ANALYSIS OF VIBRATION OF THREE-DEGREE-OF-FREEDOM DYNAMICAL SYSTEM WITH DOUBLE PENDULUM

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The nonlinear response of a three-degree-of-freedom vibratory system with a double pendulum in the neighborhood of internal and external resonances has been examined. Numerical and analytical methods have been applied for these investigations. Analytical solutions have been obtained by using the multiple scales method. This method is used to construct first-order non-linear ordinary differential equations governing the modulation of amplitudes and phases. Steady state solutions and their stability are computed for selected values of the system parameters.

Key words: nonlinear coupled oscillators, autoparametric vibrations, multiple scale method

1. Introduction

In complex three-degree-of-freedom vibrating systems with elements of pendulums suspended on a flexible element, the autoparametric excitation as a result of inertial coupling may occur (Sado, 1997). Dynamic systems of this kind with two degrees of freedom were widely discussed in the literature as autoparametric vibration eliminators (Bajaj and Johnson, 1990; Bajaj et al., 1994; Banerjee et al., 1996) or other structural components (Samaranayake and Bajaj, 1993; Sado, 2002; Shoeybi and Ghorashi, 2004). The effect of a parametric or autoparametric excitation on a three-mass system was studied by Tondl and Nabergoj (2004). Numerical simulations of a two mass system
with three degrees of freedom with pendulums hanging down from a flexibly suspended body was investigated by Sado (2004) for an elastic pendulum and by Sado and Gajos (2003) for a double pendulum.

This paper describes the analytical solution of a three-degree-of-freedom system with a double pendulum. As it is a vibrating system with changing values of amplitudes and phases, in the analytical investigation the method of multiple scales was applied (Nayfeh and Mook, 1979). This method was used by several researchers (Ertas and Chew, 1990; Ji and Leung, 2003; Moon and Kang, 2003; Çevik and Pakdemirli, 2005; Rossikhin and Shitikova, 2006). Eliminating secular terms, we can observe conditions when the phenomenon of internal and external resonances is possible. Next, for the conditions of such resonances, steady-state solutions were investigated.

2. Equations of motion

The investigated system is shown in Fig. 1. The system consists of a double pendulum and a body of mass $m_1$ suspended on a flexible element of rigidity $k$, thus $S(y) = ky$. The pendulum of length $l_1$ and mass $m_2$ hangs down from the body of mass $m_1$. The pendulum of length $l_2$ and mass $m_3$ is suspended on the body of mass $m_2$. It is assumed that a linear viscous damping force acts upon the body $m_1$ ($R(\dot{y}) = c_1\dot{y}$), and a linear damping momentum acts upon the pendulum of mass $m_2$ ($M_1(\dot{\varphi}_1) = c_2\dot{\varphi}_1$), and a linear damping
momentum applied in the hinge opposes motion of the pendulum of mass \( m_3 \) \((M_2(\dot{\phi}_1, \dot{\phi}_2) = c_3(\dot{\phi}_2 - \dot{\phi}_1))\). The body of mass \( m_1 \) is subjected to a harmonic vertical excitation \( F(t) = P_0 \cos \nu t \). This system has three degrees of freedom. As generalized coordinates, the vertical displacement \( y \) of the body of mass \( m_1 \) measured from the equilibrium position and the angles \( \varphi_1 \) and \( \varphi_2 \) of deflection of the pendulums measured from the vertical lines are assumed.

The equations of motion are derived as Lagrange’s equations

\[
\begin{align*}
(m_1 + m_2 + m_3)\ddot{y} - l_1 (m_2 + m_3)\ddot{\varphi}_1 \sin \varphi_1 - m_3 l_2 \ddot{\varphi}_2 \sin \varphi_2 + & \\
-(m_2 + m_3)l_1 \dot{\varphi}_1^2 \cos \varphi_1 - m_3 l_2 \dot{\varphi}_2^2 \cos \varphi_2 + ky + c_1 \dot{y} = P_0 \cos \nu t & \\
-(m_2 + m_3)\dot{y} \sin \varphi_1 + (m_2 + m_3)l_1 \dot{\varphi}_1 + m_3 l_2 \dot{\varphi}_2 \cos(\varphi_2 - \varphi_1) + & \\
-m_3 l_2 \dot{\varphi}_2^2 \sin(\varphi_2 - \varphi_1) + (m_2 + m_3)g \sin \varphi_1 + c_2 \dot{\varphi}_1 - c_3(\dot{\varphi}_2 - \dot{\varphi}_1) = 0 & \\
-\ddot{y} \sin \varphi_2 + l_1 \dot{\varphi}_1 \cos(\varphi_2 - \varphi_1) + l_2 \dot{\varphi}_2 + l_1 \dot{\varphi}_1^2 \sin(\varphi_2 - \varphi_1) + & \\
+g \sin \varphi_2 + c_3(\dot{\varphi}_2 - \dot{\varphi}_1) = 0 & \\
\end{align*}
\]

