The study concerns the linear elastic and viscoelastic constitutive modelling of homogeneous orthotropic solid bodies. The considerations are based on well-known coupled standard/inverse constitutive equations of elasticity. The author has derived new uncoupled standard/inverse constitutive equations of elasticity, new uncoupled standard/inverse constitutive equations of viscoelasticity, and new coupled standard/inverse constitutive equations of viscoelasticity of orthotropic materials. A homogeneous orthotropic material is described by 9 elastic and 18 viscoelastic constants, clearly interpreted physically. Simpler materials, i.e. monotropic and isotropic solid bodies, are also considered. In addition, the separation of shear and bulk strains in the uncoupled constitutive equations of elasticity has been examined numerically for exemplary materials.

Key words: solid body, orthotropy, constitutive modelling, elastic properties, viscoelastic properties

1. Introduction

Modern high-performance materials, e.g. xFRP cross-ply laminates, are usually homogenised and modelled as orthotropic solid bodies (Jones, 1975; Tsai, 1987; Daniel and Ishai, 1994). However, only the theory of linear elasticity (Timoshenko and Goodier, 1951) and viscoelasticity (Ferry, 1970; Garbarski, 1990; Klasztorny, 2004a,b) of isotropic materials is advanced in literature. Garbarski (1990) presented state-of-the-art in viscoelastic modelling of isotropic plastics exhibiting substantial viscoelastic/viscous deformations. Klasztorny (2004a,b) developed coupled/uncoupled constitutive equations of elasticity/viscoelasticity of isotropic thermohardening plastics as well as formulated
numerical algorithms for transforming standard constitutive equations into inverse constitutive equations.

In literature, viscoelastic modelling of orthotropic materials was formulated in reference to fibre-reinforced plastics (xFRP), in the form of unidirectional fibre composites or cross-ply laminates. Such materials exhibit sensible viscoelastic properties.

Sobotka (1980) presented a two-dimensional rheological model for an orthotropic viscoelastic thin plate. The basic model consists of one planar Hookean elastic and one Newtonian viscous region in the unit representative area, and is described by 4 relaxation and 2 retardation times. The writer generalizes this model via incorporating several elastic and viscous regions.

Holzapfel and Gasser (2001) developed a general viscoelastic model for the three-dimensional stress state of orthotropic materials on the assumption of finite strains. The authors have developed the expressions for the fourth-order elasticity tensor. Papers (Zaoutsos et al., 1998; Papanicolaou et al., 1999, 2004) concern unidirectional fibre-polymer matrix composites. The considerations are restricted to uniaxially tensioned samples. Zaoutsos et al. (1998) analysed a nonlinear viscoelastic response of a unidirectional CFRP, employing a one-dimensional viscoelastic model and modified Schapery’s nonlinear constitutive relationship. Creep-recovery tests in tension were executed for stress levels of 30-70% of the ultimate tensile stress. This approach has been advanced in the next papers, in which a methodology for predicting the nonlinear viscoelastic behaviour of xFRP composites was developed (Papanicolaou et al., 1999), and uniaxial tension of samples for different fibre orientations was tested (Papanicolaou et al., 2004).

To the author’s knowledge based on the literature review, a gap in the theory of linear elasticity and viscoelasticity of orthotropic materials is observed. So far, only coupled standard/inverse constitutive equations of linear elasticity of an orthotropic material have been formulated (Jones, 1975; Tsai, 1987; Daniel and Ishai, 1994). Coupled standard constitutive equations of linear viscoelasticity derived from generalisation of respective equations of elasticity can be formulated relatively easy, as shown in this study. However, analytic inversion of these equations, using well-known classic procedures, is impossible.

This paper presents a new approach to the problem of viscoelastic modelling of homogeneous orthotropic materials. Uncoupled standard constitutive equations of linear elasticity of an orthotropic solid body are derived. These equations enable one to formulate uncoupled standard constitutive equations of linear viscoelasticity. Both groups of uncoupled equations are reversed analytically in order to obtain uncoupled inverse constitutive equations of line-
ar elasticity and viscoelasticity. Finally, respective uncoupled equations are transformed into coupled standard/inverse constitutive equations of linear viscoelasticity.

2. Coupled standard constitutive equations of linear viscoelasticity of orthotropic solid bodies formulated by generalization of respective elastic equations

A homogeneous orthotropic solid body in isothermal conditions in the $x_1x_2x_3$ Cartesian co-ordinate system with the $x_1$, $x_2$, $x_3$ axes coinciding the orthotropy directions is examined. The considerations are restricted to stress levels inducing linear behaviour of the material.

