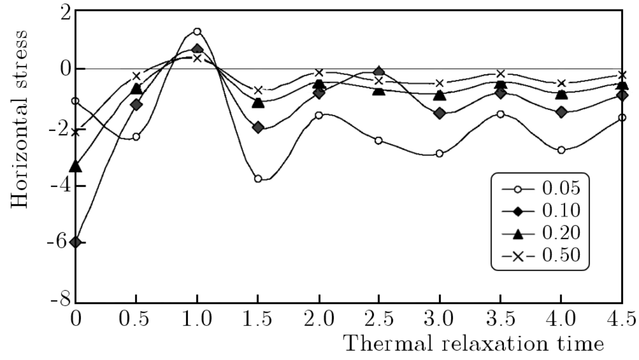

Fig. 3. Variation of the horizontal stress near the surface with distance and time

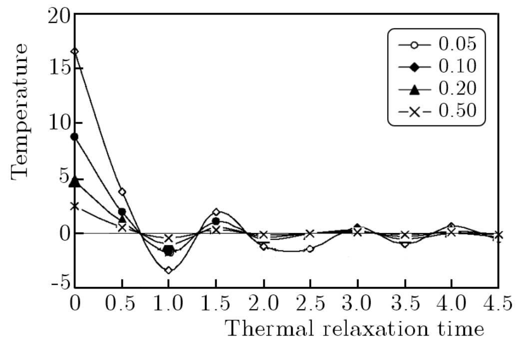

Fig. 4. Variation of the temperature near the surface with distance and time

Appendix A

$$z, k = 1, 2$$

$$A_{1k} = -\Re_1\sqrt{\frac{2\eta}{\pi r}}\exp\Big(\mathrm{i}\frac{\pi}{4}\Big)\mathbb{N}_{k0}(\eta)\frac{\alpha_{k1}^2}{2\alpha_{k0}}$$

$$A_{13} = -\Re_1\sqrt{\frac{2\eta}{\pi r}}\exp\Big(\mathrm{i}\frac{\pi}{4}\Big)\mathbb{N}_{30}(\eta)\frac{\alpha_{31}^2}{2\alpha_{30}}z\exp\Big(\frac{-z}{2\overline{k}\alpha_{30}}\Big) \qquad \Re_1 = \frac{-Q^*\varepsilon_1\eta^2}{2\pi\Delta'_{10}}$$

$$A_{2k} = -\Re_2\sqrt{\frac{2\eta}{\pi r}}\exp\Big(\mathrm{i}\frac{3\pi}{4}\Big)\mathbb{N}_{k0}(\eta)\alpha_{k0}\frac{\alpha_{k1}^2}{2} \qquad z, k = 1, 2$$

$$A_{23} = -\Re_2\sqrt{\frac{2\eta}{\pi r}}\exp\Big(\mathrm{i}\frac{3\pi}{4}\Big)\mathbb{N}_{30}(\eta)\alpha_{30}\frac{\alpha_{31}^2}{2}z\exp\Big(\frac{-z}{2\overline{k}\alpha_{30}}\Big) \qquad \Re_2 = \frac{-Q^*\varepsilon_1\eta}{2\pi\Delta'_{10}}$$

$$A_{3k} = -\Re_3\sqrt{\frac{2\eta}{\pi r}}\exp\Big(\mathrm{i}\frac{3\pi}{4}\Big)\Big\{\mathbb{N}_{k0}(\eta)b_{k0}(\eta) + \varepsilon_1\Big[\mathbb{N}_{k0}(\eta)b_{k1}(\eta) + \mathbb{N}_{k1}b_{k0}(\eta)$$
$$-\mathbb{N}_{k0}(\eta)b_{k0}(\eta)\frac{\alpha_{k1}^{*2}}{2\alpha_{k0}}z - \Big(\mathbb{N}_{k2}b_{k0}(\eta) - \mathbb{N}_{k0}(\eta)b_{k0}(\eta)\mathbb{Z}(\eta)\frac{\alpha_{k1}^2}{2\alpha_{k0}}z\Big)\Big]\Big\}$$

