JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
41, 4, pp. 735-753, Warsaw 2003

ON CONTINUUM MODELLING OF DYNAMIC PROBLEMS IN COMPOSITE SOLIDS WITH PERIODIC MICROSTRUCTURE

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In this contribution we propose an approach to the macroscopic modelling of periodic composites which is based on the periodic simplicial subdivision of the unit cell. This approach makes it possible to derive a hierarchy of continuum models which describe dynamic behaviour of periodic composites on different levels of accuracy. The obtained results are compared and applied to the analysis of a certain wave propagation problem.

Key words: composites, dynamics, modelling

1. Introduction

The investigations of wave dispersion problems in the elastodynamics of solids with a periodic microstructure can be carried out either on the basis of the Floquet-Bloch wave theory, cf. Lee (1972), Stoker (1950), Tolf (1983), or in the framework of different simplified continuum theories and models of these solids. We can mention here the effective stiffness theories, Achenbach and Sun (1972), Achenbach *et al.*(1968), Herrmann *et al.* (1976), the mixture theories, Bedford and Stern (1971), the interacting continuum theories, Hegemeier (1972), Lee (1972), the asymptotic models, Boutin and Auriault (1993), Fish and Wen Chen (2001), or the tolerance averaging models, Woźniak and Wierzbicki (2000). All aforementioned continuum theories and models describe dispersion phenomena in what are called low-frequency vibration problems. They are problems in which the macroscopic wavelength of the deformation pattern is sufficiently large when compared to the diameter of the unit cell of a periodic composite. In Rychlewska *et al.* (2000) it was proposed a discrete model of a periodic solid which can be also applied to the analysis of high-frequency vibration problems. The idea of the above approach was based on special simplicial subdivision of the unit cell and resulted in the finite-difference form of the governing equations.

The aim of this paper is to show that after introducing smoothing operations to the finite difference equations of a discrete model proposed in Rychlewska *et al.* (2000), it is possible to obtain a hierarchy of continuum models describing the macroscopic behaviour of a micro- periodic solid on different levels of accuracy. The simplest from the aforementioned models leads to the equations of a homogeneous equivalent medium which is not dispersive and can be obtained in the framework of the homogenization theory, Bensoussan *et al.* (1978), Jikov *et al.* (1994), Sanchez-Palencia (1980). It is also shown that continuum models derived in this paper on a higher level of accuracy constitute a proper tool for the analysis of dispersion phenomena in a composite medium. The general results are illustrated, compared and verified on the example of the wave propagation in a certain periodic composite medium.

To make this paper self-consistent in the subsequent section we summarise the main concepts introduced in Rychlewska *et al.* (2000).

2. Preliminaries

Let the composite solid under consideration occupies a region Ω in E^3, has perfectly bounded linear-elastic constituents and a periodic structure determined by a vector basis $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3$ in E^3. We denote by Δ a polyhedron in E^3 such that for every $\boldsymbol{x} \in \partial\Delta$ and some $\boldsymbol{d}_\alpha$, $\alpha = 1,2,3$, we have either $\boldsymbol{x} + \boldsymbol{d}_\alpha \in \partial\Delta$ or $\boldsymbol{x} - \boldsymbol{d}_\alpha \in \partial\Delta$ (but not both). Let us also assume that the diameter l of Δ is sufficiently small when compared to the smallest characteristic length dimension of the region Ω. In this case the polyhedron Δ will be referred to as the unit cell.

Let Λ be the Bravais lattice in E^3

$$\Lambda := \{\boldsymbol{z} \in E^3 : \quad \boldsymbol{z} = \eta_1\boldsymbol{d}_1 + \eta_2\boldsymbol{d}_2 + \eta_3\boldsymbol{d}_3, \quad \eta_\alpha = 0, \pm 1, \pm 2, ..., \quad \alpha = 1,2,3\}$$

and let us denote $\Delta(\boldsymbol{z}) := \boldsymbol{z} + \Delta$, $\Lambda_0 := \{\boldsymbol{z} \in \Lambda : \ \Delta(\boldsymbol{z}) \subset \Omega\}$ and $\overline{\Omega}_0 := \{\boldsymbol{x} \in \overline{\Delta}(\boldsymbol{z}) : \ \boldsymbol{z} \in \Lambda_0\}$, where Ω_0 is a regular subregion of Ω. A simplicial division of E^3 will be called Δ-periodic if it implies the simplicial subdivision of every $\overline{\Delta}(\boldsymbol{z})$, $\boldsymbol{z} \in \Lambda$, into simplexes $T^k(\boldsymbol{z})$, $k = 1, ..., m$, such that $T^k(\boldsymbol{z}) = T^k + \boldsymbol{z}$, $\boldsymbol{z} \in \Lambda$ where T^k, $k = 1, ..., m$, are simplexes in Δ. Let $\{\boldsymbol{p}_0^a \in \overline{\Delta} : a = 1, ..., n+1\}$, $n \geqslant 1$, be the smallest set of vertexes (nodal points) of T^k, $k = 1, ..., m$, such that $\{\boldsymbol{p}_0^a + \boldsymbol{z} : a = 1, ..., n+1, \boldsymbol{z} \in \Lambda\}$ is the set of all nodal points in E^3 related to a certain Δ-periodic simplicial division of E^3. We shall also introduce a system of vectors $\boldsymbol{d}_A \in \Lambda$, $A = 0, 1, ..., N$, such that $\boldsymbol{d}_0 = \boldsymbol{0}$ and every vertex related to T^k, $k = 1, ..., m$, can be uniquely represented by the sum $\boldsymbol{p}_0^a + \boldsymbol{d}_A$. It can be seen that $N = 7$ for the spatial problem and $N = 3$ for the plane problem. Setting $I := \{(a, A) \in \{1, ..., n+1\} \times \{0, 1, ..., N\} : \boldsymbol{p}_0^a + \boldsymbol{d}_A \in \overline{\Delta}\}$ and denoting $\boldsymbol{p}_A^a := \boldsymbol{p}_0^a + \boldsymbol{d}_A$ for every $(a, A) \in I$, we conclude that $\{\boldsymbol{p}_A^a : (a, A) \in I\}$ is the set of all nodal points in $\overline{\Delta}$ which is related to the Δ-periodic simplicial division of E^3. Hence, every simplex T^k can be represented by $T^k = \boldsymbol{p}_A^a \boldsymbol{p}_B^b \boldsymbol{p}_C^c \boldsymbol{p}_D^d$ where $(a, A), ..., (d, D) \in I$. Setting $I_0 := \{(a, A) \in I : A \neq 0\}$ we see that $\boldsymbol{p}_A^a \in \partial\Delta$ if and only if $(a, A) \in I_0$. Here and hereafter it is assumed that a certain Δ-periodic simplicial division of E^3 is known.

