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# STABILITY OF INELASTIC BILAYERED CONICAL SHELLS

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The paper deals with the derivation of the basic stability equations of bilayered elastic-plastic conical shells and a approximate solution to these equations, both theoretically and by numerical procedures. The subject of the analysis is a bilayered open conical shell under a combined load comprising longitudinal forces and external pressure. Kirchhoff-Love's hypotheses hold for the layers, and use is made of constitutive relations in the form of generalized Hooke's law for the elastic stability analysis, and the Prandtl-Reuss incremental plasticity theory for the stability analysis in the elastic-plastic range. The stability equations are derived using the virtual work principle, and Ritz's method is applied to solve the equations. An iterative computer algorithm has been developed which made it possible to analyse the shells in the elastic, elastic-plastic or totally plastic prebuckling state of stress. The numerical results are presented in diagrams.

*Key words:* elasto-plasticity, stability, shell theory, geometrical nonlinearity, critical load, stress-strain relations

## 1. Introduction

Multilayered thin-walled structures have many advantages as they resist comparatively large external loads, etc. Bilayered structures are constructed of two layers, firmly fixed together, made of different materials. They are commonly used in chemical apparatus and equipment. In general, the two layers fulfil different functions in the structure. The external layer (face) resists the main load acting on the shell, and the internal layer is an anti-corrosion lining in many cases (Zielnica, 2001a).

Stability analysis of modern layered shell structures requires the application of an elastic-plastic material model, which enables more realistic evaluation of the structure to resist the given set of external loads. A specific practical meaning have such problems that introduce both sources of nonlinearities, i.e. physical and geometrical. Their mutual interaction causes development of effects which are very important in proper description of the deformation process, and in the structural safety which is closer to reality (Zielnica, 2001b).

The subject of this paper is to derive the basic equations of stability of bilayered open elastic-plastic conical shells and an approximate solution to these equations, both theoretically and by numerical procedures. The subject of the analysis is a bilayered open conical shell under a complex load in the form of longitudinal forces and external pressure. Each layer can be made of a material with different physical properties and different stress hardening parameters (Ramsey, 1977). The shell consists of two thin faces with thickness $h'$ and $h''$, made of an isotropic and compressible material with some stress hardening properties.

It was assumed that the contact surface between the two layers is the reference surface of the shell. An orthogonal coordinate system $s$, $\varphi$ was set on this reference surface, where the coordinates $s$, $\varphi$ are situated along the main curvatures of the shell. The axis $z$ is perpendicular to the reference surface, and it is directed positively toward the center of the curvature. Kirchhoff-Love's hypotheses hold for the layers. It is assumed that radial displacements of both layers do not depend on the variable $z$, and variation of the displacements across the shell thickness is a linear one. With these assumptions it is also admitted that we have a plane stress state in the layers. We will consider only mechanical effects in the elastic-plastic isotropic shell. Because the basic equations of the problem are very complicated and there are great difficulties of mathematical nature, we use constitutive relations in the form of generalized Hooke's law for the elastic stability analysis, and the Prandtl-Reuss incremental plasticity theory (Maciejewski and Zielnica, 1984) with Shanley concept (Bushnell, 1982; Girgoluk, 1957) will be used for the elastic-plastic stability analysis.

The set of stability equations expressed in displacements does not have an exact solution. Thus, we use a strain energy approach is this work. If the use is made of the virtual work principle we get an equation that describes the deformed shell under compressive load. Once the approximate functions are found for the basic components of the displacement vector, and Ritz's method is used to solve the problem, we get a set of three nonlinear algebraic equations. These equations form a condition which determines the critical set of external loadings. Obtaining a solution to the problem is only possible by using a special iterative algorithm enabling numerical calculations with the use of a computer (Zielnica, 1999).

## 2. Basic assumptions

The following basic assumptions are accepted within this work:

- We consider an open bilayered conical shell supported freely at the edges, and loaded by longitudinal forces and external pressure (see Fig. 1)
- Kirchhoff-Love's hypotheses hold for particular layers of the shell
- The shell reference surface is the contact surface between the layers
- Radial displacements of particular layers do not depend on the variable $z$ (normal to the shell reference surface), and the change in displacements in the layers along the thickness is linear
- Both layers are made of an isotropic material
- We accept an arbitrary, linear or nonlinear stress-strain relation, which can be different for both layers
- Constitutive relations of the Prandtl-Reuss theory are used within the theory of small displacements
- We follow Shanley's concept of active loading in the shell.

The internal membrane forces in the shell are as follows

$$\begin{aligned} N_S &= h\sigma_S = \frac{1}{2}qs\tan\alpha\Big[\Big(\frac{s_1}{s}\Big)^2 - 1\Big] - N_a\frac{s_1}{s} \qquad && h = h' + h'' \\ N_\varphi &= h\sigma_\varphi = -qs\tan\alpha && T_S = T_\varphi = 0 \end{aligned} \tag{2.1}$$