$$
\begin{aligned}
A_{33} &= -\Re_3\sqrt{\frac{2\eta}{\pi r}}\exp\Big(\mathrm{i}\frac{3\pi}{4}\Big)\Big\{\mathbb{N}_{30}(\eta)b_{30}(\eta)+\varepsilon_1\Big[\mathbb{N}_{30}(\eta)b_{31}(\eta)+\mathbb{N}_{31}b_{30}(\eta)\\
&\quad -\mathbb{N}_{30}(\eta)b_{30}(\eta)\Big(\frac{\alpha_{31}^{*2}}{2}-\frac{\alpha_{31}^2}{2\overline{k}\alpha_{30}}\Big)z-(\mathbb{N}_{32}b_{30}(\eta)\\
&\quad -\mathbb{N}_{30}(\eta)b_{30}(\eta)\mathbb{Z}(\eta))\frac{\alpha_{31}^2}{2\alpha_{30}}z\Big]\Big\}\exp\Big(\frac{-z}{2\overline{k}\alpha_{30}}\Big)
\end{aligned}
$$

Appendix B

$$
\begin{aligned}
&k=1,2\\
&B_{1k}=\Re_3\eta\sqrt{\frac{2\eta}{\pi r}}\exp\Big(\mathrm{i}\frac{\pi}{4}\Big)\Big\{\mathbb{N}_{k0}(\eta)+\varepsilon_1\Big[\mathbb{N}_{k1}(\eta)-\mathbb{N}_{k0}(\eta)\frac{\alpha_{k1}^{*2}}{2\alpha_{k0}}z\\
&\quad -\Big(\mathbb{N}_{k2}-\mathbb{N}_{k0}(\eta)\mathbb{Z}(\eta)\frac{\alpha_{k1}^2}{2\alpha_{k0}}z\Big)\Big]\Big\} \qquad k=1,2\\
&B_{13}=\Re_3\eta\sqrt{\frac{2\eta}{\pi r}}\exp\Big(\mathrm{i}\frac{\pi}{4}\Big)\Big\{\mathbb{N}_{30}(\eta)+\varepsilon_1\Big[\mathbb{N}_{31}(\eta)-\mathbb{N}_{30}(\eta)\Big(\frac{\alpha_{31}^{*2}}{2}-\frac{\alpha_{31}^2}{2\overline{k}\alpha_{30}}\Big)z\\
&\quad +\mathbb{N}_{30}(\eta)\mathbb{Z}(\eta)\frac{\alpha_{31}^2}{2\alpha_{30}}z\Big)\Big]\Big\}\exp\Big(\frac{-z}{2\overline{k}\alpha_{30}}\Big)\\
&\Re_3=\frac{-Q^*\eta}{2\pi\varDelta_{10}'}\\
&B_{2k}=\Re_3\sqrt{\frac{2\eta}{\pi r}}\exp\Big(\mathrm{i}\frac{3\pi}{4}\Big)\Big\{\mathbb{N}_{k0}(\eta)a_{k0}(\eta)\alpha_{k0}+\varepsilon_1\Big[\mathbb{N}_{k1}(\eta)a_{k0}(\eta)\alpha_{k0}\\
&\quad +\Big(a_{k00}(\eta)\alpha_{k0}+a_{k0}(\eta)\frac{\alpha_{k1}^2}{2\alpha_{k0}}\Big)\mathbb{N}_{k0}(\eta)-\mathbb{N}_{k0}(\eta)a_{k0}(\eta)\frac{\alpha_{k1}^{*2}}{2}z\\
&\quad -[(\mathbb{N}_{k2}(\eta)a_{k0}(\eta)+\mathbb{N}_{k0}(\eta)a_{k0}^*(\eta))\alpha_{k0}-\mathbb{N}_{k1}(\eta)\mathbb{Z}(\eta)a_{k0}(\eta)\alpha_{k0}]\frac{\alpha_{k1}^2}{2\alpha_{k0}}z\Big)\Big]\Big\}\\
&B_{23}=\Re_3\sqrt{\frac{2\eta}{\pi r}}\exp\Big(\mathrm{i}\frac{3\pi}{4}\Big)\Big\{\mathbb{N}_{30}(\eta)a_{30}(\eta)\alpha_{30}+\varepsilon_1\Big[\mathbb{N}_{31}(\eta)a_{30}(\eta)\alpha_{30}\\