Next, we introduce the dimensionless time \( \tau = \omega_1 t \) and the following definitions

\[
\begin{align*}
y_1 &= \frac{y}{l_1} & y_{1st} &= \frac{y_{1st}}{l_1} & d_1 &= \frac{m_2}{m_1} \\
d_2 &= \frac{m_3}{m_1} & d_3 &= \frac{d_1}{1 + d_1 + d_2} & d_4 &= \frac{d_2}{1 + d_1 + d_2} \\
d_5 &= d_3 + d_4 & d_6 &= \frac{d_4}{d_3} & d_7 &= 1 + d_6 \\
\omega_1^2 &= \frac{k}{m_1 + m_2 + m_3} & \omega_2^2 &= \frac{g}{l_1} & \omega_3^2 &= \frac{g}{l_2} & (2.2) \\
c &= \frac{l_2}{l_1} & \beta_1 &= \frac{\omega_2}{\omega_1} & \gamma_1 &= \frac{c_1}{m_2 \omega_1} \\
\gamma_2 &= \frac{c_2}{m_2 l_1^2 \omega_1} & \gamma_3 &= \frac{c_3}{m_2 l_1^2 \omega_1} & \mu &= \frac{\nu}{\omega_1} \\
p &= \frac{P_0}{m_2 l_1 \omega_1^2} \\
\end{align*}
\]

3. The method of multiple scales

In order to find approximate solutions to equations of motion we use the method of multiple scales (Nayfeh and Mook, 1979). Partially, this problem
for a system with a double pendulum was presented by Sado and Gajos (2005). For small oscillations, after transformations the equations of motion can be written down in the form

\[ \ddot{y}_1 + y_1 - d_5 \left( \varphi_1 + \frac{\varphi_1^3}{6} \right) \dot{\varphi}_1 - d_4 c \left( \varphi_2 + \frac{\varphi_2^3}{6} \right) \dot{\varphi}_2 - \varphi_1^2 d_5 \left( 1 - \frac{\varphi_1^2}{4} \right) + \]

\[ -d_4 c \dot{\varphi}_2^2 \left( 1 - \frac{\varphi_2^2}{4} \right) = -d_3 \gamma_1 \dot{y}_1 + d_3 p \cos(\mu \tau) \]

\[ d_5 \dot{\varphi}_1 - d_5 \left( \varphi_1 + \frac{\varphi_1^3}{6} \right) \dot{y}_1 + d_4 c \left( \varphi_1 \varphi_2 + 1 - \frac{\varphi_2^2}{4} - \frac{\varphi_1^2}{4} \right) \dot{\varphi}_2 + \]

\[ +d_4 c \dot{\varphi}_2^2 \left( \varphi_1 + \frac{\varphi_2^3}{6} - \varphi_1 \frac{\varphi_2^2}{2} - \varphi_2 - \frac{\varphi_2^3}{6} - \varphi_2 \frac{\varphi_1^2}{4} \right) + d_5 \beta_1^2 \left( \varphi_1 + \frac{\varphi_1^3}{6} \right) + \]

\[ +d_3 [\gamma_2 \dot{\varphi}_1 - \gamma_3 (\dot{\varphi}_2 - \dot{\varphi}_1)] = 0 \] (3.1)

\[ c \ddot{\varphi}_2 - \left( \varphi_2 + \frac{\varphi_2^3}{6} \right) \ddot{y}_1 + \left( \varphi_1 \varphi_2 + 1 - \frac{\varphi_2^3}{4} - \frac{\varphi_1^3}{4} \right) \ddot{\varphi}_1 - \dot{\varphi}_1^2 \left( \varphi_1 + \frac{\varphi_1^3}{6} - \varphi_1 \frac{\varphi_2^2}{4} + \right. \]

\[ -\varphi_2 - \frac{\varphi_2^3}{6} + \varphi_2 \frac{\varphi_1^2}{4} \right) + \beta_1^2 \left( \varphi_2 + \frac{\varphi_2^3}{6} - \frac{d_2 c}{d_4} \gamma_2 (\dot{\varphi}_2 - \dot{\varphi}_1) = 0 \]

We introduce independent variables

\[ \{T_0, T_1, T_2, \ldots, T_n\} = \{\tau, \varepsilon \tau, \varepsilon^2 \tau, \ldots, \varepsilon^n \tau\} \] (3.2)

and parameters

\[ p_1 = \varepsilon^2 p_1 \quad \gamma_1 = \varepsilon \gamma_1 \quad \gamma_2 = \varepsilon \gamma_2 \quad \gamma_3 = \varepsilon \gamma_3 \] (3.3)

Solutions to the dimensionless equations can be represented by

\[ y_1 = \varepsilon y_{10} + \varepsilon^2 y_{11} + \ldots \]

\[ \varphi_1 = \varepsilon \varphi_{10} + \varepsilon^2 \varphi_{11} + \ldots \] (3.4)

\[ \varphi_2 = \varepsilon \varphi_{20} + \varepsilon^2 \varphi_{21} + \ldots \]