Coupled standard constitutive equations of linear viscoelasticity of an orthotropic material can be formulated by generalization of Eqs. (A.2), (A3), i.e.

$$\epsilon(t) = S(t) \otimes \sigma(t)$$

where

$$S(t) = \begin{bmatrix}
S_{11}(t) & S_{12}(t) & S_{13}(t) & 0 & 0 & 0 \\
S_{12}(t) & S_{22}(t) & S_{23}(t) & 0 & 0 & 0 \\
S_{13}(t) & S_{23}(t) & S_{33}(t) & 0 & 0 & 0 \\
S_{s4}(t) & 0 & 0 & S_{s5}(t) & 0 & \\
S_{s5}(t) & 0 & & & & \\
s_{symm.} & & & & & S_{s6}(t)
\end{bmatrix}$$

is termed as a viscoelastic compliance matrix whose elements are defined by the formulae

$$S_{ii}(t) = \frac{1}{E_{ii}} \left[ 1 + \omega_{ii} \int_0^t F_{ii}(t - \vartheta) \, d\vartheta \right] \quad ii = 11, 22, 33$$

$$S_{ij}(t) = -\frac{\nu_{ij}}{E_{jj}} \left[ 1 + \omega_{ij} \int_0^t F_{ij}(t - \vartheta) \, d\vartheta \right] \quad ij = 23, 13, 12$$

$$S_{s4}(t) = \frac{1}{2G_{23}} \left[ 1 + \omega_{s4} \int_0^t F_{s4}(t - \vartheta) \, d\vartheta \right]$$

$$S_{s5}(t) = \frac{1}{2G_{13}} \left[ 1 + \omega_{s5} \int_0^t F_{s5}(t - \vartheta) \, d\vartheta \right]$$

$$S_{s6}(t) = \frac{1}{2G_{12}} \left[ 1 + \omega_{s6} \int_0^t F_{s6}(t - \vartheta) \, d\vartheta \right]$$
where \( F_{11}(t), F_{22}(t), F_{33}(t), F_{23}(t), F_{13}(t), F_{12}(t), F_{s4}(t), F_{s5}(t), F_{s6}(t) \) are termed as generic functions or stress-history functions or memory functions of a viscoelastic material (Ferry, 1970). The generic functions are described by respective retardation times. Moreover, \( t \) is a time variable, \( \otimes \) – convolution operator.

Summing up, Eqs. (2.3) contain 9 elastic constants (set up in Appendix) and 18 viscoelastic constants, i.e. 9 long-term creep coefficients \( \omega_{11}, \omega_{22}, \omega_{33}, \omega_{23}, \omega_{13}, \omega_{12}, \omega_{s4}, \omega_{s5}, \omega_{s6} \) and 9 retardation times \( \tau_{11}, \tau_{22}, \tau_{33}, \tau_{23}, \tau_{13}, \tau_{12}, \tau_{s4}, \tau_{s5}, \tau_{s6} \). Equations (2.1)-(2.3) show high complexity of the viscoelastic modelling of orthotropic bodies. Equations (2.1) are coupled, so the analytic reversal of these equations using the available classic methods is impossible.

For a monotropic solid body, the following relationships resulting from Eqs. (A.6) are satisfied

\[
S_{33}(t) = S_{22}(t) \quad S_{13}(t) = S_{12}(t) \quad S_{55}(t) = S_{66}(t) \quad (2.4)
\]

In this case, the number of viscoelastic constants reduces itself to 12.

### 3. Uncoupled standard/inverse constitutive equations of elasticity of orthotropic solid bodies

Equations (A.2)\(_{1,2,3}\) are coupled. Uncoupled standard constitutive equations of linear elasticity of an orthotropic solid body are searched in the form of two matrix equations

\[
\begin{align*}
\varepsilon_s &= \{S_s\} \sigma_s & \varepsilon_b &= \{S_b\} \sigma_b \\
\sigma_s &= (I - B)\varepsilon & \sigma_b &= (I - A)\sigma
\end{align*}
\]

(3.1)

where

\[
\begin{align*}
\varepsilon_s &= (I - B)\varepsilon & \sigma_s &= (I - A)\sigma \\
\varepsilon_b &= B\varepsilon & \sigma_b &= A\sigma
\end{align*}
\]

(3.2)

with

\[
I = \text{diag} (1, 1, 1, 1, 1, 1) \quad A = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{3\lambda_2}{2} & \frac{3\lambda_3}{3} & 0 & 0 & 0 \\
\lambda_2 & \frac{1}{3} & \frac{5\lambda_3}{3} & 0 & 0 & 0 \\
\frac{3}{\lambda_3} & \frac{3}{\lambda_3} & \frac{3\lambda_3}{3} & 1 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(3.3)
and

\begin{align*}
\{S_s\} &= \text{diag}(S_{s1}, S_{s2}, S_{s3}, S_{s4}, S_{s5}, S_{s6}) \\
\{S_b\} &= \text{diag}(S_{b1}, S_{b2}, S_{b3}, 1, 1, 1)
\end{align*}

Matrices defined by Eqs. (3.4) may be termed as diagonal matrices of elastic compliances. The \(x_1\) direction has been privileged in order to reflect the monotropy direction for monotropic materials. Equations (3.2) result in the following relationships

\begin{align*}
\varepsilon &= \varepsilon_s + \varepsilon_b \\
\sigma &= \sigma_s + \sigma_b
\end{align*}