We introduce a parameter $\kappa$ which is equal to the ratio of the longitudinal-to-transversal load

$$\kappa = \frac{N_a}{qs_1} \tag{2.2}$$

We assume the following basic nonlinear geometrical relations:
— strains

$$\begin{aligned} \delta\epsilon_1 &= \frac{\partial u}{\partial s} + \frac{1}{2}\Big(\frac{\partial w}{\partial s}\Big)^2 \\ \delta\epsilon_2 &= \frac{1}{s\sin\alpha}\frac{\partial v}{\partial\varphi} - \frac{w}{s\tan\alpha} + \frac{1}{2s^2\sin^2\alpha}\Big(\frac{\partial w}{\partial\varphi}\Big)^2 + A_0\frac{u}{s} \\ \delta\gamma_{12} &= 2\delta\epsilon_{12} = \frac{\partial u}{\partial\varphi}\frac{1}{s\sin\alpha} + \frac{\partial v}{\partial s} + \frac{1}{s\sin\alpha}\frac{\partial w}{\partial s}\frac{\partial w}{\partial\varphi} - A_0\frac{v}{s} \end{aligned} \tag{2.3}$$

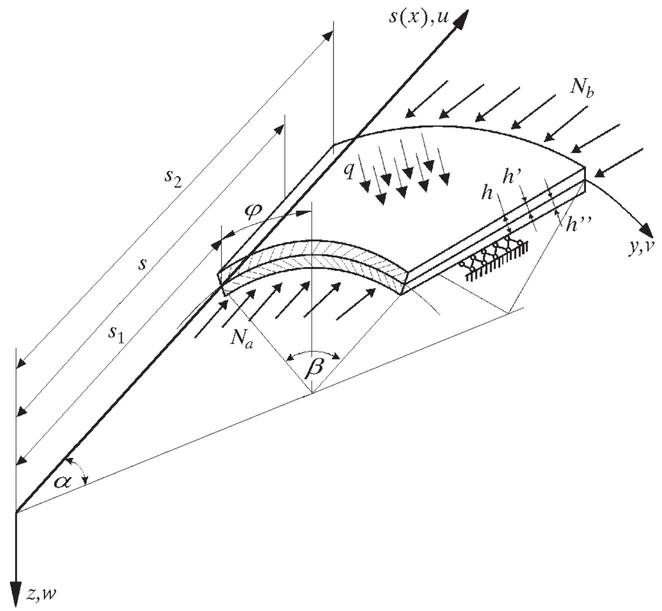


Fig. 1. Subject of consideration – bilayered open conical shell

— variations in curvature

$$
\begin{aligned}
\delta\kappa_1 &= \frac{\partial^2 w}{\partial s^2} \\
\delta\kappa_2 &= \frac{1}{s}\frac{\partial w}{\partial s} + \frac{1}{s^2\sin^2\alpha}\frac{\partial^2 w}{\partial \varphi^2} + \frac{\cos\alpha}{s^2\sin^2\alpha}\frac{\partial v}{\partial \varphi} \\
\delta\kappa_{12} &= \frac{1}{s\sin\alpha}\frac{\partial^2 w}{\partial s\partial\varphi} - \frac{1}{s^2\sin\alpha}\frac{\partial w}{\partial\varphi} + \frac{1}{2s\tan\alpha}\frac{\partial v}{\partial s} - \\
&\quad -\frac{v}{s^2\tan\alpha} - B_0\frac{1}{4s\tan\alpha}\Big(\frac{v}{s} + \frac{\partial v}{\partial s} - \frac{1}{s\sin\alpha}\frac{\partial u}{\partial\varphi}\Big)
\end{aligned}
\tag{2.4}
$$

where the coefficients $A_0$ and $B_0$ take the values 0 or 1, respectively, in order to control the influence of particular terms on the results.

The basic constitutive relations according to the Prandtl-Reuss plastic flow theory are (Zielnica, 2001b)

$$
D_{\dot\epsilon} = \lambda D_{\dot\sigma} + \frac{1}{2G}D_{\dot\sigma} \qquad\qquad \dot e_{ij} = \lambda s_{ij} + \frac{1}{2G}\dot s_{ij} \tag{2.5}
$$

where $D_{\dot\epsilon}$ and $D_{\dot\sigma}$ are the stress and strain ratio deviators, and $\lambda$ is the stress hardening parameter, which can be determined from the yield condition.

An incremental form of relations (2.5) is as follows

$$
\begin{aligned}
d\epsilon_{ij} &= \frac{1}{2G}\Big(d\sigma_{ij} - \delta_{ij}\frac{3\nu}{1+\nu}d\sigma_m\Big) + 3d\lambda\Big(\sigma_{ij} - \delta_{ij}\sigma m\Big) \\
\sigma_m &= \frac{1}{3}\sigma_{kk} \qquad\qquad d\lambda = \frac{1}{2}\frac{d\overline{\epsilon}_i^p}{\sigma_i}
\end{aligned}
\tag{2.6}
$$

We determine internal forces and moments (see Fig. 2) from the relations

$$
\begin{aligned}
\delta N_{\alpha\beta} &= \int\limits_{-h''}^{0} \delta\sigma''_{\alpha\beta}\; dx_3 + \int\limits_{0}^{h'} \delta\sigma'_{\alpha\beta}\; dx_3 \\
\delta M_{\alpha\beta} &= \int\limits_{-h''}^{0} \delta\sigma''_s x_3\; dx_3 + \int\limits_{0}^{h'} \delta\sigma'_s x_3\; dx_3
\end{aligned}
\tag{2.7}
$$

where $\delta\sigma_{\alpha\beta}$ are the variations of stress tensor components, which are determined by the reciprocal to relations (2.5). Using the respective relations and integrating, we get the constitutive relations for the internal forces and moments for a bilayered conical shell