&\quad +\Big(a_{300}(\eta)\alpha_{30}+\alpha_{30}(\eta)\frac{\alpha_{31}^2}{2\alpha_{30}}\Big)\mathbb{N}_{30}(\eta)-\mathbb{N}_{30}(\eta)a_{30}(\eta)\alpha_{30}\Big(\frac{\alpha_{31}^{*2}}{2}-\frac{\alpha_{31}^2}{4\overline{k}\alpha_{30}}\Big)\\
&\quad -\Big[\mathbb{N}_{30}(\eta)\Big((a_{30}^*(\eta)+a_{40}(\eta))\alpha_{30}+\frac{\alpha_{30}(\eta)}{2\overline{k}\alpha_{30}}\Big)\\
&\quad -\mathbb{N}_{30}(\eta)\alpha_{30}(\eta)\mathbb{Z}(\eta)\frac{\alpha_{31}^2}{2}z\Big]\Big]\Big\}\exp\Big(\frac{-z}{2\overline{k}\alpha_{30}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
B_{3k} = {} & \Re_3\sqrt{\frac{2\eta}{\pi r}}\exp\Big(\mathrm{i}\frac{3\pi}{4}\Big)\Big\{[\mathbb{N}_{k2}(\eta)-\mathbb{N}_{k0}(\eta)\mathbb{Z}(\eta)]b_{k0}(\eta)\\
& +\varepsilon_1\Big[\mathbb{N}_{k2}(\eta)b_{k1}(\eta)+\mathbb{N}_{k3}(\eta)b_{k0}(\eta)+\mathbb{N}_{k0}(\eta)b^*_{k1}(\eta)\\
& -[\mathbb{N}_{k0}(\eta)b_{k1}(\eta)+\mathbb{N}_{k1}(\eta)b_{k0}(\eta)]\mathbb{Z}(\eta)-\mathbb{N}_{k0}(\eta)b_{k0}(\eta)\mathbb{Z}_1(\eta)\\
& -[\mathbb{N}_{k2}(\eta)b_{k0}(\eta)-\mathbb{N}_{k0}(\eta)b_{k0}(\eta)\mathbb{Z}(\eta)]\frac{\alpha^2_{k1}}{2\alpha_{k0}}z-\Big(\mathbb{N}_{k4}(\eta)b_{k0}(\eta)\\
& -\mathbb{N}_{k2}(\eta)b_{k0}(\eta)\mathbb{Z}(\eta)-\mathbb{N}_{k0}(\eta)b_{k0}(\eta)\mathbb{Z}_3(\eta)\frac{\alpha^2_{k1}}{2\alpha_{k0}}z\Big)\Big]\Big\}\\
B_{33} = {} & \Re_3\sqrt{\frac{2\eta}{\pi r}}\exp\Big(\mathrm{i}\frac{3\pi}{4}\Big)\Big\{[\mathbb{N}_{30}(\eta)b_{32}(\eta)-\mathbb{N}_{30}(\eta)b_{32}(\eta)\mathbb{Z}(\eta)]\\
& +\varepsilon_1\Big[\mathbb{N}_{30}(\eta)[b^*_{31}(\eta)+b_{33}(\eta)]+\mathbb{N}_{31}(\eta)b_{32}(\eta)+\mathbb{N}_{32}(\eta)b_{30}(\eta)\\
& -[\mathbb{N}_{30}(\eta)b_{32}(\eta)-\mathbb{N}_{30}(\eta)b_{30}(\eta)\mathbb{Z}(\eta)]\Big(\frac{\alpha^{*2}_{31}}{2}-\frac{\alpha^2_{31}}{4\bar{k}\alpha_{30}}\Big)\\
& -\mathbb{N}_{30}(\eta)b_{30}(\eta)\frac{\alpha^{*2}_{31}}{4\bar{k}\alpha_{30}}z-\Big(\mathbb{N}_{30}(\eta)b_{34}(\eta)-\mathbb{N}_{30}(\eta)b_{32}(\eta)\mathbb{Z}(\eta)\\
& -\mathbb{N}_{30}(\eta)b_{30}(\eta)\mathbb{Z}_3(\eta)\frac{\alpha^2_{31}}{2\alpha_{30}}z\Big)\Big]\Big\}\exp\Big(\frac{-z}{2\bar{k}\alpha_{30}}\Big)
\end{aligned}
$$

where i, j and k are in the cyclic order $(i,j,k=1,2,3,$ and $i\neq j\neq k)$ and