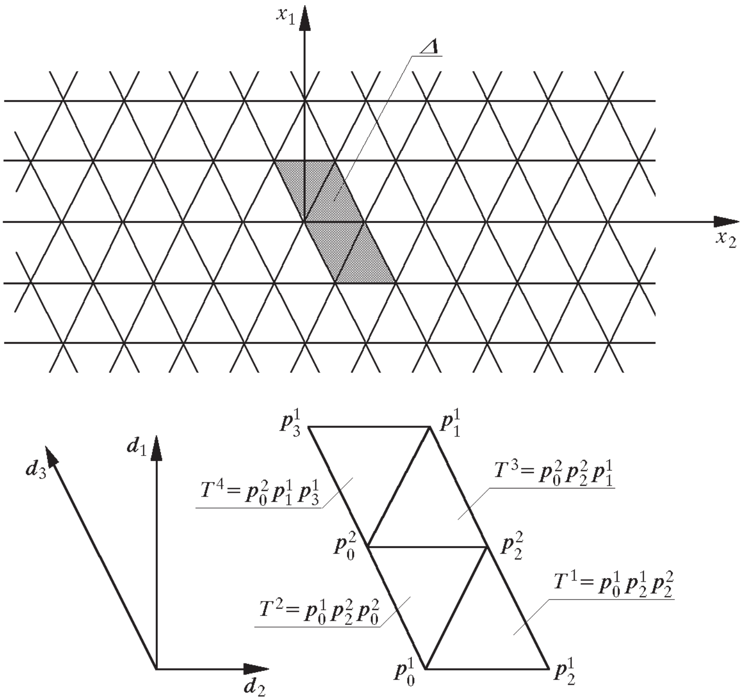

Fig. 1. Simplicial division of the $0x_1x_2$-plane with the cell Δ and a system of vectors $\boldsymbol{d}_A$ and nodal points $\boldsymbol{p}_A^a$ in $\overline{\Delta}$

In Fig. 1 a simple example of a Δ-periodic simplicial division of the plane (hence A runs over $0,1,2,3$) with points $\boldsymbol{p}_A^a$, $a=1,2$, and vectors $\boldsymbol{d}_A$, is shown.

For an arbitrary function $f(\cdot)$ defined on Λ_0 we shall define the finite differences

$$\Delta_A f(\boldsymbol{z}) = f(\boldsymbol{z}+\boldsymbol{d}_A) - f(\boldsymbol{z}) \qquad\qquad \overline{\Delta}_A f(\boldsymbol{z}) = f(\boldsymbol{z}) - f(\boldsymbol{z}-\boldsymbol{d}_A) \tag{2.1}$$

provided that $\boldsymbol{z}, \boldsymbol{z}+\boldsymbol{d}_A, \boldsymbol{z}-\boldsymbol{d}_A \in \Lambda_0$.

Throughout the paper it will be assumed that the superscripts a,b,c,d run over $1,...,n+1$, $n \geqslant 1$, and the subscripts A,B run over $0,1,...,N$, unless otherwise stated. We shall also introduce superscripts p, q which run over $1,...,n$. The summation convention with respect to all aforementioned indices holds.

Let $\boldsymbol{w}(\boldsymbol{x},t)$, $\boldsymbol{x} \in \Omega$, stand for a displacement field at time t for the solid under consideration. Let us denote

$$\boldsymbol{u}_A^a(\boldsymbol{z},t) := \boldsymbol{w}(\boldsymbol{p}_A^a(\boldsymbol{z}),t) \qquad\qquad (a,A) \in I_0 \qquad \boldsymbol{z} \in \Lambda_0$$

Subsequently, we shall interpret the simplexes T^k, $k=1,...,m$, as finite elements of the unit cell Δ which are subjected to uniform strains. Hence $\boldsymbol{p}_A^a(\boldsymbol{z})$ are nodal points of these elements. Let us also "approximate" the region Ω by Ω_0. In this case the displacement field $\boldsymbol{w}(\cdot,t)$ in every cell $\overline{\Delta}(\boldsymbol{z}), \boldsymbol{z} \in \Lambda_0$ will be uniquely determined by the displacements $\boldsymbol{u}_A^a(\boldsymbol{z},t)$ of the nodal points $\boldsymbol{p}_A^a(\boldsymbol{z})$, $(a,A) \in I$. Bearing in mind (2.1), we see that these displacements can be uniquely represented in the form

$$\boldsymbol{u}_A^a(\boldsymbol{z},t) = \boldsymbol{u}^a(\boldsymbol{z},t) + \Delta_A \boldsymbol{u}^a(\boldsymbol{z},t) \qquad\qquad (a,A) \in I$$

where for $A=0$ we obtain $\boldsymbol{u}_0^a(\boldsymbol{z},t) = \boldsymbol{u}^a(\boldsymbol{z},t)$. Let $\boldsymbol{u}$ be a certain averaged value of $\boldsymbol{u}^1,...,\boldsymbol{u}^{n+1}$, given by

$$\boldsymbol{u} = \nu_a \boldsymbol{u}^a$$

where $\nu_a > 0$ and $\nu_1 + ... + \nu_{n+1} = 1$. The values ν_a will be specified in the subsequent section. Under the above denotations, the strain and kinetic energy functions for the solid under consideration are respectively represented by

$$U = U(\Delta_A \boldsymbol{u}^a, \boldsymbol{u}^p - \nu_a \boldsymbol{u}^a) \qquad\qquad K = K(\Delta_A \dot{\boldsymbol{u}}^a, \dot{\boldsymbol{u}}^b) \tag{2.2}$$

where $(a,A) \in I_0$, $b=1,...,n+1$, $p=1,...,n$. The coefficients of forms (2.2) can be uniquely determined for every periodic solid. Introducing the differences