It follows that the derivatives with respect to \( \tau \) become expansions in terms of partial derivatives with respect to \( T_n \) as

\[ \frac{d}{d \tau} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} + \ldots = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots \] (3.5)

\[ \frac{d^2}{d \tau^2} = \frac{\partial^2}{\partial T_0^2} + \varepsilon \frac{\partial^2}{\partial T_0 \partial T_1} + \varepsilon^2 \frac{\partial^2}{\partial T_0 \partial T_2} + \varepsilon \frac{\partial^2}{\partial T_1 \partial T_0} + \varepsilon^2 \frac{\partial^2}{\partial T_1 \partial T_2} + \varepsilon^2 \frac{\partial^2}{\partial T_2 \partial T_0} + \ldots = \]

\[ = \frac{\partial^2}{\partial T_0^2} + 2 \varepsilon \frac{\partial^2}{\partial T_0 \partial T_1} + \varepsilon^2 \left( 2 \frac{\partial^2}{\partial T_0 \partial T_2} + \frac{\partial^2}{\partial T_1^2} \right) + \ldots = \]

\[ = D_0^2 + 2 \varepsilon D_0 D_1 + \varepsilon^2 (2 D_0 D_2 + D_1^2) + \ldots \]
Substituting (3.3) and (3.4) into dimensionless equations (3.1) and equating the coefficients standing at $\varepsilon^1$ and $\varepsilon^2$ on both sides, we obtain:

— for $\varepsilon^1$

\[
\begin{align*}
D_0^2 y_{10} + y_{10} &= 0 \\
D_0^2 \varphi_{10} - d_6 \beta_1^2 \varphi_{20} + d_7 \beta_1^2 \varphi_{10} &= 0 \\
D_0^2 \varphi_{20} - d_7 \beta_2^2 \varphi_{10} + d_7 \beta_2^2 \varphi_{20} &= 0
\end{align*}
\]

(3.6)

— for $\varepsilon^2$

\[
\begin{align*}
D_0^2 y_{11} + y_{11} &= -2D_0 D_1 y_{10} + d_5 (D_0 \varphi_{10})^2 + d_4 (D_0 \varphi_{20})^2 + d_3 \rho \cos(\mu \tau) + \\
&- d_3 \gamma_1 D_0 \varphi_{10} + 2d_6 d_5 \beta_1^2 \varphi_{10} \varphi_{20} - d_6 d_5 \beta_1^2 \varphi_{20} - \frac{d_5^2}{d_3} \beta_1^2 \varphi_{10} \\
D_0^2 \varphi_{11} - d_6 \beta_1^2 \varphi_{21} + d_7 \beta_1^2 \varphi_{11} &= -2D_0 D_1 \varphi_{10} - d_7 y_{10} \varphi_{10} + d_6 y_{10} \varphi_{20} + \\
&- \left(\frac{\gamma_3}{c} + \gamma_2 + \gamma_3\right) D_0 \varphi_{10} + \left(\gamma_3 + \frac{\gamma_3}{c}\right) D_0 \\
D_0^2 \varphi_{21} - \frac{1}{c} (d_7 \beta_1^2 \varphi_{11} - d_7 \beta_2^2 \varphi_{21}) &= -2D_0 D_1 \varphi_{20} + \frac{1}{c} (d_7 y_{10} \varphi_{10} - d_7 y_{10} \varphi_{20}) + \\
&- \frac{d_7 \gamma_3}{c^2} (D_0 \varphi_{20} - D_0 \varphi_{10}) + \gamma_2 D_0 \varphi_{10} - \gamma_3 (D_0 \varphi_{20} - D_0 \varphi_{10})
\end{align*}
\]

(3.7)

General solutions to equations (3.6) can be represented by

\[
\begin{align*}
y_{10}(T_0, T_1, T_2) &= A_1(T_1, T_2) e^{i \omega_1 T_0} + \bar{A}_1(T_1, T_2) e^{-i \omega_1 T_0} \\
\varphi_{10}(T_0, T_1, T_2) &= A_2(T_1, T_2) e^{i \omega_2 T_0} + \bar{A}_2(T_1, T_2) e^{-i \omega_2 T_0} + \\
&+ A_3(T_1, T_2) e^{i \omega_3 T_0} + \bar{A}_3(T_1, T_2) e^{-i \omega_3 T_0} \\
\varphi_{20}(T_0, T_1, T_2) &= A_2 A_2(T_1, T_2) e^{i \omega_2 T_0} + \bar{A}_2 A_2(T_1, T_2) e^{-i \omega_2 T_0} + \\
&+ A_3 A_3(T_1, T_2) e^{i \omega_3 T_0} + \bar{A}_3 A_3(T_1, T_2) e^{-i \omega_3 T_0}
\end{align*}
\]

(3.8)

We find natural frequencies of system (3.6) by substituting

\[
\begin{align*}
y_1 &= A_1 e^{i \omega_1 T_0} + cc \\
\varphi_1 &= A_2 e^{i \omega_1 T_0} + cc \\
\varphi_2 &= \Lambda A_2 e^{i \omega_1 T_0} + cc
\end{align*}
\]