Coefficients \(\lambda_2, \lambda_3\) and elastic compliances \(S_{s1}, S_{s2}, S_{s3}, S_{b1}, S_{b2}, S_{b3}\) will be derived from compatibility conditions related to the coupled and uncoupled constitutive equations. Comparing Eqs. (A.2) and (3.1), taking into consideration Eqs. (3.5), one obtains

\begin{align*}
S_\sigma &= \{S_s\} \sigma_s + \{S_b\} \sigma_b
\end{align*}

Substituting Eqs. (3.2) \(_2, (3.2)_4\) into Eq. (3.6) results in

\begin{align*}
S &= \{S_s\}(I - A) + \{S_b\} A
\end{align*}

Matrix equation (3.7) constitutes the compatibility conditions. Taking into account Eqs. (A.1), the explicit form of Eq. (3.7) related to sub-blocks \(i, j = 1, 2, 3\), has the following form

\begin{align*}
\begin{bmatrix}
\frac{1}{E_{11}} & -\frac{\nu_{12}}{E_{22}} & -\frac{\nu_{13}}{E_{33}} \\
-\frac{\nu_{12}}{E_{22}} & \frac{1}{E_{22}} & -\frac{\nu_{23}}{E_{33}} \\
-\frac{\nu_{13}}{E_{33}} & -\frac{\nu_{23}}{E_{33}} & \frac{1}{E_{33}}
\end{bmatrix}
&= \begin{bmatrix}
\frac{1}{3}(2S_{s1} + S_{b1}) & \frac{1}{3}(S_{b1} - S_{s1}) & \frac{1}{3\lambda_3}(S_{b1} - S_{s1}) \\
\frac{\lambda_2}{3}(S_{b2} - S_{s2}) & \frac{1}{3}(2S_{s2} + S_{b2}) & \frac{\lambda_2}{3\lambda_3}(S_{b2} - S_{s2}) \\
\frac{\lambda_3}{3}(S_{b3} - S_{s3}) & \frac{1}{3}(S_{b3} - S_{s3}) & \frac{1}{3}(2S_{s3} + S_{b3})
\end{bmatrix}
\end{align*}

Comparing respective elements in Eq. (3.8), one obtains 9 algebraic equations with one equation (optional) stating identity. The analytic solution to the remaining 8 equations has the form

\begin{align*}
\lambda_2 &= \frac{\nu_{13}}{\nu_{23}} \\
S_{s1} &= \frac{1}{E_{11}} \left(1 + \frac{\nu_{21}\nu_{13}}{\nu_{23}}\right) \\
S_{s3} &= \frac{1}{E_{33}} \left(1 + \frac{\nu_{13}\nu_{32}}{\nu_{12}}\right) \\
\lambda_3 &= \frac{\nu_{12}}{\nu_{32}} \\
S_{s2} &= \frac{1}{E_{22}} \left(1 + \frac{\nu_{12}\nu_{23}}{\nu_{13}}\right) \\
S_{b1} &= \frac{1}{E_{11}} \left(1 - 2\frac{\nu_{21}\nu_{13}}{\nu_{23}}\right)
\end{align*}
\[ S_{b2} = \frac{1}{E_{22}} \left( 1 - 2 \frac{\nu_{12} \nu_{23}}{\nu_{13}} \right) \quad S_{b3} = \frac{1}{E_{33}} \left( 1 - 2 \frac{\nu_{13} \nu_{32}}{\nu_{12}} \right) \]

The remaining equations in Eq. (3.1)\textsubscript{1}, related to shear stresses, are directly uncoupled.

Taking into account Eqs. (3.2)\textsubscript{4}, (A.9), Eq. (3.1)\textsubscript{2} can be transformed to the following form

\[ \varepsilon_b = \{S_b\} \sigma_b = \{S_b\} A \sigma = \{S_b\} A C \varepsilon = B \varepsilon \quad (3.10) \]

Therefore,

\[ B = \{S_b\} A C \quad (3.11) \]

The matrix \( B \) is of a block structure analogous to the structure of the matrix \( A \), i.e.

\[
B = \begin{bmatrix}
B_{11} & B_{12} & B_{13} & 0 & 0 & 0 \\
B_{21} & B_{22} & B_{23} & 0 & 0 & 0 \\
B_{31} & B_{32} & B_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \quad (3.12)
\]

For a monotropic material, the conditions written in Eqs. (A.5), (A.7) give the relationships

\[ \lambda_3 = \lambda_2 \quad S_{s2} = S_{s3} = S_{s4} \quad S_{s5} = S_{s6} \quad S_{b3} = S_{b2} \quad (3.13) \]