$\boldsymbol{u}^p - \boldsymbol{u}$ as arguments of the strain energy function we have taken into account the translational invariance of $U(\cdot)$. It can be shown, Woźniak (1971), that for the unknowns $\boldsymbol{u}^a(\boldsymbol{z},t)$, $a = 1,..,n+1$, $\boldsymbol{z} \in \Lambda_0$, in the absence of body forces, we obtain a system of ordinary differential equations which can be expressed in the following finite-difference form

$$\overline{\Delta}_A \boldsymbol{s}_A^a - \frac{\partial U}{\partial \boldsymbol{u}^a} = \frac{d}{dt}\Big(\frac{\partial K}{\partial \dot{\boldsymbol{u}}^a} - \overline{\Delta}_A \boldsymbol{j}_A^a\Big) \qquad a = 1,...,n+1 \tag{2.3}$$

where

$$\boldsymbol{s}_A^a = \frac{\partial U}{\partial \Delta_A \boldsymbol{u}^a} \qquad \boldsymbol{j}_A^a = \frac{\partial K}{\partial \Delta_A \dot{\boldsymbol{u}}^a} \qquad (a,A) \in I_0 \tag{2.4}$$

Equations (2.3), (2.4) are assumed to hold for every $\boldsymbol{z} \in \Lambda_0$ such that $\boldsymbol{z} \pm \boldsymbol{d}_A \in \Lambda_0$ for $A = 1,...,N$, and represent *a finite difference model* of the periodic composite under consideration. It has to be emphasised that this model has a physical sense only if the diameters l_k of simplexes T^k, $k = 1,...,m$ are small as compared to the typical wavelength of the deformation pattern in the problem under consideration. Thus, for high-frequency vibration problems, the number m of simplexes T^k and hence also the number n of the unknowns $\boldsymbol{u}^a(\boldsymbol{z},t)$ for every $\boldsymbol{z} \in \Lambda_0$ can be very large.

Equations of the form (2.3), (2.4) have been derived and applied in Rychlewska *et al.* (2000); in this paper they constitute the foundations of the subsequent analysis leading to different continuum models of the microperiodic solids under consideration.

3. Simplified finite difference models

For the given $\nu_1,...,\nu_{n+1}$ let us denote

$$\widetilde{\boldsymbol{u}}^a := \boldsymbol{u}^a - \boldsymbol{u} = \boldsymbol{u}^a - \nu_b \boldsymbol{u}^b$$

It follows that

$$\nu_a \widetilde{\boldsymbol{u}}^a = \boldsymbol{0}$$

and hence the fields $\widetilde{\boldsymbol{u}}^a(\boldsymbol{z},t)$, $\boldsymbol{z} \in \Lambda_0$, are linear dependent. In order to satisfy the above condition we shall introduce new linear independent fields $\boldsymbol{v}^q = \boldsymbol{v}^q(\boldsymbol{z},t)$, $\boldsymbol{z} \in \Lambda_0$, such that

$$\widetilde{\boldsymbol{u}}^a = l h^{aq} \boldsymbol{v}^q$$

where l is the diameter of Δ and h^{aq} are elements of the $(n+1)\times n$ matrix of an order n, satisfying conditions

$$\nu_a h^{aq} = 0$$

Hence

$$\boldsymbol{u}^a = \boldsymbol{u} + l h^{aq}\boldsymbol{v}^q \tag{3.1}$$

and we shall take $\boldsymbol{u}$ and $\boldsymbol{v}^q$ as the basic unknowns. It can be seen that the above formula represents a decomposition of the displacement field $\boldsymbol{u}^a$ into the averaged $\boldsymbol{u} = \nu_a\boldsymbol{u}^a$ and fluctuating $\widetilde{\boldsymbol{u}}^a$ parts.

Subsequently, we shall restrict ourselves to problems in which the increments $\Delta_A\widetilde{\boldsymbol{u}}^a$ of fluctuations can be neglected as small with respect to the increments $\Delta_A\boldsymbol{u}$ of the averaged displacements. Thus, we shall apply to (2.2) an approximation

$$\Delta_A\boldsymbol{u}^a \cong \Delta_A\boldsymbol{u} \tag{3.2}$$

which holds for every $(a,A)\in I_0$. The above formula states that in an arbitrary but fixed periodicity cell $\overline{\Delta}(\boldsymbol{z})$, $\boldsymbol{z}\in\Lambda_0$, the displacement fluctuations can be treated as periodic

$$\widetilde{\boldsymbol{u}}^a(\boldsymbol{z}+\boldsymbol{d}^A,t) \cong \widetilde{\boldsymbol{u}}^a(\boldsymbol{z},t) \qquad\qquad (a,A)\in I_0$$

Subsequently, for the sake of simplicity, we shall also approximate the mass distribution in the periodic medium by a periodic system of concentrated masses m^a, $a=1,...,n+1$, assigned to the nodal points. Setting $m = m^1 + ... + m^{n+1}$ we shall assume that $\nu_a = m^a/m$. Hence, the kinetic energy function will take the form

$$K = \frac{1}{2|\Delta|}\sum_{a=1}^{n+1} m^a\dot{\boldsymbol{u}}^a\cdot\dot{\boldsymbol{u}}^a$$

where $|\Delta|$ is the measure of the cell Δ. Taking into account formula (3.1) we obtain the kinetic energy function in the form

$$\widetilde{K} = \frac{1}{2}\rho\dot{\boldsymbol{u}}\cdot\dot{\boldsymbol{u}} + \frac{1}{2}l^2M^{pq}\dot{\boldsymbol{v}}^p\cdot\dot{\boldsymbol{v}}^q \tag{3.3}$$

where

$$\rho = \frac{m}{|\Delta|} \qquad\qquad M^{pq} = \frac{1}{|\Delta|}\sum_{a=1}^{n+1} m^a h^{aq}h^{ap}$$

Taking into account (3.1) and (3.2), we obtain from (2.2) the strain energy function

$$\widetilde{U} = \frac{1}{2}a_{AB}\Delta_A\boldsymbol{u}\cdot\Delta_B\boldsymbol{u} + \frac{1}{2}b^{pq}\boldsymbol{v}^p\cdot\boldsymbol{v}^q + c_A^q\boldsymbol{v}^q\cdot\Delta_A\boldsymbol{u} \tag{3.4}$$

which is the density per unit measure of Δ. Because of $\Delta_A \boldsymbol{u} \in O(l)$ we have $a_{AB} \in O(l^{-2})$, $c^q_A \in O(l^{-1})$ and $b^{pq} \in O(1)$, i.e., all terms in (3.4) are of the same order.