$$
\begin{aligned}
\delta N_1 &= C_{11}\delta\epsilon_1 + C_{12}\delta\epsilon_2 - C_{13}\delta\gamma_{12} - C_{14}\delta\kappa_1 - C_{15}\delta\kappa_2 + C_{16}\delta\kappa_{12} \\
\delta N_2 &= C_{21}\delta\epsilon_1 + C_{22}\delta\epsilon_2 - C_{23}\delta\gamma_{12} - C_{24}\delta\kappa_1 - C_{25}\delta\kappa_2 + C_{26}\delta\kappa_{12} \\
\delta T &= -C_{31}\delta\epsilon_1 - C_{32}\delta\epsilon_2 + C_{33}\delta\gamma_{12} + C_{34}\delta\kappa_1 + C_{35}\delta\kappa_2 - C_{36}\delta\kappa_{12} \\
\delta M_1 &= C_{41}\delta\epsilon_1 + C_{42}\delta\epsilon_2 - C_{43}\delta\gamma_{12} - C_{44}\delta\kappa_1 - C_{45}\delta\kappa_2 + C_{46}\delta\kappa_{12} \\
\delta M_2 &= C_{51}\delta\epsilon_1 + C_{52}\delta\epsilon_2 - C_{53}\delta\gamma_{12} - C_{54}\delta\kappa_1 - C_{55}\delta\kappa_2 + C_{56}\delta\kappa_{12} \\
\delta H &= C_{61}\delta\epsilon_1 + C_{62}\delta\epsilon_2 - C_{63}\delta\gamma_{12} - C_{64}\delta\kappa_1 - C_{65}\delta\kappa_2 + C_{66}\delta\kappa_{12}
\end{aligned}
\tag{2.8}
$$

The coefficients in the above relations can be expressed as follows

$$
C_{ij} = \frac{m}{n}\Big(B'_{ij}h^{k\prime} - B''_{ij}h^{l\prime\prime}\Big)
\tag{2.9}
$$

where $i,j,k,l,m,n = 0,1,...,6$.

The coefficients of the local stiffness matrix $B_{ij}$ are given in Zielnica (2001b).

## 3. Stress-strain relations

We assume two basic stress-strain relations of the shell material behaviour:

(a) linear stress hardening
(b) nonlinear stress hardening.

The diagrams $sigma_i$ - $\epsilon_i$ for these two models are as in Fig. 2 and Fig. 3.

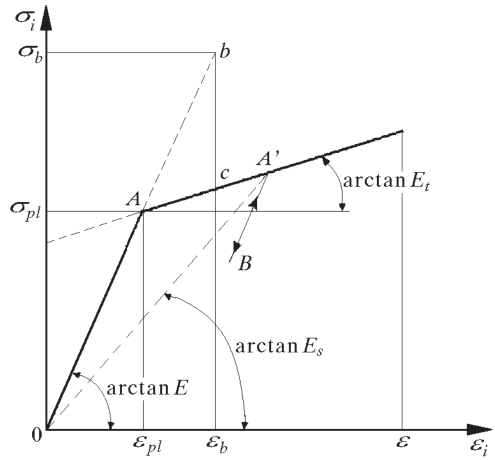


Fig. 2. Bilinear stress-strain relation

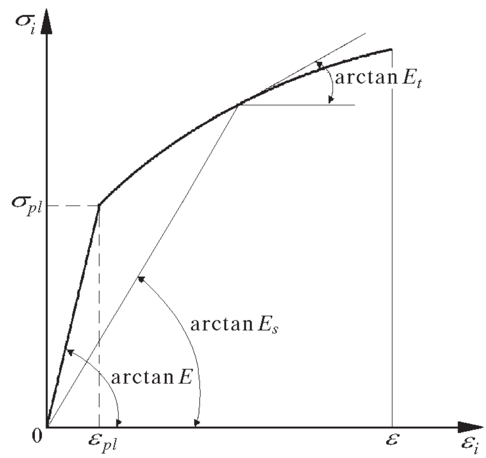


Fig. 3. Nonlinear stress-strain relation

## 4. Mathematical model

The stability equations expressed in the displacements do not have an exact solution. Any approximate solution, found e.g. by Galerkin's method is complicated because complex displacement functions have to be taken, and approximate calculations are time consuming. In order to avoid the difficulties the Ritz method is applied here.

The principle of stationary potential energy states that the necessary condition for the equilibrium of any given state is that the variation of the total potential energy of the considered system is equal to zero. Thus, we have the following relation

$$\delta U_p = \delta(W + L_z) = 0 \tag{4.1}$$

where $W$ is the change in the strain energy stored in the shell, and $L_z$ is the potential energy of the external load. Equation (4.1) is correct both for the pre- and postcritical deformation state.