In this case, Eqs. (3.4) simplify themselves to the form

\[
\{S_s\} = \text{diag} (S_{s1}, S_{s4}, S_{s4}, S_{s4}, S_{s6}, S_{s6}) \\
\{S_b\} = \text{diag} (S_{b1}, S_{b2}, S_{b2}, 1, 1, 1)
\quad (3.14)
\]

where

\[
S_{s1} = \frac{1}{E_{11}} \left( 1 + \frac{\nu_{21} \nu_{12}}{\nu_{32}} \right) \quad S_{s4} = \frac{1}{2G_{23}} \quad S_{s6} = \frac{1}{2G_{12}} \\
S_{b1} = \frac{1}{E_{11}} \left( 1 - 2 \frac{\nu_{21} \nu_{12}}{\nu_{32}} \right) \quad S_{b2} = \frac{1}{3B_{22}} \quad (3.15)
\]

with

\[
G_{23} = \frac{E_{22}}{2(1 + \nu_{32})} \quad B_{22} = \frac{E_{22}}{3(1 - 2\nu_{32})} \quad \nu_{12} = \nu_{21} \frac{E_{22}}{E_{11}} \quad (3.16)
\]
Equations (A.8) for an isotropic material yield

\[
\begin{align*}
\lambda_2 &= \lambda_3 = 1 & \mathbf{B} &= \mathbf{A} \tag{3.17}
\end{align*}
\]

In this case, Eqs. (3.2) take the matrix form equivalent to the classic results (Timoshenko and Goodier, 1951; Klasztorny, 2004a)

\[
\begin{align*}
\varepsilon_s &= (\mathbf{I} - \mathbf{A})\varepsilon & \sigma_s &= (\mathbf{I} - \mathbf{A})\sigma \\
\varepsilon_b &= \mathbf{A}\varepsilon & \sigma_b &= \mathbf{A}\sigma 
\end{align*} \tag{3.18}
\]

where

\[
\mathbf{A} = \begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & & & \\
0 & 0 & 0 & & & \\
symm. & & & & & 0
\end{bmatrix} \tag{3.19}
\]

and

\[
\begin{align*}
\{\mathbf{S}_s\} &= \text{diag}(S_s, S_s, S_s, S_s, S_s, S_s) & \{\mathbf{S}_b\} &= \text{diag}(S_b, S_b, S_b, 1, 1, 1) \\
S_s &= \frac{1}{2G} & S_b &= \frac{1}{3B} \tag{3.20}
\end{align*}
\]

\[
G = \frac{E}{2(1 + \nu)} \quad & \quad B = \frac{E}{3(1 - 2\nu)}
\]

According to the classic theory of elasticity (Timoshenko and Goodier, 1951), \(S_s\) is termed as an elastic shear compliance, \(S_b\) – elastic bulk compliance, \(G\) – Kirchhoff’s modulus, \(B\) – Helmholtz’s modulus of an isotropic medium.

The reversal of Eqs. (3.1) gives uncoupled inverse constitutive equations of elasticity of an orthotropic solid body, i.e.

\[
\begin{align*}
\sigma_s &= \{\mathbf{C}_s\}\varepsilon_s & \sigma_b &= \{\mathbf{C}_b\}\varepsilon_b \tag{3.21}
\end{align*}
\]

where

\[
\begin{align*}
\{\mathbf{C}_s\} &= \{\mathbf{S}_s\}^{-1} & \{\mathbf{C}_b\} &= \{\mathbf{S}_b\}^{-1} \\
C_{si} &= S_{si}^{-1} & i &= 1, 2, 3, 4, 5, 6 \tag{3.22} \\
C_{bj} &= S_{bj}^{-1} & j &= 1, 2, 3
\end{align*}
\]

Matrices defined by Eqs. (3.22)\(_{1,2}\) may be termed as diagonal matrices of the elastic stiffness.
For a monotropic material, Eqs. (3.22) take the following form

\[
\{ C_s \} = \text{diag}(C_{s1}, 2G_{23}, 2G_{23}, 2G_{12}, 2G_{12}) \quad (3.23)
\]

\[
\{ C_b \} = \text{diag}(C_{b1}, 3B_{22}, 3B_{22}, 1, 1, 1)
\]

with

\[
C_{s1} = S_{s1}^{-1} \quad C_{b1} = S_{b1}^{-1} \quad (3.24)
\]

For an isotropic material, diagonal matrices of the elastic stiffness in Eqs.
(3.21) reduce themselves to the following form (see Eqs. (3.20) and (3.22))

\[
\{ S_s \} = \text{diag}(C_s, C_s, C_s, C_s, C_s, C_s) \quad C_s = 2G \quad (3.25)
\]

\[
\{ S_b \} = \text{diag}(C_b, C_b, C_b, 1, 1, 1) \quad C_b = 2B
\]

According to the classic theory of elasticity (Timoshenko and Goodier, 1951),
\( C_s \) is termed as the elastic shear stiffness, and \( C_b \) – elastic bulk stiffness of an
isotropic material.