Using (3.3) and (3.4), we shall transform equations (2.3), (2.4) to the form

$$\begin{aligned} &\overline{\Delta}_A \boldsymbol{s}_A + c^q_A \overline{\Delta}_A \boldsymbol{v}^q = \rho \ddot{\boldsymbol{u}} \qquad\qquad \boldsymbol{s}_A = a_{AB} \Delta_B \boldsymbol{u} \\ &l^2 M^{pq} \ddot{\boldsymbol{v}}^q + b^{pq} \boldsymbol{v}^q + c^p_A \Delta_A \boldsymbol{u} = 0 \end{aligned} \tag{3.5}$$

The above equations hold for every $\boldsymbol{z} \in \Lambda_0$ and time t, and represent *a simplified finite difference model* of the periodic composite medium under consideration.

Let us observe that $l^2 M^{pq} \ddot{\boldsymbol{v}}^q \in O(l^2)$ and the values of all other terms in (3.5) are independent of l. Hence, for a sufficiently small l we can apply the limit passage $l \to 0$. In this case, the first term in the second from equations (3.5) will be neglected and we arrive at the equations

$$b^{pq} \boldsymbol{v}^q = -c^p_A \Delta_A \boldsymbol{u}$$

Since b^{pq} represent a non-singular $n \times n$ matrix, then denoting by B^{pq} the elements of the inverse matrix and setting

$$a^0_{AB} := a_{AB} - c^q_A B^{qp} c^p_B$$

the first from equations (3.5) yields

$$a^0_{AB} \overline{\Delta}_A \Delta_B \boldsymbol{u} = \rho \ddot{\boldsymbol{u}} \tag{3.6}$$

Thus, we have arrived at the single equation for the averaged displacement field $\boldsymbol{u}(\boldsymbol{z}, t)$, $\boldsymbol{z} \in \Lambda_0$. The above equation together with the formulae

$$\boldsymbol{v}^q = -B^{qp} c^p_A \Delta_A \boldsymbol{u} \tag{3.7}$$

represent what will be called the asymptotic discrete finite element model of the periodic composite under consideration. Let us observe that for stationary problems the last from equations (3.5) coincide with equations (3.7).

Discrete models governed by equations (3.5) and (3.6) will be treated in subsequent analysis as a basis for the formulation of continuum models. The main advantage of the aforementioned equations is that they involve finite differences with respect to only one unknown field $\boldsymbol{u}$, in contrast to equations (2.3), (2.4). This fact will imply a relatively simple form of pertinent continuum model equations which will be derived in the subsequent section. It has to be remembered that equations (3.5), (3.6) can be applied exclusively to the analysis of the long wave propagation problems.

4. Formulation of continuum models

Let $\boldsymbol{u}(\cdot,t)$, $\boldsymbol{s}_A(\cdot,t)$ be arbitrary sufficiently smooth fields defined on Ω, which after restricting their domain Ω to Λ_0 reduce to fields $\boldsymbol{u}(\boldsymbol{z},t)$, $\boldsymbol{s}_A(\boldsymbol{z},t)$, $\boldsymbol{z} \in \Lambda_0$, occurring in (3.5). In order to obtain a continuum model of the periodic solid under consideration, we shall assume that for every $\boldsymbol{z} \in \Lambda_0$ and every $\boldsymbol{y}$ such that $|\boldsymbol{y}| \leqslant l$ and $\boldsymbol{z}+\boldsymbol{y} \in \Omega$, the aforementioned smooth fields can be approximated by means of the formulae

$$\boldsymbol{w}(\boldsymbol{z}+\boldsymbol{y},t) \cong \boldsymbol{w}(\boldsymbol{z},t) + \boldsymbol{y}\cdot\nabla\boldsymbol{w}(\boldsymbol{z},t) + \frac{1}{2}(\boldsymbol{y}\otimes\boldsymbol{y}) : (\nabla\otimes\nabla)\boldsymbol{w}(\boldsymbol{z},t)$$

where $\boldsymbol{w}$ stands for $\boldsymbol{u}$ and $\boldsymbol{s}_A$. From the above approximation we also obtain

$$\boldsymbol{w}(\boldsymbol{z},t) \cong \boldsymbol{w}(\boldsymbol{z}-\boldsymbol{y},t) + \boldsymbol{y}\cdot\nabla\boldsymbol{w}(\boldsymbol{z},t) - \frac{1}{2}(\boldsymbol{y}\otimes\boldsymbol{y}) : (\nabla\otimes\nabla)\boldsymbol{w}(\boldsymbol{z},t)$$

Hence, under denotations (no summation over A!)

$$\boldsymbol{e}_A := \boldsymbol{d}_A l^{-1} \qquad\qquad \boldsymbol{E}_A := \frac{1}{2}\boldsymbol{e}_A\otimes\boldsymbol{e}_A$$

we conclude that the following approximations

$$\begin{aligned}
\Delta_A\boldsymbol{u}(\boldsymbol{z},t) &\cong l\boldsymbol{e}_A\cdot\nabla\boldsymbol{u}(\boldsymbol{z},t) + l^2\boldsymbol{E}_A : (\nabla\otimes\nabla)\boldsymbol{u}(\boldsymbol{z},t)\\
\overline{\Delta}_A\boldsymbol{s}_A(\boldsymbol{z},t) &\cong l\boldsymbol{e}_A\cdot\nabla\boldsymbol{s}_A(\boldsymbol{z},t) - l^2\boldsymbol{E}_A : (\nabla\otimes\nabla)\boldsymbol{s}_A(\boldsymbol{z},t)
\end{aligned} \tag{4.1}$$

hold for every $\boldsymbol{z} \in \Lambda_0$. Substituting the right-hand sides of the above formulae into (3.5) and denoting

$$\begin{aligned}
\boldsymbol{G} &:= a_{AB}\boldsymbol{E}_A\otimes\boldsymbol{E}_B l^2 & \boldsymbol{C} &:= a_{AB}\boldsymbol{e}_A\otimes\boldsymbol{e}_B l^2\\
\boldsymbol{H}^q &:= c_A^q\boldsymbol{E}_A l & \boldsymbol{h}^q &:= c_A^q\boldsymbol{e}_A l
\end{aligned} \tag{4.2}$$

after simple manipulations we obtain

$$\begin{aligned}
&-l^2(\nabla\otimes\nabla) : [\boldsymbol{G} : (\nabla\otimes\nabla)\boldsymbol{u}] + \nabla\cdot(\boldsymbol{C}\cdot\nabla\boldsymbol{u}) - l\boldsymbol{H}^q : (\nabla\otimes\nabla)\boldsymbol{v}^q + \boldsymbol{h}^q\cdot\nabla\boldsymbol{v}^q = \rho\ddot{\boldsymbol{u}}\\
&l^2 M^{pq}\ddot{\boldsymbol{v}}^q + b^{pq}\boldsymbol{v}^q + \boldsymbol{h}^q\cdot\nabla\boldsymbol{u} + l\boldsymbol{H}^q : (\nabla\otimes\nabla)\boldsymbol{u} = 0 \qquad\qquad q = 1,...,n
\end{aligned} \tag{4.3}$$