The term in (4.1) related with the strain energy is equal to the sum of the strain energy of particular layers

$$W = W' + W'' \tag{4.2}$$

where $W'$ is the strain energy accumulated in the external layer, $W''$ is the strain energy accumulated in the internal layer of the bilayered conical shell

$$L = -q \sin\alpha \iint\limits_A ws\; ds\, d\varphi - \frac{1}{2} N_a s_1 \sin\alpha \iint\limits_A \Big(\frac{\partial w}{\partial s}\Big)^2\; ds\, d\varphi \tag{4.3}$$

The strain energy of the internal forces and moments developed by stability loss for the shell element $h\,ds\,r\,d\varphi$ is as follows

$$\delta\overline{W} = \delta N_1 \epsilon_1 + \delta N_2 \epsilon_2 + \delta T \gamma_{12} + \delta M_1 \kappa_1 + \delta M_2 \kappa_2 + \delta H \kappa_{12} \tag{4.4}$$

When constitutive relations (2.8) are introduced into the above equation, we can represent the right hand side of this equation in the form of the total variation

$$\begin{aligned}\delta\overline{W} = \delta\Big[\frac{1}{2}(&C_{11}\epsilon_1^2 + 2C_{12}\epsilon_1\epsilon_2 + C_{22}\epsilon_2^2 + C_{33}\gamma_{12}^2 - \\ &-C_{44}\kappa_1^2 - 2C_{45}\kappa_1\kappa_2 - C_{55}\kappa_2^2 + 2C_{66}\delta\kappa_{12}^2)\Big]\end{aligned} \tag{4.5}$$

The term $L_z$ is the potential energy of the external loads, and it is given by

$$L_z = -q\sin\alpha \int\limits_{s_1}^{s_2}\int\limits_{0}^{\beta} ws\;ds\,d\varphi - \frac{1}{2}N_a s_1 \sin\alpha \int\limits_{s_1}^{s_2}\int\limits_{0}^{\beta} \Big(\frac{\partial w}{\partial s}\Big)^2\;ds\,d\varphi \tag{4.6}$$

If we use Eqs. (2.8) and (2.9) and integrate over the whole surface of the considered shell, we get the following expression for the strain energy

$$\begin{aligned} U_p = W + L_z = \frac{1}{2}\int\limits_{s_1}^{s_2}\int\limits_{0}^{\beta} & s\sin\alpha(C_{11}\epsilon_1^2 + 2C_{12}\epsilon_1\epsilon_2 + C_{22}\epsilon_2^2 + \\ & +C_{33}\gamma_{12}^2 - C_{44}\kappa_1^2 - 2C_{45}\kappa_1\kappa_2 - C_{55}\kappa_2^2 + 2C_{66}\delta\kappa_{12}^2)\;ds\,d\varphi - \\ & -q\sin\alpha \int\limits_{s_1}^{s_2}\int\limits_{0}^{\beta} ws\;ds\,d\varphi - \frac{1}{2}N_a s_1 \sin\alpha \int\limits_{s_1}^{s_2}\int\limits_{0}^{\beta} \Big(\frac{\partial w}{\partial s}\Big)^2\;ds\,d\varphi \end{aligned} \tag{4.7}$$

The base functions are assumed in the form

$$\begin{aligned} w(s,\varphi) &= A_1 r^2(s)\sin(k\psi)\sin(t\varphi) \\ u(s,\varphi) &= A_2 r^2(s)\cos(k\psi)\sin(t\varphi) \\ v(s,\varphi) &= A_3 r^2(s)\sin(k\psi)\cos(t\varphi) \end{aligned} \tag{4.8}$$

where $A_1$, $A_2$, $A_3$ are free parameters to be determined, and

$$\begin{array}{lll} k = \dfrac{m\pi}{l} & t = \dfrac{n\pi}{\beta} & \psi = s - s_1 \\ l = s_2 - s_1 & r(s) = s\sin\alpha & \end{array}$$

Here $m$ and $n$ are parameters. Approximate functions (4.8) satisfy the kinematic boundary conditions of the simply supported shell edges

$$\begin{array}{ll} w\Big|_{\substack{s=s_0\\ s=s_1}} = 0 & w\Big|_{\substack{\varphi=0\\ \varphi=\beta}} = 0 \\ u\Big|_{\substack{\varphi=0\\ \varphi=\beta}} = 0 & v\Big|_{\substack{s=s_0\\ s=s_1}} = 0 \end{array} \tag{4.9}$$

Once the expression for the total potential energy of the shell is determined, and the displacements $u$, $v$, $w$ are expressed by displacement functions (4.8), following (4.1) we get

$$\delta U_p = \sum_{i=1}^{k}\frac{\delta U_p}{\partial A_i}\delta A_i = 0 \qquad\qquad i = 1,2,...,k \tag{4.10}$$

Keeping in mind that the variation of the parameters $\delta A_i$ in (4.10) is arbitrary, the following condition has to be satisfied

$$\frac{\partial U_p}{\partial A_i} = 0 \tag{4.11}$$

It is relatively easy to integrate analytically the functions in (4.11) over the variable $\varphi$. The integrals are as follows

$$\begin{array}{ll} I_1 = \int\limits_0^\beta \sin^2(t\varphi)\; d\varphi & I_2 = \int\limits_0^\varphi \sin(t\varphi)\; d\varphi \\ I_3 = \int\limits_0^\beta \cos(t\varphi)\; d\varphi \quad ... & I_6 = \int\limits_0^\beta \cos^4(t\varphi)\; d\varphi \quad ... \\ I_9 = \int\limits_0^\beta \sin^2(t\varphi)\cos(t\varphi)\; d\varphi \quad ... & I_{12} = \int\limits_0^\beta \sin^4(t\varphi)\; d\varphi \end{array} \tag{4.12}$$