4. Analysis of separation of shear and bulk strains in uncoupled
equations of elasticity of orthotropic solid bodies

Shear strains of an orthotropic material are fully expressed in terms of the
vector \( \varepsilon_s \). This statement results from Eqs. (3.2), (3.12). The bulk (volumetric)
strains are expressed in terms of the vector \( \varepsilon_b \), but a minor part of these strains
is included into the vector \( \varepsilon_s \). The following quantities constitute the measures
of the bulk strains corresponding to the vectors \( \varepsilon_b, \varepsilon_s \), respectively

\[
e_b = \varepsilon_b1 + \varepsilon_b2 + \varepsilon_b3 = \rho_1 \varepsilon_{11} + \rho_2 \varepsilon_{22} + \rho_3 \varepsilon_{33}
\]

\[
e_s = e - e_b = (1 - \rho_1) \varepsilon_{11} + (1 - \rho_2) \varepsilon_{22} + (1 - \rho_3) \varepsilon_{33}
\]

where

\[
e = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}
\]

is termed as dilatation (the full volumetric strain of a unit element) (Timoshenko and Goodier, 1951). From Eqs. (3.2) and (3.12), one obtains

\[
\rho_1 = B_{11} + B_{21} + B_{31} \quad \rho_2 = B_{12} + B_{22} + B_{32} \quad \rho_3 = B_{13} + B_{23} + B_{33}
\]

(4.3)
Taking into consideration Eqs. (3.18) and (3.19) for an isotropic material, one obtains
\[
\rho_1 = 1 \quad \rho_2 = 1 \quad \rho_3 = 1 \\
e_b = e \quad e_s = 0
\] (4.4)
hence, the shear and bulk strains are fully separated, as expected.

In this study, the separation of shear and bulk strains is examined for the following materials:
a) monotropic materials
\[
E_{11} = 10 - 100 \text{ GPa} \quad \quad E_{22} = 10 \text{ GPa} \quad \quad E_{33} = 10 \text{ GPa} \\
\nu_{32} = 0.4 \quad \quad \nu_{31} = 0.4 \quad \quad \nu_{21} = 0.4
\]
b) orthotropic materials
\[
E_{11} = 10 - 100 \text{ GPa} \quad \quad E_{22} = 40 \text{ GPa} \quad \quad E_{33} = 10 \text{ GPa} \\
\nu_{32} = 0.4 \quad \quad \nu_{31} = 0.4 \quad \quad \nu_{21} = 0.1
\]
The calculations were performed using author’s computer programme.

Diagrams of the coefficients \(\rho_1, \rho_2, \rho_3\) are presented in Fig. 1 and Fig. 2 for monotropic and orthotropic materials, respectively. Full separation of the shear and bulk strains in the isotropic material has been confirmed. Values of the coefficients \(\rho_1, \rho_2, \rho_3\) depend on the orthotropy level. The vector \(\varepsilon_b\) incorporates \(80-100\%\) of the bulk strains, whereas the remaining part of these strains is included in the vector \(\varepsilon_s\). For a monotropic material, one obtains \(\rho_2 = \rho_3\).

Fig. 1. Values of coefficients \(\rho_1, \rho_2, \rho_3\) vs. longitudinal Young’s modulus \(E_{11}\) for monotropic materials
In this study, uncoupled standard constitutive equations of linear viscoelasticity of an orthotropic material are derived from generalisation of Eqs. (3.1) and, in matrix notation, take the following form
\[
\varepsilon_s(t) = \{S_s(t)\} \otimes \sigma_s(t) \quad \varepsilon_b(t) = \{S_b(t)\} \otimes \sigma_b(t)
\] (5.1)

where \(\varepsilon_s, \sigma_s, \varepsilon_b, \sigma_b\) are defined by Eqs. (3.2), and
\[
\{S_s(t)\} = \text{diag} [S_{s1}(t), S_{s2}(t), S_{s3}(t), S_{s4}(t), S_{s5}(t), S_{s6}(t)]
\]
\[
\{S_b(t)\} = \text{diag} [S_{b1}(t), S_{b2}(t), S_{b3}(t), 1, 1, 1]
\] (5.2)

The matrices defined by Eqs. (5.2) may be termed as a diagonal viscoelastic quasi-shear compliance matrix and a diagonal viscoelastic quasi-bulk compliance matrix, respectively. General formulae for these compliances have the form
\[
S_{si}(t) = S_{si} \left[ 1 + \omega_{si} \int_{0}^{t} F_{si}(t-\vartheta) \, d\vartheta \right] \quad i = 1, 2, 3, 4, 5, 6
\]
\[
S_{bj}(t) = S_{bj} \left[ 1 + \omega_{bj} \int_{0}^{t} F_{bj}(t-\vartheta) \, d\vartheta \right] \quad j = 1, 2, 3
\] (5.3)

where \(F_{si}(t), F_{bj}(t)\) are termed as quasi-deviatoric and quasi-axiatoric stress-history functions, respectively. For each function, a retardation time is to be specified.

Summing up, an orthotropic material is described by 9 elastic constants and 18 viscoelastic constants, i.e. 9 long-term creep coefficients \(\omega_{si}\),
\[ i = 1, 2, 3, 4, 5, 6; \ \omega_{bj}, j = 1, 2, 3 \text{ and } 9 \text{ retardation times } \tau_{si}, i = 1, 2, 3, 4, 5, 6; \ \tau_{bj}, j = 1, 2, 3. \]

For a monotropic solid body, the following relationships resulting from Eqs. (3.13) are valid

\[ S_{s2}(t) = S_{s3}(t) = S_{s4}(t) \quad S_{s5}(t) = S_{s6}(t) \quad S_{b2}(t) = S_{b3}(t) \] (5.4)

In this case, the number of viscoelastic constants drops to 10.