Because $\boldsymbol{u}(\cdot,t)$, $\boldsymbol{v}^q(\cdot,t)$ are functions defined for every time t on the region Ω we have arrived at the system of $n+1$ differential equations (4.3) for $n+1$ unknown vector fields $\boldsymbol{u}$, $\boldsymbol{v}^q$. The aforementioned equations represent what will be called *the second order continuum model* of the periodic composite medium under consideration. It has to be emphasized that for the averaged displacement field $\boldsymbol{u}$ we have obtained the partial differential equation and for the displacement fluctuations $\boldsymbol{v}^q$ the system of n ordinary differential equations. It follows that the boundary conditions can be imposed only on the averaged displacement field; we deal here with a situation similar to that occurring in the tolerance averaging model equations, Woźniak and Wierzbicki (2000).

Applying approximations (4.1) to equation (3.6) and denoting

$$\boldsymbol{G}_0 := a^0_{AB}\boldsymbol{E}_A \otimes \boldsymbol{E}_B l^2 \qquad\qquad \boldsymbol{C}_0 := a^0_{AB}\boldsymbol{e}_A \otimes \boldsymbol{e}_B l^2 \tag{4.4}$$

we obtain

$$-l^2(\nabla \otimes \nabla) : [\boldsymbol{G}_0 : (\nabla \otimes \nabla)\boldsymbol{u}] + \nabla \cdot (\boldsymbol{C}_0 \cdot \nabla \boldsymbol{u}) = \rho\ddot{\boldsymbol{u}} \tag{4.5}$$

The above equation represent *the asymptotic second order continuum model* of the medium under consideration.

Now assume that instead of (4.1) we introduce the linear approximations

$$\varDelta_A \boldsymbol{u}(\boldsymbol{z},t) \cong l\boldsymbol{e}_A \cdot \nabla \boldsymbol{u}(\boldsymbol{z},t) \qquad\qquad \overline{\varDelta}_A \boldsymbol{s}_A(\boldsymbol{z},t) \cong l\boldsymbol{e}_A \cdot \nabla \boldsymbol{s}_A(\boldsymbol{z},t) \tag{4.6}$$

In this case equations (3.5) reduce to the form

$$\begin{aligned} &\nabla \cdot (\boldsymbol{C} \cdot \nabla \boldsymbol{u}) + \boldsymbol{h}^q \cdot \nabla \boldsymbol{v}^q = \rho\ddot{\boldsymbol{u}} \\ &l^2 M^{pq}\ddot{\boldsymbol{v}}^q + b^{pq}\boldsymbol{v}^q + \boldsymbol{h}^q \cdot \nabla \boldsymbol{u} = 0 \qquad\qquad q = 1,...,n \end{aligned} \tag{4.7}$$

where $\boldsymbol{C}$ and $\boldsymbol{h}^q$ are defined by formulae (4.2). The above equations represent *the first order continuum model* of the periodic composite medium. Similarly, from (3.6) we derive the equation

$$\nabla \cdot (\boldsymbol{C}_0 \cdot \nabla \boldsymbol{u}) = \rho\ddot{\boldsymbol{u}} \tag{4.8}$$

representing *the asymptotic first order continuum model* of the medium under consideration.

By introducing higher order derivatives into approximations of the form (4.1), it is possible to formulate higher-order continuum models of the periodic medium under consideration; these models have a rather complicated form and will not be discussed here. Subsequently, we shall apply the obtained model equations only to the analysis of the wave propagation in an unbounded medium; that is why in this paper we shall not discuss the physical meaning of boundary conditions related to equations (4.3), (4.5), (4.7) and (4.8). It can be shown that the aforementioned equations together with pertinent natural boundary conditions can also be derived from the principle of stationary action for the action functional

$$\int_{t_0}^{t_1}\int_{\Omega}(\overline{K}-\overline{U})\,d\boldsymbol{x}dt$$

where $\overline{K}$ and $\overline{U}$ are obtained from (3.3) and (3.4), respectively, by using the approximations introduced at the beginning of this section.

Summarising the obtained results, we shall state that the macroscopic dynamic behaviour of the elastic composites with a periodic microstructure can be analysed in the framework of different continuum models described by independent systems of equations (4.3), (4.5), (4.7) and (4.8). The above equations have constant coefficients which depend on the geometric and material structure of the unit cell, i.e. on the vectors $\boldsymbol{e}_A$, $A = 1, ..., N$, and coefficients of the quadratic forms (3.3), (3.4) related to the discrete model. Solutions to these equations have a physical sense only if approximation formulae (4.1) or (4.6) are satisfied with a sufficient accuracy. Obviously, the derived continuum models describe the dynamic behaviour of the composite on different levels of accuracy. Thus, the problem arises what is the interrelation between these models and their accuracy when compared to the discrete model given by equations (2.3), (2.4). The above problem will be discussed in the subsequent section.

5. Comparison and reliability of continuum models

The aim of the subsequent analysis is to compare the results obtained from the second and first order continuum models (represented respectively by equations (4.3), (4.5) and (4.7), (4.8)) with the results derived from the discrete models described by equations (2.3), (2.4). This comparison will be carried out on the example of the analysis of a harmonic wave propagating in an unbounded linear elastic homogeneous medium which is reinforced by two parallel families of fibres. We shall assume that the axes of all fibres are parallel to the x_3-axis of the Cartesian orthogonal coordinate system $0x_1x_2x_3$, and the cross-sections of fibres belonging to the first and second family are periodically distributed on the $0x_1x_2$-plane as shown in Fig. 2.