It is not possible to procede with analytical integration over the variable $s$ because this variable exists among all the local stiffnes matrix elements $C_{11}$, $C_{12}$, $C_{21}$, $C_{22}$, $C_{33}$. Thus, numerical integration will be realised. To simplify the apropriate relations, the following notation will be used

$$\begin{array}{ll} D_1 = s\sin^2 k\psi & D_2 = s^2\sin^2(k\psi) \\ D_3 = \sin^2(k\psi) & D_4 = s^3\sin^2(k\psi) \quad .... \\ D_{11} = s^2\sin(k\psi)\cos(k\psi) \quad ... & D_{31} = s^4\sin(k\psi)\cos^2(k\psi) \\ D_{32} = \sin(k\psi) \quad ... & D_{37} = s^9\cos^4(k\psi) \quad ... \\ D_{41} = s^7\sin(k\psi)\cos^2(k\psi) & \end{array} \tag{4.13}$$

Carrying out the prescribed transformations according to (4.11), we get a nonlinear and nonhomogeneous set of algebraical equations with respect to the free parameters $A_i$. The equations represent the actual critical set of the external loadings acting on the shell being in an elastic-plastic state of stress

$$\begin{aligned} &(a_{11} + N_a\widetilde{a}_{11})A_1 + a_{12}A_2 + a_{13}A_3 = \\ &= b_{11}A_1^2 + b_{12}A_1^3 + b_{13}A_1A_2 + b_{14}A_1A_3 + qb_{15} \\ &a_{21}A_1 + a_{22}A_2 + a_{23}A_3 = b_{21}A_1^2 \\ &a_{31}A_1 + a_{32}A_2 + a_{33}A_3 = b_{13}A_1^2 \end{aligned} \tag{4.14}$$

The coefficients $a_{ij}$, $b_{ij}$ have a very complex form, and the following expressions are implemented in these coefficients

$$\int\limits_{s_1}^{s_2} C_{ij}(s)D_n(s)\;ds \tag{4.15}$$

Here, the quantities $C_{ij}$ are the elements of the local stiffness matrix, and $D_n$ are given by (2.9).

Further simplifications of the formulas required in the numerical calculations introduce a function that is very valuable in the numerical procedure

$$V_{m,n}(s) = \int\limits_{s_1}^{s_2} F_m(s)D_n(s)\;ds \qquad\qquad \begin{array}{l} m = 1,...,9 \\ n = 1,...,41 \end{array} \tag{4.16}$$

Now, be making use of the above equations, the coefficients in the stability equation of the considered bilayered shell are as follows

$$\begin{aligned}
a_{11} &= g^3h^2I_1V_{2,4} - \\
&\quad -g^5I_1(4V_{4,1} + 16k^2V_{4,23} + k^4V_{4,8} + 16kV_{4,11} - 4k^2V_{4,4} - 8k^3V_{4,13}) - \\
&\quad -g^5I_1(8V_{8,1} + 20kV_{8,11} - 4k^2V_{8,4} + 8k^2V_{8,23} - 2k^3V_{8,13}) - \\
&\quad -t^2g^3I_1(-4V_{8,1} - 8kV_{8,11} + 2k^2V_{8,4}) - g^5I_1(4V_{5,1} + 4kV_{5,11} + k^2V_{5,23}) - \\
&\quad -t^4gI_1V_{5,1} + t^2g^3I_4(8V_{6,1} + 8kV_{6,11} + 2k^2V_{6,23}) + 2t^2g^3I_4V_{6,1} - \\
&\quad -t^2g^3I_4(8V_{6,1} + 4kV_{6,11}) - t^2g^3I_1(4V_{5,1} + 2kV_{5,11}) \quad .... \\
\widetilde{a}_{11} &= -N_as_1g^5I_1(4V_{9,2} + 4kV_{9,12} + k^2V_{9,24}) \quad ... \\
b_{11} &= g^6hI_{11}(6V_{7,7} + 6kV_{7,16} + 1.5k^2V_{7,17}) - 1.5t^2g^4hI_7V_{2,7} \quad ... \\
b_{15} &= qg^3I_2V_{9,34}
\end{aligned} \tag{4.17}$$

where $g = \sin\alpha$, $h = \cos\alpha$.

It is worth to notice that only two of these coefficients $(\widetilde{a}_{11}, b_{15})$ comprise the external loads $N_a$ and $q$. The other coefficients depend on the geometrical and physical parameters, and on the parameters $m$ and $n$ in deflection functions (4.8).

The set of equations (4.13) resolved with respect to the parameter $A_1$ (i.e. the parameter of the deflection function $w$) takes the final form

$$q = \frac{\widetilde{e}_1A_1 + \widetilde{e}_2A_1^2 + \widetilde{e}_3A_1^3}{\widetilde{e}_4\kappa A_1 + \widetilde{e}_5} \tag{4.18}$$

with

$$N_a = \kappa q s_1 \tag{4.19}$$

where

$$
\begin{aligned}
\widetilde{e}_1 &= a_{11} + a_{12}\frac{a_{23}a_{31} - a_{21}a_{33}}{a_{22}a_{33} - a_{23}a_{32}} - a_{13}\Big(\frac{a_{31}}{a_{33}} + \frac{a_{32}}{a_{33}}\frac{a_{23}a_{31} - a_{21}a_{33}}{a_{22}a_{33} - a_{23}a_{32}}\Big)\\
&\vdots\\
\widetilde{e}_4 &= \frac{\widetilde{a}_{11}}{q\kappa} = -\frac{\widetilde{a}_{11}s_1}{N_a} = -s_1^2 \sin\alpha I_1(4V_{9,2} + 4kV_{9,12} + k^2V_{9,24})\\
\widetilde{e}_5 &= \frac{b_{15}}{q} = \sin^3\alpha I_2V_{9,34}
\end{aligned}
\tag{4.20}
$$