$$
\begin{aligned}
& \mathbb{N}_{i0}(\eta)=T_{j00}(\eta)S_{k0}(\eta)-S_{j0}(\eta)T_{k00}(\eta)\\
& \mathbb{N}_{i1}(\eta)=T_{j0}(\eta)S_{k0}(\eta)+S_{k00}(\eta)T_{j00}(\eta)-[T_{k0}(\eta)S_{j0}(\eta)+S_{j00}(\eta)T_{k00}(\eta)]\\
& \mathbb{N}_{12}(\eta)=S_{31}(\eta)T_{200}(\eta)-S_{20}(\eta)T_{301}(\eta)\\
& \mathbb{N}_{22}(\eta)=S_{10}(\eta)T_{301}(\eta)-S_{31}(\eta)T_{100}(\eta)\\
& \mathbb{N}_{32}(\eta)=T^*_{10}(\eta)S_{20}(\eta)+S^*_{200}(\eta)T_{100}(\eta)-[T^*_{20}(\eta)S_{10}(\eta)+S^*_{100}(\eta)T_{200}(\eta)]\\
& \mathbb{N}_{13}(\eta)=T_{20}(\eta)S_{31}(\eta)+S^*_{301}(\eta)T_{200}(\eta)+S_{30}(\eta)T^*_{20}(\eta)\\
& \quad -[T^*_{30}(\eta)S_{20}(\eta)+S_{200}(\eta)T_{301}(\eta)+S^*_{200}(\eta)T_{300}(\eta)]\\
& \mathbb{N}_{25}(\eta)=T_{302}(\eta)S_{300}(\eta)+S_{10}(\eta)T^{**}\mathbb{N}_{30}(\eta)+S_{100}(\eta)T_{301}(\eta)\\
& \quad -[T_{10}(\eta)S_{32}(\eta)+S^*_{302}(\eta)T_{100}(\eta)+S_{31}(\eta)T^*_{10}(\eta)]\\
& \mathbb{N}_{14}(\eta)=S_{32}(\eta)T_{200}(\eta)-S_{20}(\eta)T_{302}(\eta)\\
& \mathbb{N}_{24}(\eta)=S_{10}(\eta)T_{302}(\eta)-S_{32}(\eta)T_{100}(\eta)\\
& \mathbb{N}_{15}(\eta)=T^*_{20}(\eta)S_{31}(\eta)+S^*_{302}(\eta)T_{200}(\eta)+S_{32}(\eta)T_{20}(\eta)\\
& \quad -[T_{301}(\eta)S^*_{200}(\eta)+S_{200}(\eta)T_{302}(\eta)+S_{20}(\eta)T^{**}\mathbb{N}_{30}(\eta)]\\
& \mathbb{N}_{23}(\eta)=T^*_{30}(\eta)S_{10}(\eta)+S_{100}(\eta)T_{301}(\eta)+S^*_{100}(\eta)T_{300}(\eta)\\
& \quad -[T^*_{10}(\eta)S_{30}(\eta)+S_{31}(\eta)T_{10}(\eta)+S^*_{301}(\eta)T_{100}(\eta)]
\end{aligned}
$$

$$S_{k0}(\eta)=\frac{D_{k0}(\eta)\alpha_{k0}}{\Delta_k(\eta)}\qquad S_{k00}(\eta)=\frac{L_{k0}(\eta)}{\Delta_k(\eta)}\qquad S^*_{k00}(\eta)=\frac{L^*_{k0}(\eta)}{\Delta_k(\eta)}$$

$$S_{31}(\eta)=\frac{D'_{30}(\eta)\overline{k}\Delta_3(\eta)-(c_1-c_3\overline{\beta})D_{30}(\eta)\alpha_{30}}{\overline{k}\Delta_k^2(\eta)}$$

$$S_{32}(\eta)=\frac{1}{\Delta_3(\eta)}\Big[\frac{D''_{30}(\eta)\overline{k}\Delta_3(\eta)-(c_1-c_3\overline{\beta})D'_{30}(\eta)}{\Delta_3(\eta)}+\frac{(c_1-c_3\overline{\beta})^2D_{30}(\eta)\alpha_{30}}{\overline{k}\Delta_3^2(\eta)}\Big]$$

$$S^*_{301}(\eta)=\frac{1}{\Delta_3(\eta)}\frac{[L^*_{30}(\eta)+L_{40}(\eta)]\overline{k}\Delta_3(\eta)-(c_1-c_3\overline{\beta})L_{30}(\eta)}{\overline{k}\Delta_3(\eta)}$$

$$S^*_{302}(\eta)=\frac{1}{\Delta_3(\eta)}\Big[\frac{[L^*_{40}(\eta)+L_{50}(\eta)]\overline{k}\Delta_3(\eta)-[L^*_{30}(\eta)+L_{40}(\eta)](c_1-c_3\overline{\beta})L_{30}(\eta)}{\overline{k}\Delta_3(\eta)}$$
$$+\frac{L_{30}(\eta)(c_1-c_3\overline{\beta})^2}{\overline{k}^2\Delta_3^2(\eta)}\Big]$$