Coupled standard constitutive equations of linear viscoelasticity of an orthotropic solid body are obtained by summing matrix equations (5.1), i.e.

\[ \varepsilon(t) = S(t) \otimes \sigma(t) \] (5.5)

where

\[
S(t) = \{S_s(t)\}(I - A) + \{S_b(t)\}A = \\
\begin{bmatrix}
S_{11}(t) & S_{12}(t) & S_{13}(t) & 0 & 0 & 0 \\
S_{21}(t) & S_{22}(t) & S_{23}(t) & 0 & 0 & 0 \\
S_{31}(t) & S_{32}(t) & S_{33}(t) & 0 & 0 & 0 \\
0 & 0 & 0 & S_{s4}(t) & 0 & 0 \\
0 & 0 & 0 & 0 & S_{s5}(t) & 0 \\
0 & 0 & 0 & 0 & 0 & S_{s6}(t) \\
\end{bmatrix}
\]

is named as in Section 2, i.e. the viscoelastic compliance matrix.

### 6. Uncoupled/coupled inverse constitutive equations of viscoelasticity of orthotropic solid bodies

The exact analytic reversal of uncoupled standard constitutive equations of viscoelasticity (Eqs. (5.1)) is possible for a number of generic functions. As a result, one obtains uncoupled inverse constitutive equations of viscoelasticity of an orthotropic material in the form

\[ \sigma_s(t) = \{C_s(t)\} \otimes \varepsilon_s(t) \quad \sigma_b(t) = \{C_b(t)\} \otimes \varepsilon_b(t) \] (6.1)

where

\[
\{C_s(t)\} = \text{diag}[C_{s1}(t), C_{s2}(t), C_{s3}(t), C_{s4}(t), C_{s5}(t), C_{s6}(t)] \\
\{C_b(t)\} = \text{diag}[C_{b1}(t), C_{b2}(t), C_{b3}(t), 1, 1, 1]
\]
The matrices defined by Eqs. (6.2) may be termed as a diagonal viscoelastic quasi-shear stiffness matrix and a diagonal viscoelastic quasi-bulk stiffness matrix, respectively. Quantities $K_{si}(t)$, $K_{bj}(t)$ may be termed as quasi-shear and quasi-bulk strain-history functions, respectively. For each function, a relaxation time is to be specified.

From the point of view of inverse equations, an orthotropic material is still described by 9 elastic constants and 18 viscoelastic constants, i.e. 9 long-term relaxation coefficients $\kappa_{si}$, $i = 1, 2, 3, 4, 5, 6$; $\kappa_{bj}$, $j = 1, 2, 3$ and 9 relaxation times $\theta_{si}$, $i = 1, 2, 3, 4, 5, 6$; $\theta_{bj}$, $j = 1, 2, 3$.

For some generating functions, formulae transforming quantities related to the standard equations (Eqs. (5.1)) into quantities related to the inverse equations (Eqs. (6.1)) are known. For example, normal exponential generic functions

\[
F_{si}(t) = \alpha_{si} e^{-\alpha_{si} t} \quad \alpha_{si} = \frac{1}{\tau_{si}} \quad i = 1, 2, 3, 4, 5, 6
\]
\[
F_{bj}(t) = \alpha_{bj} e^{-\alpha_{bj} t} \quad \alpha_{bj} = \frac{1}{\tau_{bj}} \quad j = 1, 2, 3
\] (6.3)

result in (Timoshenko and Goodier, 1951; Klasztorny, 2004a,b)

\[
\begin{align*}
\kappa_{si} &= \frac{\omega_{si}}{1 + \omega_{si}} & K_{si}(t) &= \beta_{si} e^{-\beta_{si} t} \\
\beta_{si} &= (1 + \omega_{si}) \alpha_{si} & \beta_{si} &= \frac{1}{\theta_{si}} \\
\kappa_{bj} &= \frac{\omega_{bj}}{1 + \omega_{bj}} & K_{bj}(t) &= \beta_{bj} e^{-\beta_{bj} t} \\
\beta_{bj} &= (1 + \omega_{bj}) \alpha_{bj} & \beta_{bj} &= \frac{1}{\theta_{bj}}
\end{align*}
\] (6.4)

Summing Eqs. (6.1)1, (6.2)2 and taking into account formulae (3.2), one obtains coupled inverse constitutive equations of linear viscoelasticity of an orthotropic solid body in the form
\[ \sigma(t) = C(t) \otimes \varepsilon(t) \]  \hspace{1cm} (6.5)

where

\[ C(t) = \{ C_s(t) \}(I - B) + \{ C_b(t) \}B = \]

\[
\begin{bmatrix}
C_{11}(t) & C_{12}(t) & C_{13}(t) & 0 & 0 & 0 \\
C_{21}(t) & C_{22}(t) & C_{23}(t) & 0 & 0 & 0 \\
C_{31}(t) & C_{32}(t) & C_{33}(t) & 0 & 0 & 0 \\
0 & 0 & 0 & C_{s4}(t) & 0 & 0 \\
0 & 0 & 0 & 0 & C_{s5}(t) & 0 \\
0 & 0 & 0 & 0 & 0 & C_{s6}(t)
\end{bmatrix}
\]

is termed as a viscoelastic stiffness matrix.