The Lamé moduli and mass density of the medium will be denoted by λ, μ and ρ_0, respectively. The mass densities and areas of cross sections of fibres belonging to the first and second family of fibres are denoted by ρ_1, A_1 and ρ_2,

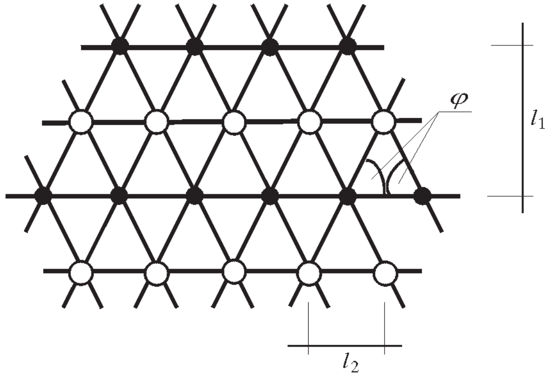

Fig. 2. Cross section of a composite reinforced by two families of parallel fibres

A_2, respectively. The values A_1, A_2 are assumed to be small when compared with the area A of the unit cell, and hence the masses $\rho_1 A_1$ and $\rho_2 A_2$ will be treated as concentrated masses on the $0x_1x_2$-plane. To simplify considerations, we shall deal with the longitudinal wave propagating in the x_1-axis direction. Introducing the periodic simplicial division of the plane as shown in Fig. 1, with $a = 1, 2$ and $A = 0, 1, 2, 3$, we obtain the scheme of displacements of nodal points of $\overline{\varDelta}$ as in Fig. 3, where the finite-difference operator $\varDelta_1$ is denoted by $\varDelta$. We also denote $l := l_1$. For the sake of simplicity we have introduced here the simplest $\varDelta$-periodic simplicial division of the plane, and hence all subsequent results can be treated only as a rough approximation of the problem under consideration.

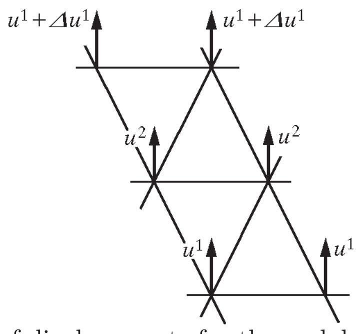

Fig. 3. Scheme of displacements for the nodal points of cell $\overline{\varDelta}$

We begin with the discrete model governed by equations (2.3), (2.4). Under denotations

$$\eta := \lambda + 2\mu \qquad\qquad \varepsilon_1 := \frac{A_1}{A} \qquad\qquad \varepsilon_2 := \frac{A_2}{A}$$

the strain energy function is given by

$$U = \frac{1}{2}\alpha(\varDelta u^1)^2 + \frac{1}{2}\beta(u^2 - u^1)^2 + \gamma(u^2 - u^1)\varDelta u^1$$

where

$$\alpha = \frac{2\eta}{l^2} \qquad \beta = \frac{4\eta}{l^2} \qquad \gamma = -\frac{2\eta}{l^2}$$

and the kinetic energy function has the form

$$K = \frac{1}{2}\psi(\varDelta\dot{u}^1)^2 + \frac{1}{2}\nu(\dot{u}^1)^2 + \frac{1}{2}\xi(\dot{u}^2)^2 + \zeta\dot{u}^1\varDelta\dot{u}^1 + \delta\dot{u}^2\varDelta\dot{u}^1 + \vartheta\dot{u}^1\dot{u}^2$$

where

$$\begin{array}{lll} \psi = \dfrac{1}{6}\rho_0 & \nu = \dfrac{1}{3}\rho_0 + \rho_1\varepsilon_1 & \xi = \dfrac{1}{3}\rho_0 + \rho_2\varepsilon_2 \\ \zeta = \dfrac{1}{6}\rho_0 & \delta = \dfrac{1}{12}\rho_0 & \vartheta = \dfrac{1}{6}\rho_0 \end{array}$$

Governing equations (2.3), (2.4) are given by

$$\begin{aligned} &\overline{\varDelta}s^1 + \beta(u^2 - u^1) + \gamma\varDelta u^1 = \nu\ddot{u}^1 + \zeta\varDelta\ddot{u}^1 + \vartheta\ddot{u}^2 - \frac{d}{dt}\overline{\varDelta}j^1 \\ &-\beta(u^2 - u^1) - \gamma\varDelta u^1 = \xi\ddot{u}^2 + \delta\varDelta\ddot{u}^1 + \vartheta\ddot{u}^1 \end{aligned} \tag{5.1}$$

where

$$s^1 = \alpha\varDelta u^1 + \gamma(u^2 - u^1) \qquad j^1 = \psi\varDelta\dot{u}^1 + \delta\dot{u}^2 + \zeta\dot{u}^1$$

In order to obtain equations (3.5), we assign the concentrated masses m_1, m_2 to points $\boldsymbol{p}_0^1$, $\boldsymbol{p}_0^2$, respectively, where

$$m_1 = \frac{1}{2}\rho_0 A + \rho_1 A_1 \qquad m_2 = \frac{1}{2}\rho_0 A + \rho_2 A_2$$

Setting

$$h^1 = \frac{m_2}{m_1 + m_2} \qquad h^2 = -\frac{m_1}{m_1 + m_2}$$

decomposition (3.1) will be taken in the form

$$u^1 = u + lh^1 v \qquad u^2 = u + lh^2 v \tag{5.2}$$

Hence, the kinetic energy function is given by

$$K = \frac{1}{2}\nu(\dot{u}^1)^2 + \frac{1}{2}\xi(\dot{u}^2)^2 + \vartheta\dot{u}^1\dot{u}^2$$

Using (5.2) we obtain (3.3) in the form

$$\widetilde{K} = \frac{1}{2}\rho(\dot{u})^2 + \frac{1}{2}l^2 M(\dot{v})^2$$

where

$$\begin{aligned}
\rho &= \rho_0 + \rho_1\varepsilon_1 + \rho_2\varepsilon_2 \\
M &= \frac{(\frac{1}{2}\rho_0 + \rho_1\varepsilon_1)(\frac{1}{2}\rho_0 + \rho_2\varepsilon_2)}{\rho_0 + \rho_1\varepsilon_1 + \rho_2\varepsilon_2} - \frac{1}{6}\rho_0 = \\
&= \frac{\rho_0^2 + 4\rho_0(\rho_1\varepsilon_1 + \rho_2\varepsilon_2) + 12\rho_1\varepsilon_1\rho_2\varepsilon_2}{12(\rho_0 + \rho_1\varepsilon_1 + \rho_2\varepsilon_2)}
\end{aligned}$$