$$D_{k0}(\eta)=c_2\overline{\beta}\alpha_{k0}^2-(c_3-\overline{\beta})\eta^2-\overline{\beta}$$
$$\Delta\mathbb{N}_{k0}(\eta)=(c_1-c_3\overline{\beta})\alpha_{k0}^2-c_2\eta^2-1$$

$$D'_{30}(\eta)=\frac{D_{30}(\eta)}{2\overline{k\beta}\alpha_{30}}+\frac{c_2\overline{\beta}^2\alpha_{30}}{\overline{k}}\qquad D''_{30}(\eta)=\frac{c_2\overline{\beta}^2}{2\overline{k\beta}\alpha_{30}}$$

$$(L_{k0}(\eta),L^*_{k0}(\eta))=\Big(\frac{D_{k0}(\eta)}{2\alpha_{k0}}+c_2\overline{\beta}\alpha_{k0}-\frac{D_{k0}(\eta)(c_1-c_3\overline{\beta})\alpha_{k0}}{\Delta_k(\eta)}\Big)(\alpha_{k1}^2,\alpha_{k1}^{*2})$$

$$L_{40}(\eta)=\Big\{\frac{1}{2\overline{k}\alpha_{30}}\Big(2c_2\overline{\beta}-\frac{D_{30}(\eta)}{\Delta_3(\eta)\alpha_{30}^2}\Big)$$
$$-\frac{c_1-c_3\overline{\beta}}{\overline{k}\Delta_3(\eta)}\Big[\frac{D_{30}(\eta)}{2\alpha_{30}}+\alpha_{30}\Big(c_2\overline{\beta}-\frac{D_{30}(\eta)}{\Delta_3(\eta)}\Big)\Big]\Big\}\alpha_{31}^2$$

$$L^*_{40}(\eta)=\frac{1}{2\overline{k}^2\alpha_{30}^2}\Big(\frac{D_{30}(\eta)}{\Delta_3(\eta)\alpha_{30}^2}-c_2\overline{\beta}\Big)$$
$$+\Big(\frac{D_{30}(\eta)}{\Delta_3(\eta)}-c_2\overline{\beta}\Big)\Big(\frac{c_1-c_3\overline{\beta}}{2\overline{k}^2\Delta_3(\eta)\alpha_{30}}-\frac{(c_1-c_3\overline{\beta})^2\alpha_{30}}{\overline{k}^2\Delta_3^2(\eta)}\Big)$$

$$(a_{k0}(\eta),a^*_{k0}(\eta))=\frac{D_{10}(\eta)}{\Delta_k(\eta)}(\alpha_{k1}^2,a_{k1}^{*2})$$

$$(a_{k00}(\eta),a^*_{k00}(\eta))=\Big(c_2\overline{\beta}-\frac{D_{k0}(\eta)}{\Delta_k(\eta)}(c_1-c_3\overline{\beta})\Big)\Big(\frac{\alpha_{k1}^2}{\Delta_k(\eta)},\frac{a_{k1}^{*2}}{\Delta_k(\eta)}\Big)$$

$$(a_{40}(\eta),a^*_{40}(\eta))=\Big(\frac{c_2\overline{\beta}^2}{\overline{k}}-\frac{D_{30}(\eta)}{\Delta_3(\eta)\overline{k}}(c_1-c_3\overline{\beta})\Big)\Big(\frac{\alpha_{31}^2}{\Delta_3(\eta)},\frac{a_{31}^{*2}}{\Delta_3(\eta)}\Big)$$

$$a_{400}(\eta) = -c_2\overline{\beta} + \frac{D_{30}(\eta)}{\Delta_3(\eta)}(c_1 - c_3\overline{\beta})^2\Big(1 - \frac{1}{\overline{\overline{k}}}\Big)$$

$$a^*_{400}(\eta) = a_{400}(\eta)\frac{\alpha^{*2}_{31}}{\Delta_3(\eta)}$$

$$a_{50}(\eta) = \Big(\frac{D_{30}(\eta)(c_1 - c_3\overline{\beta})}{\Delta_3(\eta)} - c_2\overline{\beta}^2\Big)\frac{\alpha^2_{31}(c_1 - c_3\overline{\beta})}{\Delta^2_3(\eta)\overline{k}^2}$$