7. Conclusions

The study concerns the linear elastic/viscoelastic constitutive modelling of homogeneous orthotropic solid bodies. Coupled standard/inverse constitutive equations of elasticity have constituted the basis for deriving the following equations:

- uncoupled standard constitutive equations of elasticity,
- uncoupled inverse constitutive equations of elasticity,
- uncoupled standard constitutive equations of viscoelasticity,
- uncoupled inverse constitutive equations of viscoelasticity,
- coupled standard constitutive equations of viscoelasticity,
- coupled inverse constitutive equations of viscoelasticity.

The uncoupled/coupled constitutive equations of viscoelasticity of a homogeneous orthotropic material are described by 9 elastic and 18 viscoelastic constants. These constants have been clearly interpreted physically. Two particular cases have been considered, i.e. monotropic and isotropic solid bodies.

In addition, the separation of shear and bulk strains in the uncoupled constitutive equations of elasticity has been examined numerically for selected monotropic and orthotropic materials.
A. Coupled standard/inverse constitutive equations of linear elasticity of orthotropic solid bodies

This appendix is based on Jones (1975), Tsai (1987), Daniel and Ishai (1994) and slightly develops respective equations via unification of Poisson’s ratios. A homogeneous orthotropic solid body in isothermal conditions in the $x_1x_2x_3$ Cartesian co-ordinate system with the $x_1$, $x_2$, $x_3$ axes coinciding the orthotropy directions is examined. The considerations are restricted to stress levels inducing linear behaviour of the material. The stress and strain states are described by the following vectors (reflecting stress and strain tensors, respectively)

$$\sigma = \text{col}(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12})$$

$$\varepsilon = \text{col}(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{23}, \varepsilon_{13}, \varepsilon_{12})$$

where, for $i, j = 1, 2, 3$, $\sigma_{ii}$ is the normal stress, $\sigma_{ij}$ – shear stress ($i \neq j$), $\varepsilon_{ii}$ – relative elongation, $\varepsilon_{ij}$ – half of the shear strain angle of the $dx_1dx_2dx_3$ differential element ($i \neq j$).

A homogeneous orthotropic material is described by 9 independent elastic constants, i.e. $E_{11}$, $E_{22}$, $E_{33}$ – Young’s longitudinal moduli of elasticity; $\nu_{32}$, $\nu_{31}$, $\nu_{21}$ – Poisson’s ratios; $G_{23}$, $G_{13}$, $G_{12}$ – Kirchhoff’s shear moduli. The remaining Poisson’s ratios ($\nu_{23}, \nu_{13}, \nu_{12}$) are derived from the symmetry conditions, i.e.

$$\frac{\nu_{ij}}{E_{jj}} = \frac{\nu_{ji}}{E_{ii}} \quad i \neq j; \quad i, j = 1, 2, 3 \quad (A.1)$$

where $(-\nu_{ji})$ denotes shortening in the $x_j$ direction induced by tensioning in the $x_i$ direction.

Coupled standard constitutive equations of linear elasticity of an orthotropic solid body, written in matrix notation, have the following form

$$\varepsilon = S\sigma \quad (A.2)$$

where

$$S = \begin{bmatrix}
S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\
S_{22} & S_{23} & 0 & 0 & 0 \\
S_{33} & 0 & 0 & 0 & 0 \\
S_{s4} & 0 & 0 & S_{s5} & 0 \\
\text{symm.} & & & & S_{s6}
\end{bmatrix} \quad (A.3)$$
is termed as an elastic compliance matrix, whose elements are defined by the following formulae

\begin{align*}
S_{11} &= \frac{1}{E_{11}} & S_{12} &= -\frac{\nu_{12}}{E_{22}} & S_{13} &= -\frac{\nu_{13}}{E_{33}} \\
S_{21} &= -\frac{\nu_{21}}{E_{11}} & S_{22} &= \frac{1}{E_{22}} & S_{23} &= -\frac{\nu_{23}}{E_{33}} \\
S_{31} &= -\frac{\nu_{31}}{E_{11}} & S_{32} &= -\frac{\nu_{32}}{E_{22}} & S_{33} &= \frac{1}{E_{33}} \\
S_{s4} &= \frac{1}{2G_{23}} & S_{s5} &= \frac{1}{2G_{13}} & S_{s6} &= \frac{1}{2G_{12}}
\end{align*}

\hspace{1cm} (A.4)

A monotropic material, also termed as a transverse isotropic material, satisfies the following relationships

\begin{align*}
E_{33} &= E_{22} & G_{13} &= G_{12} & \nu_{13} &= \nu_{12} & G_{23} &= \frac{E_{22}}{2(1+\nu_{32})}
\end{align*}

\hspace{1cm} (A.5)

resulting in

\begin{align*}
S_{33} &= S_{22} & S_{13} &= S_{12} & S_{55} &= S_{66}
\end{align*}