(3.9)

where $cc$ represents the complex conjugate, and using the condition that the determinant of the matrix of coefficients is zero. In this case

\[
\omega_1 = 1
\]

and

\[
\omega_{2,3}^2 = \frac{1}{2} \left[ -d_7 \beta_1^2 \left(1 + \frac{1}{c}\right) \pm \beta_1^2 \sqrt{d_2^2 \left(1 + \frac{1}{c}\right)^2 - \frac{4d_5 d_6}{c}} \right]
\]
Amplitudes and phases can be found by substituting (3.8) into (3.7). We obtain a system of equations

\[ D_0^2 y_{11} + y_{11} = -2iA_1 e^{iT_0} + d_5(-\omega_2^2 A_2 e^{2i\omega_2 T_0} - \omega_3^2 A_3 e^{2i\omega_3 T_0} + \omega_2^2 A_3 e^{i(\omega_2 + \omega_3) T_0} + 2\omega_2 A_2 A_2 + 2\omega_2 A_3 A_3 + \omega_2^2 A_2 A_3 e^{i(\omega_2 - \omega_3) T_0} + \omega_3^2 A_2 A_2 + 2\omega_2 A_2 A_3 + 2\omega_3^2 A_3 A_3 A_3 + \frac{1}{2}d_3 e^{i\mu T_0} - 3\gamma_1 \bar{\omega}_1 A_1 e^{iT_0} + 2d_5 d_6 A_2^2 e^{2i\omega_2 T_0} + A_3 e^{2i\omega_3 T_0} + (3.10) + (A_2 + \bar{A}_2) A_2 A_2 + (A_3 + \bar{A}_3) A_3 A_3 + (A_2 + A_3) A_2 A_3 e^{i(\omega_2 + \omega_3) T_0} + \bar{A}_2 + \bar{A}_3) A_2 A_3 e^{i(\omega_2 - \omega_3) T_0} - d_5 d_6 A_3^2 e^{2i\omega_3 T_0} + A_3 e^{2i\omega_3 T_0} + \omega_3^2 A_3 A_3 e^{i(\omega_3 - \omega_2) T_0} + 2\omega_2 A_2 A_3 e^{i(\omega_2 + \omega_3) T_0} + 2\omega_2 A_2 A_2 + 2\omega_2 A_3 e^{i(\omega_3 - \omega_2) T_0} \right) \]

\[ D_0^2 \varphi_{11} - d_6 \beta_1^2 \varphi_{11} + d_7 \beta_1^2 \varphi_{11} = -2i\omega_2 A_2 e^{i\omega_2 T_0} - 2i\omega_3 A_3 e^{i\omega_3 T_0} + d_7 (A_1 A_2 e^{i(1+\omega_2) T_0} + A_1 \bar{A}_2 e^{i(1-\omega_2) T_0} + A_1 A_3 e^{i(1+\omega_3) T_0} + A_1 A_3 e^{i(-1+\omega_3) T_0}) + d_6 (A_2 A_2 e^{i(1+\omega_2) T_0} + A_2 A_3 e^{i(1+\omega_3) T_0} + A_3 A_3 e^{i(1+\omega_3) T_0} + A_3 A_3 e^{i(1-\omega_3) T_0} + \bar{A}_3 A_3 e^{i(1+\omega_3) T_0}) = \left( \frac{\gamma_3}{c} + \gamma_2 + \gamma_3 \right) (i\omega_2 A_2 e^{i\omega_2 T_0} + i\omega_3 A_3 e^{i\omega_3 T_0}) + \left( \frac{\gamma_3}{c} \right) (i\omega_3 A_3 e^{i\omega_3 T_0} + i\omega_3 A_3 e^{i\omega_3 T_0}) \]

\[ D_0^2 \varphi_{21} - \frac{d_7 \beta_1^2}{c} \varphi_{11} + \frac{d_7 \beta_1^2}{c} \varphi_{21} = -2i\omega_2 A_2 A_2 e^{i\omega_2 T_0} - 2i\omega_3 A_3 A_3 e^{i\omega_3 T_0} + \frac{d_7}{c} (A_1 A_2 e^{i(1+\omega_2) T_0} + A_1 \bar{A}_2 e^{i(1-\omega_2) T_0} + A_1 A_3 e^{i(1+\omega_3) T_0} + A_1 A_3 e^{i(-1+\omega_3) T_0}) + \frac{d_7}{c} (A_2 A_2 e^{i(1+\omega_2) T_0} + A_2 A_3 e^{i(1+\omega_3) T_0} + A_3 A_3 e^{i(1+\omega_3) T_0} + A_3 A_3 e^{i(-1+\omega_3) T_0}) \]
In this work, we analyze one combination of internal resonances and the external resonance

\[ \mu = 1 \quad 2\omega_2 = 1 \quad \omega_3 = 3\omega_2 \]

We introduce detuning parameters \( \sigma_1, \sigma_2, \sigma_3 \) defined by

\[ 2\omega_2 + \varepsilon\sigma_1 = 1 \quad \omega_3 = 3\omega_2 + \varepsilon\sigma_2 \quad \mu = 1 + \varepsilon\sigma_3 \quad (3.13) \]

Substituting (3.13) into equation (3.10) and eliminating terms that produce secular terms, we obtain

\[-2iA'_1 + 2\left[d_5 + d_4c\Lambda_2A_3 + d_5d_6\beta_1^2(\Lambda_2 + A_3) - d_5d_6\beta_1^2\Lambda_2A_3 + \right.
\]
\[\left. \frac{d^2}{d_3}\beta_1^2\right]2\omega_2\omega_3^2\Lambda_2A_3e^{iT_1(-\sigma_1+\sigma_2)} - \left(d_5 + d_4cA_2^2 - 2d_5d_6\beta_1^2A_2 + \right.
\]
\[\left. +d_5d_6\beta_1^2A_2^2 + \frac{d^2}{d_3}\beta_1^2\right)\omega_2^2A_2^2e^{-iT_1\sigma_3} + \frac{1}{2}d_3pe^{i\sigma_3T_1} - d_3\gamma_1i\omega_1A_1 = 0 \quad (3.14) \]