$$a_{500}(\eta) = \Big(c_2\overline{\beta} - \frac{D_{30}(\eta)}{\Delta_3(\eta)\overline{k}^2}(c_1 - c_3\overline{\beta})^3\Big)\frac{\alpha^2_{31}}{\Delta_3(\eta)}$$
$$-\Big(\frac{c_2\overline{\beta}^2}{\overline{k}} - \frac{D_{30}(\eta)}{\Delta_3(\eta)\overline{k}}(c_1 - c_3\overline{\beta})\Big)\frac{(c_1 - c_3\overline{\beta})^3}{\Delta^2_3(\eta)\overline{k}}\alpha^2_{31}$$
$$+\Big(\frac{D_{30}(\eta)(c_1 - c_3\overline{\beta})^3}{\Delta^2_3(\eta)\overline{k}^2} - \frac{c_2\overline{\beta}^2(c_1 - c_3\overline{\beta})}{\overline{k}^2}\Big)(c_1 - c_3\overline{\beta})\alpha^2_{31}$$

$$b_{k0}(\eta) = \frac{E_{k0}(\eta)}{\Delta_k(\eta)} \qquad (b_{k1}, b^*_{k1}) = \frac{F_{k0}(\eta) - b_{k0}(\eta)(c_1 - c_3\overline{\beta})(\alpha^2_{k1}, \alpha^{*2}_{k1})}{\Delta_k(\eta)}$$

$$b_{31}(\eta) = \frac{F_{30}(\eta) - b_{30}(\eta)(c_1 - c_3\overline{\beta})}{\Delta_k(\eta)}\alpha^2_{31}$$

$$b_{32}(\eta) = \Big(\frac{-2c_1c_2}{\overline{k}} - \frac{b_{30}(\eta)(c_1 - c_3\overline{\beta})}{\Delta_k(\eta)\overline{k}}\Big)\alpha^2_{31}$$

$$b_{33}(\eta) = \frac{-2c_1c_2}{\overline{k}} - b_{32}(\eta)(c_1 - c_3\overline{\beta}) - \frac{b_{31}(\eta)(c_1 - c_3\overline{\beta})}{\overline{k}\alpha^2_{31}}$$

$$b_{34}(\eta) = \Big(\frac{F_{30}(\eta)(c_1 - c_3\overline{\beta})^2}{\Delta^2_3(\eta)\overline{k}^2} + \frac{c_1c_2}{\overline{k}} - E'_{30}(\eta)\frac{c_1 - c_3\overline{\beta}}{\overline{k}\Delta_3(\eta)}\Big)\frac{1}{\Delta_3(\eta)}$$
$$\Big(\frac{c_2\overline{\beta}^2}{\overline{k}} - \frac{D_{30}(\eta)}{\Delta_3(\eta)\overline{k}}(c_1 - c_3\overline{\beta})\Big)\Big(\frac{\alpha^2_{31}}{\Delta_3(\eta)}, \frac{a^{*2}_{31}}{\Delta_3(\eta)}\Big)$$

$$b^*_{31}(\eta) = b_{31}(\eta)\frac{\alpha^{*2}_{31}}{\alpha^2_{31}} \qquad b^*_{33}(\eta) = b_{33}(\eta)\frac{\alpha^{*2}_{31}}{\alpha^2_{31}}$$

$$b_{35}(\eta) = \Big[\frac{b_{31}(\eta)(c_1 - c_3\overline{\beta})^2}{\Delta^2_3(\eta)\overline{k}^2} - \frac{c_1 - c_3\overline{\beta}}{\overline{k}\Delta_3(\eta)}\Big(\frac{c_1c_2}{\overline{k}^2} - b_{32}(\eta)(c_1 - c_3\overline{\beta})\Big)$$
$$-\frac{c_1 - c_3\overline{\beta}}{\overline{k}\Delta^2_3(\eta)}\Big(\frac{F_{30}(\eta)(c_1 - c_3\overline{\beta})^2}{\Delta^2_3(\eta)\overline{k}^2} - \frac{E'_{30}(\eta)}{k\Delta_3(\eta)} + \frac{c_1c_2}{\overline{k}^2}\Big)\Big]\alpha^2_{31}$$

$$E_{k0}(\eta) = G_{k0}(\eta)H_{k0}(\eta) + H'_{k0}(\eta) \qquad E'_{30}(\eta) = \frac{E_{30}(\eta)}{\overline{k}}$$