\hspace{1cm} (A.6)

with \( x_1 \) being the direction of monotropy, \( x_2x_3 \) – plane of isotropy. A monotropic material is described by 5 independent elastic constants, i.e.: \( E_{11} \) – Young’s longitudinal modulus, \( E_{22} \) – Young’s transverse modulus \( (E_{22} \leq E_{11}) \), \( \nu_{32} \) – Poisson’s ratio in the \( x_2x_3 \) plane, \( \nu_{21} \) – greater Poisson’s ratio in the \( x_1x_2 \) plane, \( G_{12} \) – Kirchhoff’s shear modulus in the \( x_1x_2 \) plane. The remaining 7 elastic constants for a monotropic material depend on the constants \( E_{11}, E_{22}, \nu_{32}, \nu_{21}, G_{12} \) according to Eqs. (A.1), (A.5). In this case, one obtains

\begin{align*}
\nu_{12} &= \nu_{21} \frac{E_{22}}{E_{11}} \leq \nu_{21} & \nu_{31} &= \nu_{21} & \nu_{23} &= \nu_{32}
\end{align*}

\hspace{1cm} (A.7)

The coefficient \( \nu_{12} \) is termed as a smaller Poisson’s ratio in the \( x_1x_2 \) plane.

An isotropic material is described by two classic elastic constants, \( E, \nu, \) and satisfies the following relationships

\begin{align*}
E_{ii} &= E & ii &= 11, 22, 33 \\
\nu_{ij} &= \nu_{ji} = \nu & ij &= 23, 13, 12 \\
G_{ij} &= G = \frac{E}{2(1+\nu)} & j &= 23, 13, 12
\end{align*}

\hspace{1cm} (A.8)
The coupled inverse constitutive equations describing behaviour of an orthotropic solid body, written in matrix notation, have the following form

$$\sigma = C \varepsilon$$  \hspace{1cm}  (A.9)

where

$$C = \begin{bmatrix}
    C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
    C_{22} & C_{23} & 0 & 0 & 0 & 0 \\
    C_{33} & 0 & 0 & 0 & C_{s4} & 0 \\
    0 & 0 & C_{s5} & 0 & C_{s6} & \text{symm.}
\end{bmatrix}$$  \hspace{1cm}  (A.10)

is termed as an elastic stiffness matrix, which elements are defined by the following formulae

$$C_{11} = E_{11} \frac{1 - \nu_{23} \nu_{32}}{\Delta}$$
$$C_{33} = E_{33} \frac{1 - \nu_{12} \nu_{21}}{\Delta}$$
$$C_{13} = E_{33} \frac{\nu_{31} + \nu_{21} \nu_{32}}{\Delta}$$
$$C_{s4} = 2G_{23}$$

$$C_{22} = E_{22} \frac{1 - \nu_{13} \nu_{31}}{\Delta}$$
$$C_{12} = E_{22} \frac{\nu_{21} + \nu_{23} \nu_{31}}{\Delta}$$
$$C_{23} = E_{33} \frac{\nu_{32} + \nu_{12} \nu_{31}}{\Delta}$$
$$C_{s5} = 2G_{13}$$
$$C_{s6} = 2G_{12}$$

with

$$\Delta = 1 - \nu_{23} \nu_{32} - \nu_{13} \nu_{31} - \nu_{12} \nu_{21} - \nu_{12} \nu_{23} \nu_{31} - \nu_{21} \nu_{32} \nu_{13}$$  \hspace{1cm}  (A.12)

and $C = S^{-1}$.

For a monotropic material, one obtains

$$C_{33} = C_{22} \hspace{1cm} C_{13} = C_{12} \hspace{1cm} C_{55} = C_{66}$$  \hspace{1cm}  (A.13)

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**References**


Sprzężone i niesprzężone równania konstytutywne liniowej sprężystości i lepkosprężystości materiałów ortotropowych

**Streszczenie**

Praca dotyczy modelowania konstytutywnego jednorodnych ortotropowych ciał stałych w zakresie liniowym, sprężystym i lepkosprężystym. Podstawą rozważań są
znane sprzężone standardowe/odwrotne równania konstytutywne liniowej sprężystości tych materiałów. Wyznaczono niesprężone standardowe/odwrotne równania konstytutywne liniowej sprężystości, sformułowano niesprężone standardowe/odwrotne równania konstytutywne liniowej lepkosprężystości, a następnie wyznaczono sprzężone standardowe/odwrotne równania konstytutywne liniowej lepkosprężystości ortotropowych ciał stałych. Jednorodny materiał ortotropowy opisano za pomocą 9 stałych sprężystości i 18 stałych lepkosprężystości z podaniem przejrzystej interpretacji fizycznej tych stałych. Rozważono również przypadki szczególne materiału monotropowego i izotropowego. Dodatkowo, przetestowano numerycznie rozdzielanie odkształceń postaciowych i objętościowych w przypadku niesprężonych równań konstytutywnych sprężystości materiałów ortotropowych i monotropowych.

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