## 5. Solution procedure

The coefficients $\widetilde{e}_i$ in the stability equations are variable, and they depend on unknown functions in which the external load is included. Thus, this equation is a transcendental one. Appropriate numerical calculations will be conducted iteratively with the help of special algorithm. A simplified block diagram of the algorithm is presented in Fig. 4. The procedure includes controlling the yield condition on every step of external load increment, which makes it possible to analyse the shells being in a different state of yielding or even fully elastic shells where the yield stress is not reached at the critical loads. The objective of the following numerical calculations is the analysis of postcritical equilibrium paths for an arbitrary combination of lateral-to-longitudinal loads, and also the study of the influence of geometrical and physical parameters on the critical load and form of stability loss.

The procedure required in the determination process of the upper $(q^+, N_a^+)$ and lower $(q^-, N_a^-)$ critical loads is as follow:

- We assume geometrical and material data for the considered shell
- We assume a definite value of the coefficient $\kappa$
- We adopt a series of values for the parameters $m$, and $n$ (number of half-waves in the longitudinal and circumferential directions)
- We assume, starting from zero, a sequence of increasing values $A_1$ for fixed $m$ and $n$
- We determine the maximum deflection for the respective $A_1$ from Eq. $(4.8)_1$, and the resulting loadings $q$ (Eq. (4.18)), and $N_a$ (Eq. (4.19))
- In the system of coordinates $(q, \widetilde{w})$ or $(N_a, \widetilde{w})$ we obtain a two-parameter family of the curves $q(\widetilde{w}, m, n)$ or $N_a(\widetilde{w}, m, n)$

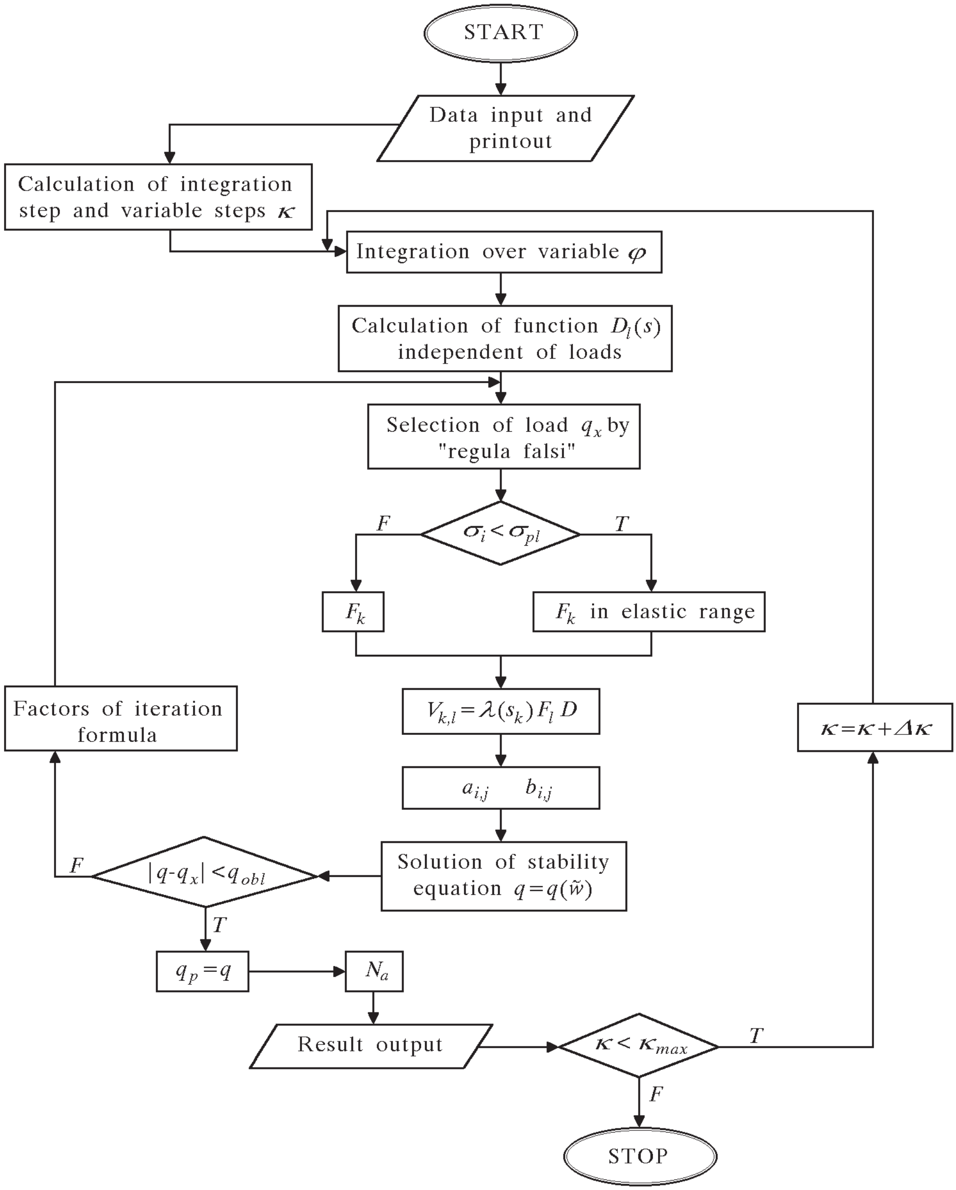


Fig. 4. Simplified block diagram of the algorithm for numerical calculation

- From the family of curves we chose the points of less values of $q$ or $N_a$ with specified values of the variable $\widetilde{w}$, and we obtain a curve which constitutes the solution.