$$E''_{30}(\eta) = \frac{c_1 c_2}{\overline{k}^2} \qquad F_{k0}(\eta) = c_3\eta^2 - c_1 GN_{k0}(\eta) - c_2 H_{k0}(\eta)$$

$$G_{k0}(\eta) = \eta^2 + 1 - c_2\alpha_{k0}^2 \qquad H_{k0}(\eta) = c_2\eta^2 + 1 - c_1\alpha_{k0}^2$$

$$H'_{k0}(\eta) = c_3\eta^2\alpha_{k0}^2 \qquad \mathbb{Z}(\eta) = \frac{\Delta'_{20}(\eta)}{\Delta'_{10}(\eta)}$$

$$\mathbb{Z}_1(\eta) = \frac{\Delta'_{21}(\eta) - 2(\eta)\Delta_{11}(\eta)}{\Delta'_{10}(\eta)} \qquad \mathbb{Z}_2(\eta) = \frac{\Delta'_{30}(\eta) - \mathbb{Z}(\eta)\Delta'_{20}(\eta)}{\Delta'_{10}(\eta)}$$

$$\mathbb{Z}_3(\eta) = \frac{\Delta'_{31}(\eta) - \frac{\Delta'_{30}(\eta)}{\Delta'_{10}(\eta)}\Delta'_{11}(\eta) + 2\mathbb{Z}(\eta)[\mathbb{Z}(\eta)\Delta'_{11}(\eta) - \Delta'_{21}(\eta)] - \Delta'_{11}(\eta)\mathbb{Z}_2(\eta)}{\Delta'_{10}(\eta)}$$

$$\Delta'_{10}(\eta) = -\sum_{j=1}^{3}\mathbb{N}_{j0}(\eta)a_{j0}(\eta)\alpha_{j0}$$

$$\begin{aligned}\Delta'_{20}(\eta) = &-\Big[\sum_{j=1}^{2}[\mathbb{N}_{j0}(\eta)a^*_{j0}(\eta) + \mathbb{N}_{j2}(\eta)a_{j0}(\eta)]\alpha_{j0}\\ &+\mathbb{N}_{30}(\eta)\Big([a^*_{30}(\eta) + a_{40}(\eta)]\alpha_{30} + \frac{a_{30}(\eta)}{2\overline{k}\alpha_{30}}\Big)\Big]\end{aligned}$$

$$\begin{aligned}\Delta'_{30}(\eta) = &-\Big[\sum_{j=1}^{2}[\mathbb{N}_{j2}(\eta)a^*_{j0}(\eta) + \mathbb{N}_{j4}(\eta)a_{j0}(\eta)]\alpha_{j0}\\ &+\mathbb{N}_{30}(\eta)\Big([a^*_{40}(\eta) + a_{50}(\eta)]\alpha_{30} + \frac{a_{40}(\eta) + a^*_{30}(\eta)}{2\overline{k}\alpha_{30}} - \frac{a_{30}(\eta)\alpha^{*2}_{31}}{4\overline{k}\alpha^2_{30}}\Big)\Big]\end{aligned}$$

$$\Delta'_{11}(\eta) = -\Big[\sum_{j=1}^{2}\Big[\mathbb{N}_{j1}(\eta)a_{j0}(\eta)\alpha_{j0} + \Big(a_{j00}(\eta)\alpha_{j0} + a_{j0}(\eta)\frac{\alpha^2_{j1}}{2\alpha_{j0}}\Big)\mathbb{N}_{j0}(\eta)\Big]\Big]$$