By introducing

\[ A_1 = \frac{1}{2}a_1e^{i\alpha_1} \quad A_2 = \frac{1}{2}a_2e^{i\alpha_2} \quad A_3 = \frac{1}{2}a_3e^{i\alpha_3} \quad (3.15) \]

and

\[ \theta_1 = 2\alpha_2 - \alpha_1 - T_1\sigma_1 \quad \theta_2 = \alpha_3 - \alpha_2 - \alpha_1 - T_1\sigma_1 + T_1\sigma_2 \]
\[ \theta_3 = -\alpha_1 + T_1\sigma_3 \quad (3.16) \]

into (3.14), we obtain the first modulation equation

\[-ia'_1 + a_1\alpha' + \frac{1}{4}f_1a_2a_3e^{-i\theta_2} + \frac{1}{4}f_2a_2^2e^{i\theta_1} + \frac{1}{2}d_3pe^{i\theta_3} - \frac{1}{2}id_3\gamma_1a_1 = 0 \quad (3.17) \]

where

\[ f_1 = 2d_5\omega_2\omega_3 + 2d_4c\omega_2\omega_3\Lambda_2A_3 + 2d_6d_5\beta_1^2(\Lambda_2 + A_3 - \Lambda_2A_3) - \frac{2d_5^2\beta_1^2}{d_3} \]
\[ f_2 = -d_5\omega_2^2 - d_4c\omega_2^2A_2 + d_5d_6\beta_1^2(2A_2 - A_2^2) + \frac{d^2\beta_1^2}{d_3} \]

To determine the solvability conditions of (3.11) and (3.12), we seek for particular solutions in the form

\[ \varphi_{11} = P_{11}e^{\varphi_{22}T_0} + P_{12}e^{\varphi_{33}T_0} \quad \varphi_{21} = P_{21}e^{\varphi_{22}T_0} + P_{22}e^{\varphi_{33}T_0} \quad (3.18) \]
Substituting particular solutions (3.18) into equations (3.11), (3.12) and using resonant conditions (3.13) and equaling the coefficients of \( \exp(i\omega_2 T_0) \) and \( \exp(i\omega_3 T_0) \) on both sides, we obtain system of four equations

\[
-\omega_2^2 P_{11} - d_6 \beta_1^2 P_{21} + d_7 \beta_1^2 P_{11} = R_{11} \tag{3.19}
\]

\[
-\omega_2^2 P_{21} - \frac{d_7 \beta_1^2}{c} (P_{11} + P_{21}) = R_{21}
\]

and

\[
-\omega_3^2 P_{12} - d_6 \beta_1^2 P_{22} + d_7 \beta_1^2 P_{12} = R_{12} \tag{3.20}
\]

\[
-\omega_3^2 P_{22} - \frac{d_7 \beta_1^2}{c} (P_{12} + P_{22}) = R_{22}
\]

where

\[
R_{11} = -2i\omega_2 A_2' - d_7 (A_1 \overline{A}_2 e^{iT_1 \sigma_1} + \overline{A}_1 A_3 e^{iT_1 (\sigma_2 - \sigma_1)}) +
+ d_6 (A_1 \overline{A}_2 A_2 e^{iT_1 \sigma_1} + \overline{A}_1 A_3 A_3 e^{iT_1 (\sigma_2 - \sigma_1)}) +
- \left( \frac{\gamma_3}{c} + \gamma_2 + \gamma_3 \right) (i\omega_2 A_2 + \left( \frac{\gamma_3}{c} + \gamma_3 \right) (i\omega_2 A_2 A_2)
\]

\[
R_{21} = -2i\omega_2 A_2' \delta A_2 - d_7 (A_1 \overline{A}_2 e^{iT_1 \sigma_1} + \overline{A}_1 A_3 e^{iT_1 (\sigma_2 - \sigma_1)}) +
+ \frac{d_7}{c} (A_1 \overline{A}_2 A_2 e^{iT_1 \sigma_1} + \overline{A}_1 A_3 A_3 e^{iT_1 (\sigma_2 - \sigma_1)}) +
+ \left( \frac{d_7 \gamma_3}{c^2} + \frac{\gamma_2}{c} + \frac{\gamma_3}{c} \right) (i\omega_2 A_2 - \left( \frac{d_7 \gamma_3}{c^2} + \frac{\gamma_3}{c} \right) (i\omega_2 A_2 A_2)
\]

\[
R_{12} = -2i\omega_3 A_3' - d_7 A_1 A_2 e^{iT_1 (-\sigma_2 + \sigma_1)} + d_6 A_1 A_2 A_2 e^{iT_1 (-\sigma_2 + \sigma_1)} +
+ \left( \frac{\gamma_3}{c} + \gamma_2 + \gamma_3 \right) (i\omega_3 A_3 + \left( \frac{\gamma_3}{c} + \gamma_3 \right) (i\omega_3 A_3 A_3)
\]

\[
R_{22} = -2i\omega_3 A_3' A_3 + d_7 A_1 A_2 A_2 e^{iT_1 (-\sigma_2 + \sigma_1)} - \frac{d_7}{c} A_1 A_2 A_2 e^{iT_1 (-\sigma_2 + \sigma_1)} +
+ \left( \frac{d_7 \gamma_3}{c^2} + \frac{\gamma_2}{c} + \frac{\gamma_3}{c} \right) (i\omega_3 A_3 - \left( \frac{d_7 \gamma_3}{c^2} + \frac{\gamma_3}{c} \right) (i\omega_3 A_3 A_3)
\]