The strain energy function (3.4), reduces to the form

$$\widetilde{U} = \frac{1}{2}a(\Delta u)^2 + \frac{1}{2}b(v)^2 + cv\Delta u$$

where

$$a = \frac{2\eta}{l^2} \qquad b = 4\eta \qquad c = \frac{2\eta}{l}$$

In order to formulate continuum models for the problem under consideration we shall use the denotation

$$\partial^k(\cdot) \equiv \frac{\partial^k(\cdot)}{\partial x^k} \qquad k = 1,2,3,4$$

Second order continuum model equations (4.3) yield

$$\begin{aligned}
&\underline{-l^2 G\partial^4 u} + C\partial^2 u - \underline{lH\partial^2 v} + h\partial v = \rho\ddot{u} \\
&l^2 M\ddot{v} + bv + h\partial u + \underline{lH\partial^2 u} = 0
\end{aligned} \tag{5.3}$$

where by means of (4.2) we obtain

$$G = \frac{1}{2}\eta \qquad C = 2\eta \qquad H = \eta \qquad h = 2\eta$$

The asymptotic second order continuum model equation reduces to the form

$$\underline{-l^2 G_0\partial^4 u} + C_0\partial^2 u = \rho\ddot{u} \tag{5.4}$$

where

$$G_0 = \frac{1}{4}\eta \qquad C_0 = \eta$$

For the pertinent first order continuum model equations we have to neglect the underlined terms in equations (5.3), (5.4).

In order to compare the results obtained from the proposed models, let us investigate propagation of a harmonic wave, which for discrete model (5.1) is given by

$$u^1 = A_1 \exp[ik(nl - ct)] \qquad u^2 = A_2 \exp[ik(nl - ct)] \qquad n = 0, \pm 1, \pm 2, ...$$

where A_1, A_2 are the wave amplitudes, $k = 2\pi/L$ is the wave number and c is the propagation speed. Substituting the right-hand sides of the above formulae into (5.1), we obtain the dispersion relation between the wave number k and the vibration frequency $\omega = kc$ in the form

$$\Big(-\frac{\rho_0^2}{72}\cos kl + r + \frac{\rho_0^2}{72}\Big)\omega^4 + \alpha\Big(-\frac{\rho_0}{3}\cos kl - 2\rho + \frac{\rho_0}{3}\Big)\omega^2 - 2\alpha^2\cos kl + 2\alpha^2 = 0 \tag{5.5}$$

where

$$r_1 := \frac{1}{2}\rho_0 + \rho_1\kappa_1 \qquad r_2 := \frac{1}{2}\rho_0 + \rho_2\kappa_2 \qquad r := r_1 r_2 - \frac{1}{6}\rho_0\rho$$

Introducing the nondimensional wave number $q = kl$ and bearing in mind that the analysis is restricted to the wavelengths L sufficiently large when compared to the microstructure length l, we have to assume that $q \ll 1$. Hence q can be treated as a small parameter, and we may set

$$\cos q \cong 1 - \frac{1}{2}q^2$$

In this case, we obtain from (5.5) the following asymptotic formulae for lower and higher free vibration frequencies

$$\begin{aligned}(\omega_-)^2 &= \frac{\alpha}{2\rho}q^2 - \alpha\rho_0\frac{12r + \rho_0\rho}{288r\rho^2}q^4 + \mathcal{O}(q^6)\\ (\omega_+)^2 &= \frac{2\alpha\rho}{r} - \frac{\alpha r_1^2 r_2^2}{2r^2\rho}q^2 + \alpha\rho_0\frac{36r^2 + \rho_0^2\rho^2 + 3\rho_0\rho r}{864\rho^2 r^2}q^4 + \mathcal{O}(q^6)\end{aligned} \tag{5.6}$$

where the terms $\mathcal{O}(q^6)$ are small and can be neglected.

On passing to the analysis of harmonic waves in the framework of the proposed continuum models we assume

$$u = A\exp[ik(x - ct)] \qquad v = B\exp[ik(x - ct)]$$

Substituting the above formulae into second order continuum model equations (5.3), we obtain the dispersion relation of the form

$$r\omega^4 - \alpha\Big[\frac{r}{\rho}q^2\Big(1+\frac{1}{4}q^2\Big) + 2\rho\Big]\omega^2 + \alpha^2 q^2\Big(1+\frac{1}{4}q^2\Big) = 0 \tag{5.7}$$

where $q = kl$ is the nondimensional wave number. By means of $q \ll 1$, we obtain from (5.7) the asymptotic formulae

$$\begin{aligned}
(\omega_-)^2 &= \frac{\alpha}{2\rho}q^2 + \frac{\alpha}{8\rho}\Big(1-\frac{r}{\rho^2}\Big)q^4 + \mathcal{O}(q^6)\\
(\omega_+)^2 &= \frac{2\alpha\rho}{r} + \frac{\alpha}{2\rho}q^2 + \frac{\alpha}{8\rho}\Big(1+\frac{r}{\rho^2}\Big)q^4 + \mathcal{O}(q^6)
\end{aligned} \tag{5.8}$$

The pertinent formulae obtained in the framework of the first order continuum model equations are

$$\begin{aligned}
(\omega_-)^2 &= \frac{\alpha}{2\rho}q^2 - \frac{\alpha r}{8\rho^3}q^4 + \mathcal{O}(q^6)\\
(\omega_+)^2 &= \frac{2\alpha\rho}{r} + \frac{\alpha}{2\rho}q^2 + \frac{\alpha r}{8\rho^3}q^4 + \mathcal{O}(q^6)
\end{aligned} \tag{5.9}$$

For the asymptotic model equations, we assume

$$u = A\exp[\mathrm{i}k(x-ct)]$$

and after substituting the right-hand side of this equation into (5.4), we obtain

$$\omega^2 = \frac{\alpha}{2\rho}\Big(q^2 + \frac{1}{4}q^4\Big) \tag{5.10}$$

The pertinent formula obtained in the framework of the first order asymptotic model equations is

$$\omega^2 = \frac{\alpha}{2\rho}q^2 \tag{5.11}$$

At the end of this contribution we present some numerical results. We introduce the following dimensionless coefficients

$$\kappa_1 := \frac{\rho_1}{\rho_0} \qquad\qquad \kappa_2 := \frac{\rho_2}{\rho_0}$$

where ρ_0, ρ_1, ρ_2 are mass densities of the medium and fibres belonging to the first and the second family of fibres, respectively. The free vibrations frequencies for discrete model (5.6), second order continuum model (5.8), first order continuum model (5.9), asymptotic second order continuum model (5.10) and asymptotic first order continuum model (5.11), we will write in the dimensionless form

$$\widetilde{\omega}^2 := \frac{\rho_0}{\alpha}(\omega_0)^2$$

where $\alpha = 2\eta l^{-2}$ and ω_0 is the frequency described by (5.6), (5.8)-(5.11). The calculations were carried out for $\kappa_1 = 5$, $\lambda_1 = 0.1$, $\lambda_2 = 0.05$ and $\kappa_2 = 5; 10; 15$. The diagrams of spectral lines obtained in the framework of the proposed models coincide if $q \leqslant 0.1$; for lower frequencies are shown in Fig. 4, and for higher frequencies in Fig. 5. Diagrams of spectral lines for frequencies in the asymptotic first and second order continuum models also coincide for $q \leqslant 0.1$ and are presented in Fig. 6.