The local maximum and minimum of the curve determine the "upper" $(q^+, N_a^+)$ and the lower $(q^-, N_a^-)$ critical loads, respectively.

As it has been already said, the equation of elastic-plastic stability is a transcendental function, where the local stiffnes matrix coefficients $B_{ij}$ and the integrated functions depend on the load $q$, thus the function $q = f(\widetilde{w})$ cannot be determined directly.

The analysis of the shell in a elastic-plastic or fully plastic state of stress starts from a control value of the initial load $q_{p(n)}$ based on the values $q_{p(n-1)}$, $q_{p(n-2)}$ from previous steps of the iteration process. This enables determination of the coefficients of the local stiffness matrix $B_{ij}$ and obtaining of the stability equation with the known coefficients $q = f(q_p)$, where the deflection exists as a parameter and $f(q_p)$ is the right-hand side of the stability equation.

Basing on the difference $|q - q_p|$ a new (corrected) value $q_p$ is taken in the next load step, and the calculations are repeated until the condition $|q - q_p| < \epsilon_p$ is satisfied, where $\epsilon_p$ is a parameter of the assumed calculation accuracy.

## 6. Basic data and numerical examples

The calculations are made for an open bilayered conical shell under longitudinal forces and lateral pressure. The basic geometrical and material data admitted in numerical calculations are as follows:

| | | |
|---|---|---|
| $L = 1.0\,\mathrm{m}$ | $r_s = 1.4\,\mathrm{m}$ | $w_u = 2$ |
| $tt = 0.001\,\mathrm{m}$ | $cc = 0.001\,\mathrm{m}$ | $\kappa = 1000$ |
| $E_{tt} = E_{cc} = 71\,900\,\mathrm{MPa}$ | $\alpha = 45°$ | $\beta = 35°$ |
| $E_c = E_t = 10\,000\,\mathrm{MPa}$ | $\nu = 0.3$ | $R_e = 70\,\mathrm{MPa}$ |
| $m = n = 1$ | $A_0 = 1$ | $B_0 = 1$ |

where

| | | |
|---|---|---|
| $L$ | – | shell generatrix length |
| $r_s$ | – | mean radius of the shell |
| $w_u$ | – | ratio of the maximum shell deflection to the total shell thickness |
| $tt$ | – | thickness of the external layer of the shell |

| | | |
|---|---|---|
| $cc$ | – | thickness of the internal layer of the shell |
| $\kappa$ | – | ratio of the longitudinal force $N_a$ to lateral pressure $q$ |
| $\alpha$ | – | alpha angle |
| $\beta$ | – | beta angle |
| $\nu$ | – | Poisson's coefficient of the shell layers |
| $E_c$ | – | modulus of linear stress hardening of the external shell layer material |
| $E_t$ | – | modulus of linear stress hardening of the internal shell layer material |
| $R_e$ | – | yield stress |
| $E_{tt}$ | – | Young's modulus of the external layer |
| $E_{cc}$ | – | Young's modulus of the internal layer |
| $m$ | – | half-wave number along the shell generatrix |
| $n$ | – | wave number in the circumferential direction |
| $A_0, B_0$ | – | feature constants in geometrical relations. |

Numerical results in the form of the external force (longitudinal or lateral) versus shell deflection, representing nonlinear equilibrium paths, are shown in Figures 5-7.