We reduce the problem of determination of the solvability conditions of equations (3.11), (3.12) to finding solvability conditions of equations (3.19) and (3.20). The determinants of the coefficient matrices of equations (3.19) and (3.20) are the same and equal 0 according to conditions on the natural frequencies of system (3.6).
Then the solvability conditions are

\[
\begin{vmatrix}
R_{11} & -d_6\beta_1^2 \\
R_{21} & -\omega_2^2 + \frac{d_7\beta_1^2}{c}
\end{vmatrix} = 0
\] (3.21)

for equations (3.19) and

\[
\begin{vmatrix}
R_{12} & -d_6\beta_1^2 \\
R_{22} & -\omega_3^2 + \frac{d_7\beta_1^2}{c}
\end{vmatrix} = 0
\] (3.22)

for equations (3.20).

Substituting (3.15) and (3.16) and after some transformations, we obtain two modulation equations

\[
-ia_2' + a_2\alpha_2' + \frac{f_4}{4f_3\omega_2} a_1 a_2 e^{-i\theta_1} + \frac{f_5}{4f_3\omega_2} a_1 a_3 e^{i\theta_2} + \frac{f_6}{2f_3} ia_2 = 0
\] (3.23)

\[
-ia_3' + a_3\alpha_3' + \frac{f_8}{4f_7\omega_3} a_1 a_2 e^{-i\theta_2} + \frac{f_9}{2f_7} ia_3 = 0
\] (3.24)

where

\[
\begin{align*}
f_3 &= -\omega_2^2 + \frac{d_7\beta_1^2}{c} + d_6\Lambda_2\beta_1^2 \\
f_4 &= \left(-\omega_2^2 + \frac{d_7\beta_1^2}{c}\right)(-d_7 + d_6\Lambda_2) + \frac{d_6d_7\beta_1^2}{c}(1 - \Lambda_2) \\
f_5 &= \left(-\omega_2^2 + \frac{d_7\beta_1^2}{c}\right)(-d_7 + d_6\Lambda_3) + \frac{d_6d_7\beta_1^2}{c}(1 - \Lambda_3) \\
f_6 &= \left(-\omega_2^2 + \frac{d_7\beta_1^2}{c}\right)\left[\frac{-\gamma_3}{c} - \gamma_3 - \gamma_2 + \Lambda_2\left(\frac{\gamma_3}{c} + \gamma_3\right)\right] + \\
&\quad + \frac{d_6\beta_1^2}{c}\left[\frac{d_7\gamma_3}{c} + \gamma_3 + \gamma_2 - \Lambda_2\left(\frac{d_7\gamma_3}{c} + \gamma_3\right)\right] \\
f_7 &= -\omega_3^2 + \frac{d_7\beta_1^2}{c} + d_6\Lambda_3\beta_1^2 \\
f_8 &= \left(-\omega_3^2 + \frac{d_7\beta_1^2}{c}\right)(-d_7 + d_6\Lambda_2) + \frac{d_6d_7\beta_1^2}{c}(1 - \Lambda_2) \\
f_9 &= \left(-\omega_3^2 + \frac{d_7\beta_1^2}{c}\right)\left[\frac{-\gamma_3}{c} - \gamma_3 - \gamma_2 + \Lambda_3\left(\frac{\gamma_3}{c} + \gamma_3\right)\right] + \\
&\quad + \frac{d_6\beta_1^2}{c}\left[\frac{d_7\gamma_3}{c} + \gamma_3 + \gamma_2 - \Lambda_3\left(\frac{d_7\gamma_3}{c} + \gamma_3\right)\right]
\end{align*}
\]
To separate the real and imaginary parts of modulation equations (3.17), (3.23) and (3.24), we have to transform \( \exp(i\theta) \) into a complex form
\[
\exp(i\theta) = \cos \theta + i\sin \theta.
\]
We obtain six modulation equations
\[
a_1' = a_2a_3f_1 \sin \theta_2 + a_2^2 f_2 \sin \theta_1 + \frac{1}{2} d_3 p \sin \theta_3 - \frac{1}{2} d_3 \gamma_1 a_1
\]
\[
a_1 a_1' = -a_2a_3f_1 \cos \theta_2 - a_2^2 f_2 \cos \theta_1 + \frac{1}{2} d_3 p \cos \theta_3
\]
\[
a_2' = -\frac{f_4}{4f_3\omega_2} a_1 a_2 \sin \theta_1 + \frac{f_5}{4f_3\omega_2} a_1 a_3 \sin \theta_2 + \frac{f_6}{2f_3} a_2
\]
\[
a_2 a_2' = -\frac{f_4}{4f_3\omega_2} a_1 a_2 \cos \theta_1 - \frac{f_5}{4f_3\omega_2} a_1 a_3 \cos \theta_2
\]
\[
a_3' = -\frac{f_8}{4f_7\omega_3} a_1 a_2 \sin \theta_2 + \frac{f_9}{2f_7} a_3
\]
\[
a_3 a_3' = -\frac{f_8}{4f_7\omega_3} a_1 a_2 \cos \theta_2
\]
(3.25)