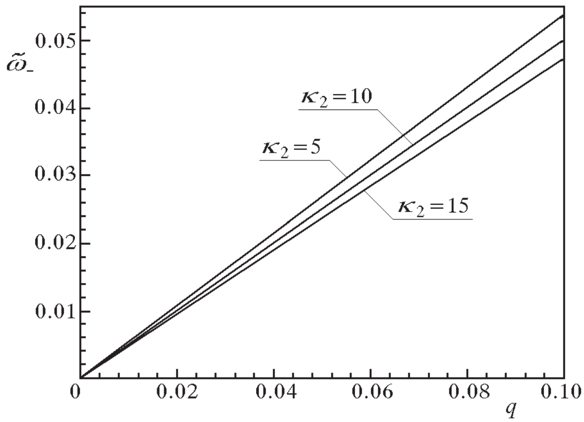

Fig. 4. Diagrams of spectral lines for lower frequencies in the finite difference models and the first and second order continuum models

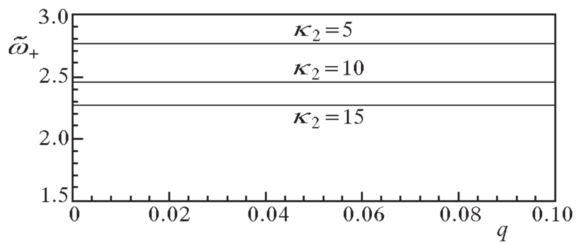

Fig. 5. Diagrams of spectral lines for higher frequencies in the finite difference models and the first and second order continuum models

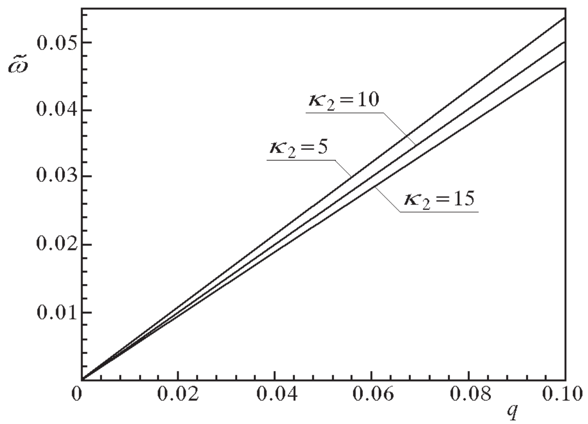

Fig. 6. Diagrams of spectral lines for frequencies in the asymptotic first and second order continuum models

6. Concluding remarks

The aim of this contribution was to derive a certain hierarchy of continuum models for the dynamics of a linear elastic composite solid with a periodic microstructure and to compare different results obtained in the framework of these models with the results derived from the discrete finite-difference models given by equations (2.3), (2.4). Restricting the considerations to the low-frequency vibration problems and introducing successively a series of approximations into the aforementioned discrete model, we formulated models given by independent equations (4.3), (4.5), (4.7), (4.8) which describe the dynamic problems on different levels of accuracy. The simplest from these models is represented by equation (4.8) which can be also obtained by the asymptotic homogenization of a periodic medium.

Among new results obtained in this contribution, the following ones seem to be most important.

- The proposed discrete model makes it possible to obtain independent systems of equations for displacement fluctuations $\boldsymbol{v}^q(\boldsymbol{z}, t)$, $q = 1, ..., n$, in every cell $\Delta(\boldsymbol{z})$, $\boldsymbol{z} \in \Lambda_0$.
- The proposed continuum models are governed by partial differential equations only for the mean displacement field $\boldsymbol{u}(\cdot)$. The displacement fluctuation fields $\boldsymbol{v}^q(\cdot)$ are governed by ordinary differential equations involving only time derivatives of $\boldsymbol{v}^q(\cdot)$. It follows that in stationary problems the fields $\boldsymbol{v}^q(\cdot)$ are governed by linear algebraic equations and can be eliminated from the governing equations.

- The displacement fluctuations $\boldsymbol{v}^q$ are governed by a system of linear algebraic equations also in dynamic problems provided that we apply the asymptotic approximation both to discrete and continuum model equations.
- Apart from the asymptotic first order continuum model, all proposed models take into account the effect of the microstructure size on the dynamic behaviour of a composite solid, which plays an important role in the dispersive analysis of dynamic problems.
- From a formal point of view, the second order continuum model, (4.3), corresponds to that obtained in the framework of the tolerance averaging method, Woźniak and Wierzbicki (2000).
- Comparing formulae (5.6) related to the discrete model and formulae (5.8), (5.9) obtained in the framework of the second and first order continuum models, it can be seen that the first terms coincide for lower and higher frequencies, respectively.
- Differences between the values of free vibration frequencies calculated within the first and second order continuum models are negligible.

It has to mentioned that in most engineering problems the number n of displacement fluctuations can be large, and solution to these problems requires applications of computational methods.

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Modele ciągłe zagadnień dynamiki kompozytów z periodyczną mikrostrukturą

Streszczenie

W pracy zaproponowano nowe podejście do modelowania zagadnień dynamiki w liniowo-sprężystych mikroperiodycznych ośrodkach ciągłych, które za punkt wyjścia przyjmuje periodyczny podział symplicjalny elementu reprezentatywnego struktury. Podejście to umożliwia wyprowadzenie pewnej hierarchii modeli ciągłych, które opisują zagadnienia dynamiki z różną dokładnością. Otrzymane wyniki zastosowano do analizy propagacji fali w nieograniczonym ośrodku o budowie mikroperiodycznej.

Manuscript received February 4, 2003; accpeted for print April 15, 2003