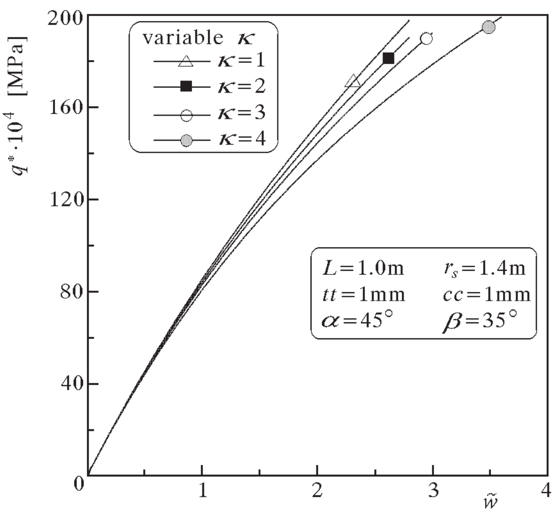


Fig. 5. Equilibrium paths; pressure $q$ versus deflection $\widetilde{w}$

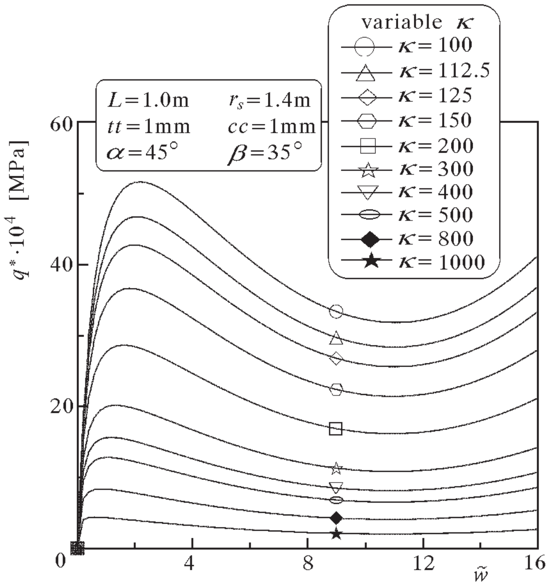


Fig. 6. Equilibrium paths; pressure $q$ versus deflection $\widetilde{w}$

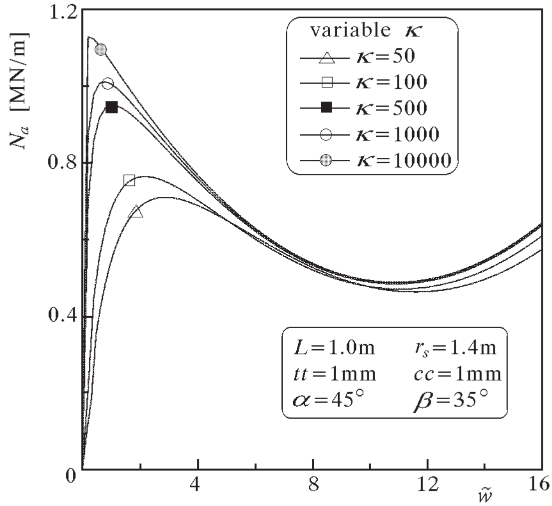


Fig. 7. Longitudinal force $N_a$ versus deflection $\widetilde{w}$

## 7. Conclusions

The nonlinear equilibrium paths of the analyzed shells are shown in Fig. 5 - Fig. 7. They show the results of analysis of the influence of the load ratio coefficient $\kappa$ on the critical loads and forms of postcritical equilibrium paths. The parameter $\kappa$ represents the ratio of the longitudinal force $N_a$ over the lateral pressure $q$. It can be seen that the coefficient $\kappa$ has a substantial influence on the form of postcritical equilibrium of the shell. For small values of the parameter $\kappa$, where the lateral pressure dominates, the curves $q(\widetilde{w})$ exhibit a change in the curvature sign. Moreover:

- Instability region appears at a certain value of the coefficient $\kappa \approx 10$ where $q^+ \approx q^-$. This region expands when the coefficient $\kappa$ increases, as it begins consecutively at smaller deflections.
- Critical loads $q^{+/-}$ decrease, and the difference between the upper $q^+$ and lower $q^-$ critical load increases nonlinearly up to the maximum value at $\kappa \approx 100$. Then, the difference decreases.
- Critical longitudinal forces $N_a^{+/-}$ increase when the $\kappa$ coefficient increases. When $\kappa$ increases in the region from 100 to $10^8$ the load $q^+$ decreases several times, whereas the longitudinal force $N_a^+$ increases inconsiderably from 0.8 up to 1.2 MPa.
- Above $\kappa = 10^5$ the equilibrium paths do not substantially differ from the limit curve that corresponds to the case when the longitudinal force itself acts on the shell ($\kappa = \infty$). Thus, we can conclude that if the shell is loaded almost exclusively by the longitudinal force, the change in the lateral pressure does not substantially influence the form of the equilibrium paths.

The above given results are quite general and can be used, among others to test the results obtained on the basis of the finite element method. Also, the results can be helpful for the optimal design of bilayered shell structures.

## References


1. Bushnell D., 1982, Plastic buckling of various shells, *Trans. ASME, Journal of Pressure Vessel Technology*, **105**, 5, 51-72
2. Grigoluk E.I., 1957, Equations of sandwich shells with soft-type core (in Russian), *Izw. AN SSSR*, 1

3. MACIEJEWSKI J., ZIELNICA J., 1984, Nonolinear stability problem of open elastic-plastic conical shell (in Polish), *Engineering Transactions*, **32**, 361-380

4. RAMSEY H., 1977, Plastic buckling of conical shells under axial compression, *Int. J. Mech. Sci.*, **19**, 252-272

5. ZIELNICA J., 1999, Large deformations and stability of bilayered elastic-plastic open conical shells, *Archives of Civil Engineering*, **45**, 3, 521-536

6. ZIELNICA J., 2001a, Stability of elastic-plastic sandwich structures (in Polish), in: *Stability and optimisation of layered structures*, Edited by Magnucki K., Ostwald M., Ed. 1. Poznań-Zielona Góra

7. ZIELNICA J., 2001b, *Stability of Elastic-Plastic Shells* (in Polish), PUT Edition, Poznań, 1-258

## Stateczność sprężysto-plastycznych powłok stożkowych

Streszczenie

Tematem pracy jest wyprowadzenie podstawowych równań stateczności dwuwarstwowych otwartych sprężysto-plastycznych powłok stożkowych i przybliżone rozwiązanie tych równań w sposób analityczny i numeryczny. Przedmiotem analizy jest swobodnie podparta dwuwarstwowa powłoka w kształcie wycinka stożka poddana działaniu obciążenia złożonego w postaci sił podłużnych i ciśnienia poprzecznego. Równania stateczności wyprowadzono na podstawie zasady minimum energii potencjalnej powłoki, a do rozwiązania zastosowano metodę Ritza. Wyniki numeryczne otrzymano za pomocą specjalnie zbudowanej iteracyjnej procedury numerycznej z wykorzystaniem komputera.

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