From these equations, we look for steady-state motion. In this case, we have
\[
a_1' = 0 \quad a_2' = 0 \quad a_3' = 0
\]
\[
\theta_1' = 0 \quad \theta_2' = 0 \quad \theta_3' = 0
\]
(3.26)

We obtain a system of equations
\[
a_2a_3f_1 \sin \theta_2 + a_2^2 f_2 \sin \theta_1 + \frac{1}{2} d_3 p \sin \theta_3 - \frac{1}{2} d_3 \gamma_1 a_1 = 0
\]
\[
-a_2a_3f_1 \cos \theta_2 - a_2^2 f_2 \cos \theta_1 + \frac{1}{2} d_3 p \cos \theta_3 - a_1 \sigma_3 = 0
\]
\[
-\frac{f_4}{4f_3\omega_2} a_1 a_2 \sin \theta_1 + \frac{f_5}{4f_3\omega_2} a_1 a_3 \sin \theta_2 + \frac{f_6}{2f_3} a_2 = 0
\]
\[
-\frac{f_4}{4f_3\omega_2} a_1 a_2 \cos \theta_1 - \frac{f_5}{4f_3\omega_2} a_1 a_3 \cos \theta_2 - a_2 \sigma_3 + \frac{\sigma_1}{2} = 0
\]
\[
-\frac{f_8}{4f_7\omega_3} a_1 a_2 \sin \theta_2 + \frac{f_9}{2f_7} a_3 = 0
\]
\[
-\frac{f_8}{4f_7\omega_3} a_1 a_2 \cos \theta_2 - a_3 \frac{3\sigma_3 - 2\sigma_2 + 3\sigma_1}{2} = 0
\]
(3.27)
After transformations, we get amplitude equations

\[
\begin{align*}
    & f_8^2 a_1^2 a_2^2 - 4[f_9^2 + f_7^2(3\sigma_3 - 2\sigma_2 + 3\sigma_1)]\omega_3^2 a_3^2 = 0 \\
    & f_4^2 f_8^2 a_1^2 a_2^2 - (2f_5 f_9 \bar{\omega}_3 a_3^2 + 2f_6 f_8 \bar{\omega}_2 a_2^2)^2 + \\
    & - 2f_5 f_7 \bar{\omega}_3(3\sigma_3 - 2\sigma_2 + 3\sigma_1) a_3^2 - 2f_3 f_8 \bar{\omega}_2(\sigma_3 + \sigma_1) a_2^2 = 0 \\
    & f_4^2 f_8^2 d_3^2 a_1^2 - [4f_9 \bar{\omega}_3(f_4 f_1 + f_5 f_2) a_3^2 + 4f_6 f_8 f_9 \bar{\omega}_2 a_2^2 - f_4 f_8 d_3 \gamma_1 a_1^2]^2 + \\
    & + [4f_7 \bar{\omega}_3(f_4 f_1 - f_5 f_2)(3\sigma_3 - 2\sigma_2 + 3\sigma_1) a_3^2 + 4f_3 f_8 f_2 \bar{\omega}_2(\sigma_3 + \sigma_1) a_2^2 + \\
    & - f_4 f_8 \sigma_3 a_1^2]^2 = 0
\end{align*}
\] (3.28)

From equations (3.28), we obtain

\[
a_2^4 \left[- \frac{h_1^2 h_4}{h_3^2} a_1^4 + \left(h_3 - \frac{h_6 h_1}{h_3}\right) a_1^2 - h_5\right] = 0
\] (3.29)

We have two types of solutions, and these possibilities are examined in turn:

— case I – one-frequency solution

\[
a_2 = 0 \quad \text{then} \quad a_3 = 0 \quad \text{and} \quad a_2^2(h_1 a_1^2 - h_7) = 0
\]

so

\[
a_1 = 0 \quad \text{or} \quad a_1 = \sqrt{\frac{h_7}{h_{10}}}
\] (3.30)

— case II – multi-frequency solution

\[
\frac{h_1^2 h_4}{h_3^2} a_1^4 - \left(h_3 - \frac{h_6 h_1}{h_3}\right) a_1^2 + h_5 = 0
\] (3.31)

so

\[
a_1 = \sqrt{h_3 - \frac{h_6 h_1}{h_3} \pm \sqrt{\Delta_1}}
\] (3.32)

where

\[
\Delta_1 = \left(h_3 - \frac{h_6 h_1}{h_3}\right)^2 - 4\frac{h_4 h_5 h_1^2}{h_3^2}
\]

and from (3.28)

\[
a_2 = \sqrt{-\frac{\left(h_3 h_1 a_1^2 + h_{11} a_1^2\right)}{2\left(h_3^2 a_1^4 + h_{12} a_1^2 + h_9\right)}} \quad \text{and} \quad a_3 = \sqrt{\frac{h_1}{h_3} a_1 a_2}
\] (3.33)
where 

\[ \Delta_2 = \left( \frac{h_{13}h_1}{h_3}a_1^4 + h_{11}a_1^2 \right)^2 - 4\left( \frac{h_{12}h_1}{h_3}a_1^4 + \frac{h_{12}h_1}{h_3}a_1^2 + h_9 \right)(h_{10}a_1^4 - h_7a_1^2) \]