$$\begin{aligned}\Delta'_{21}(\eta) = &-\Big[\sum_{j=1}^{2}\Big[\mathbb{N}_{j3}(\eta)a_{j0}(\eta)\alpha_{j0} + \Big(a_{j00}(\eta)\alpha_{j0} + a_{j0}(\eta)\frac{\alpha^2_{j1}}{2\alpha_{j0}}]\Big)\mathbb{N}_{j2}(\eta)\Big]\Big]\\ &+\sum_{j=1}^{2}\Big[\mathbb{N}_{j1}(\eta)a^*_{j0}(\eta)\alpha_{10} + \mathbb{N}_{j0}(\eta)\Big(a^*_{j00}(\eta)\alpha_{j0} + a^*_{j0}(\eta)\frac{\alpha^2_{j1}}{2\alpha_{j0}} + a_{j0}(\eta)\frac{\alpha^{*2}_{j1}}{2\alpha_{j0}}\Big)\Big]\\ &+\mathbb{N}_{30}(\eta)\Big[a_{40}(\eta) + a^*_{30}(\eta)\frac{\alpha^2_{31}}{2\alpha_{30}} + [a^*_{300}(\eta) + a_{400}(\eta)]\alpha_{30}\\ &+\frac{a_{300}(\eta)}{2\overline{k}\alpha_{30}} + \Big(\frac{\alpha^{*2}_{31}}{2\alpha_{30}} - \frac{\alpha^2_{31}}{4\overline{k}\alpha^2_{30}}\Big)a_{30}(\eta)\Big]\\ &+\mathbb{N}_{31}(\eta)\Big(a_{40}(\eta) + a^*_{30}(\eta)\alpha_{30} + \frac{\alpha_{30}}{2\overline{k}\alpha_{30}}\Big) + \mathbb{N}_{32}(\eta)a_{30}(\eta)\alpha_{30}\end{aligned}$$

$$\begin{aligned}
\Delta'_{31}(\eta) = &-\Big[\sum_{j=1}^{2}\Big[\mathbb{N}_{j3}(\eta)a^*_{j0}(\eta)\alpha_{j0}\\
&+\Big(a^*_{j00}(\eta)\alpha_{j0}+\frac{a^*_{j0}(\eta)\alpha^2_{j1}+a_{j0}(\eta)\alpha^{*2}_{j1}}{2\alpha_{j0}}\Big)\mathbb{N}_{j2}(\eta)\Big]\Big]\\
&+\sum_{j=1}^{2}\Big[\mathbb{N}_{j5}(\eta)a_{j0}(\eta)\alpha_{10}+\mathbb{N}_{j4}(\eta)\Big(a_{j00}(\eta)\alpha_{j0}+a_{j0}(\eta)\frac{\alpha^2_{j1}}{2\alpha_{j0}}\Big)\\
&+\mathbb{N}_{j0}(\eta)a^*_{j0}(\eta)\frac{\alpha^{*2}_{j1}}{2\alpha_{j0}}\Big]+\mathbb{N}_{30}(\eta)\Big[\mathbb{N}_{500}(\eta)+a^*_{400}(\eta)\alpha_{30}+[a^*_{40}(\eta)+a_{50}(\eta)]\frac{\alpha^2_{31}}{2\alpha_{30}}\\
&+\frac{a_{400}(\eta)+a^*_{300}(\eta)}{2\overline{k}\alpha_{30}}+\Big(\frac{\alpha^{*2}_{31}}{2\alpha_{30}}-\frac{\alpha^2_{31}}{4\overline{k}\alpha^2_{30}}\Big)a_{30}(\eta)\Big]+\mathbb{N}_{31}(\eta)\Big(a_{50}(\eta)+a^*_{40}(\eta)\alpha_{30}\\
&+\frac{a_{40}(\eta)+a^*_{30}(\eta)}{2\overline{k}\alpha_{30}}-\frac{a_{30}(\eta)\alpha^{*2}_{31}}{4\overline{k}\alpha^2_{30}}\Big)+\mathbb{N}_{32}(\eta)\Big(a_{50}(\eta)+a^*_{40}(\eta)\alpha_{30}+\frac{a_{30}(\eta)}{2\overline{k}\alpha_{30}}\Big)
\end{aligned}$$

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Termosprężyste zaburzenia w poprzecznie izotropowej półprzestrzeni wywołane punktowym obciążeniem termicznym

Streszczenie

Celem pracy jest zaprezentowanie zaburzeń wywołanych punktowym obciążeniem termicznym przyłożonym do jednorodnej, poprzecznie izotropowej półprzestrzeni w ogólnym sformułowaniu zagadnieniu termosprężystości. Do wyznaczenia równań układu zastosowano kombinację transformaty Fouriera i Hankela. Przy odwracaniu tak otrzymanych transformat użyto metody Cagniarda dla krótkich przedziałów czasowych. Rezultaty analizy pod kątem wyznaczenia temperatury i naprężeń otrzymano

w drodze symulacji numerycznej dla przypadku cynku jako materiału badawczego. Wykazano, że oscylacje poziomu naprężeń i temperatury są szczególnie wyraźne dla krótkich przedziałów czasu i gasną z jego upływem. Wyniki badań zilustrowano graficznie dla różnych czasów relaksacji termicznej.

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