and 

\[
\begin{align*}
  h_1 &= f_8^2 \\
  h_2 &= 4f_7^2(f_7^2 + f_5^2)(3\sigma_3 - 2\sigma_2 + 3\sigma_1)^2 \\
  h_3 &= f_4f_8^2 \\
  h_4 &= 4f_5^2\omega_3^2(f_9^2 + f_7^2(3\sigma_3 - 2\sigma_2 + 3\sigma_1)^2) \\
  h_5 &= 4f_8^2\omega_3^2[f_6^2 + f_3^2(\sigma_3 + \sigma_1)^2] \\
  h_6 &= 8f_5f_8\omega_3[f_6f_9 - f_3f_7(3\sigma_3 - 2\sigma_2 + 3\sigma_1)(\sigma_3 + \sigma_1)] \\
  h_7 &= f_4^2f_8^2d_3^2p^2 \\
  h_{10} &= f_4^2f_8^2(d_3^2\gamma_1 + \sigma_3^2) \\
  h_8 &= 16f_9^2\omega_3^2(f_4f_1 + f_5f_2)^2 + 16f_7^2\omega_3^2(f_4f_1 - f_5f_2)^2(3\sigma_3 - 2\sigma_2 + 3\sigma_1)^2 \\
  h_9 &= 16f_6^2f_8^2f_9^2\omega_2^2 + 16f_2^2f_3^2f_8^2\omega_2^2(\sigma_3 + \sigma_1)^2 \\
  h_{11} &= -8f_4f_8^2\omega_2[f_6f_9d_3^2\gamma_1 + f_2f_3\sigma_3(\sigma_3 + \sigma_1)] \\
  h_{12} &= 32f_8\omega_2\omega_3[f_6f_9^2(f_4f_1 + f_5f_2) + \\
  & \quad + f_1f_3f_7(f_4f_1 - f_5f_2)(3\sigma_3 - 2\sigma_2 + 3\sigma_1)(\sigma_3 + \sigma_1)] \\
  h_{13} &= -8f_4f_8\omega_3[f_9d_3^2\gamma_1(f_4f_1 + f_5f_2) + f_7\sigma_3(f_4f_1 - f_5f_2)(3\sigma_3 - 2\sigma_2 + 3\sigma_1)]
\end{align*}
\]

Both cases of solutions (one-frequency and multi-frequency) are presented in Figs. 2-5. In Fig. 2 and Fig. 3 amplitudes \(a_1, a_2, a_3\) are plotted as functions of the amplitude of excitation \(p\). We can see the jump phenomenon associated with the varying amplitude \(p\). We have regions where two of the three solutions are stable. The initial conditions determine which of these solutions gives the response. We can clearly see the saturation phenomenon, when the amplitude \(a_1\) assumes its maximum value for stable solutions.

In Fig. 4 and Fig. 5, these amplitudes are presented versus the detuning parameter \(\sigma_1\). We can see the jump phenomenon associated with the varying frequency \(\omega_1\) according with the amplitude \(a_1\).

4. Conclusions

The multiple scales method can be used to find an approximate solution for a system with three degrees of freedom with variable amplitudes and phases. We can find resonance conditions (sometimes the resonance area is very narrow
Fig. 2. Amplitudes of the response as functions of the amplitude of the excitation; \( d_1 = 0.9, d_2 = 1.6, c = 1, \beta_1 = 0.67082, \mu = 1, \gamma_1 = 0.0001, \gamma_2 = 0.00001, \gamma_3 = 0.00001, \sigma_1 = \sigma_2 = \sigma_3 = 0 \)

Fig. 3. Amplitudes of the response as functions of the amplitude of the excitation; \( d_1 = 0.9, d_2 = 1.6, c = 1, \beta_1 = 0.67082, \mu = 1, \gamma_1 = 0.0001, \gamma_2 = 0.00001, \gamma_3 = 0.00001, \sigma_1 = \sigma_2 = \sigma_3 = 1 \)
Fig. 4. Frequency-response curves; $d_1 = 0.9$, $d_2 = 1.6$, $c = 1$, $\beta_1 = 0.67082$, $\mu = 1$, $p = 4.4$, $\gamma_1 = 0.0001$, $\gamma_2 = 0.00001$, $\gamma_3 = 0.00001$, $\sigma_2 = 0$, $\sigma_3 = 1$

and difficult to find numerically). It is possible to investigate steady state solutions for different combinations of external and internal resonances. We can observe regions where the solutions are stable or unstable, and can clearly see the saturation phenomenon.
References


**Analiza drgań dynamicznego układu z podwójnym wahadłem o trzech stopniach swobody**

**Streszczenie**

W pracy przebadano drgania nieliniowego układu o trzech stopniach swobody z podwójnym wahadłem w otoczeniu rezonansów wewnętrznych i zewnętrznych. Badania przeprowadzono analitycznie i numerycznie. Rozwiązanie analityczne uzyskano przy użyciu metody wielu skali czasowych. Metoda posłużyła do zbudowania nieliniowych równań różniczkowych pierwszego rzędu opisujących modulację amplitud i faz. Rozwiązanie ustalone i jego stabilność zostały przedstawione dla wybranych wartości parametrów układu.

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