

GEOMETRICALLY NONLINEAR STATIC ANALYSIS OF SANDWICH PLATES AND SHELLS

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Sandwich shells composed of three layers: two thin and very strong faces, and the soft and comparatively weaker core which is much thicker than their neighbouring layers covering it from up and down, are considered in the paper. The proposal of a finite element adequate to such a kind of structural members is presented. In the paper the finite element based principally on the Ahmad original element and on the author's adaptation of this very element to the nonlinear range is presented. It was assumed that in the faces and in the core the materials exhibit orthotropic properties, and only elastic deformations are taken into account. The static analysis is performed within fully geometrically nonlinear range. The ultimate purpose of the work is determination of the critical value of the load intensity factor in a quasi static stability analysis. All procedures related to tracing of nonlinear equilibrium paths are adopted from the codes prepared before for homogeneous shells. Two verification problems are inserted. They confirm correctness of the adopted approach.

Key words: sandwich shells, orthotropy, nonlinear stability analysis, finite element method

1. Introduction

Sandwich shells and plates exhibit many valuable properties and this is the reason that structures made in that technology are so often used in engineering practice. The most important advantages of sandwich structures can be listed as follows: good thermal and acoustic isolation, good vibration damping, good strength to weight ratio, good local and global buckling resistance. These

physical and mechanical properties of sandwich structural elements have caused that they are applied not only as finishing elements but also as structural members (cf. Hop, 1980; Romanów, 1995).

The mechanical behaviour of this kind of structures is analysed in this work, and particular emphasis is put on static analysis with reference to the equilibrium stability phenomenon within the range of large displacements. The main purpose of the present work can be explained with the help of Fig. 1. The critical value of the load intensity factor for one parameter loading is searched. This particular value can be found in fully nonlinear, quasi static analysis in which nonlinear equilibrium paths are determined in the whole load displacement space. The critical points in the form of limit points or bifurcation points can appear on these paths, and the primary critical point defines the searched value of load parameter. The procedure of calculation of the nonlinear equilibrium paths must be comprehensive enough to determine all mentioned objects no matter how complicated the primary or secondary paths could be.

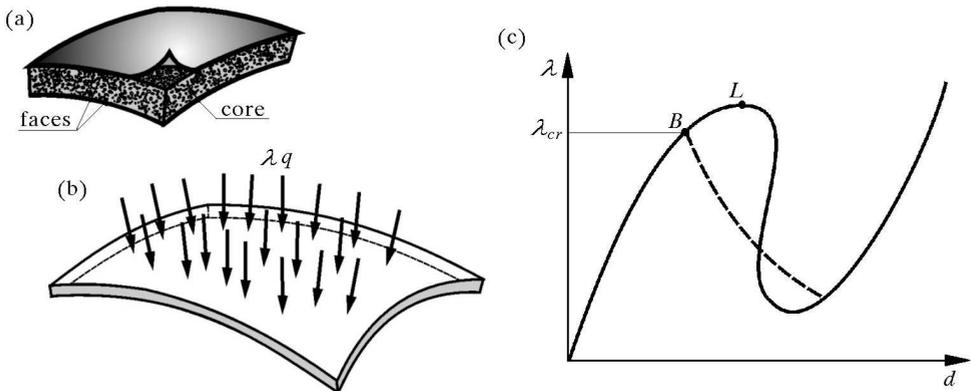


Fig. 1. On parameter loading and equilibrium path

The problem is important and this is the reason that so many authors have been trying to build better and better numerical tools to describe the mechanical behaviour of these kinds of structural members. In the literature there exist many proposals of numerical modelling of mechanics of sandwich shells by FEM. In this context, it is worthy to mention the work of Rammerstorfer *et al.* (1992). The proposed models are more and more comprehensive and take into account phenomena observed in experiments. Ferreira *et al.* (2000) present the approach in which the plasticity is taken into account and degenerated conception of finite element is used. Riks and Rankin (1995) proposed

a manner of description of face delamination. Works of Ding and Hou (1995) and Moita *et al.* (1999) refer to initial buckling of sandwich structures.

In the present work the conception of a degenerated finite element is used to model the mechanical behaviour of sandwich, three layered shells within the range of assumptions formulated in the next section.

2. Main assumptions

A three layered sandwich shell or plate is considered. The curvature of the shell can be arbitrary. The structure is composed of a thick and soft core covered from both sides by very thin and strong faces (comp. Fig. 1a). Materials of both are linearly elastic and exhibit orthotropy properties. The directions of orthotropy can be different in the core and in faces.

The main assumptions refer to the mode of core deformation. It has been assumed that the straight normal to the middle surface of the core remains straight but can rotate independently with respect to the deformed middle surface. It means that the severe Kirchhoff's assumption is dropped.

The faces can deform only within their planes and these membrane deformations are equal to their counterparts in the core fulfilling in this way the strain continuity conditions. Local deformations of the faces, independent of the mode of deformation of the core are excluded from considerations, which is the obvious drawback of the presented approach.

The modes of deformations taken into account are shown in Fig. 2a,b. The mode shown in Fig. 2c is an example of the mode excluded from considerations.

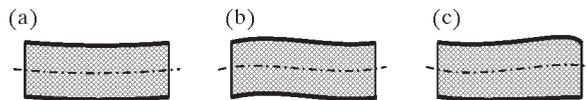


Fig. 2. Modes of deformation

Generally, arbitrary displacements are taken into account but as far as rotations are concerned they should be moderate in the presented version of the program. Strains are small, so linear stress-strain relations can be adopted. The strain energy corresponding to the stress normal to the middle surface is ignored. It means that the plane stress state is assumed within every layer parallel to the middle surface. The constant, plane, membrane stress state occurs within both faces.

The formulated assumptions correspond to the assumptions of the Mindlin-Reissner theory of shells extended to the case of sandwich shells.

3. Finite element

In the present approach the original conception of degeneration due to Ahmad *et al.* (1970) is adopted. The idea is based on such geometry and displacement approximation that the reduction of 3D solid to the 2D curved surface fulfils all assumptions of the Mindlin-Reissner shell theory.

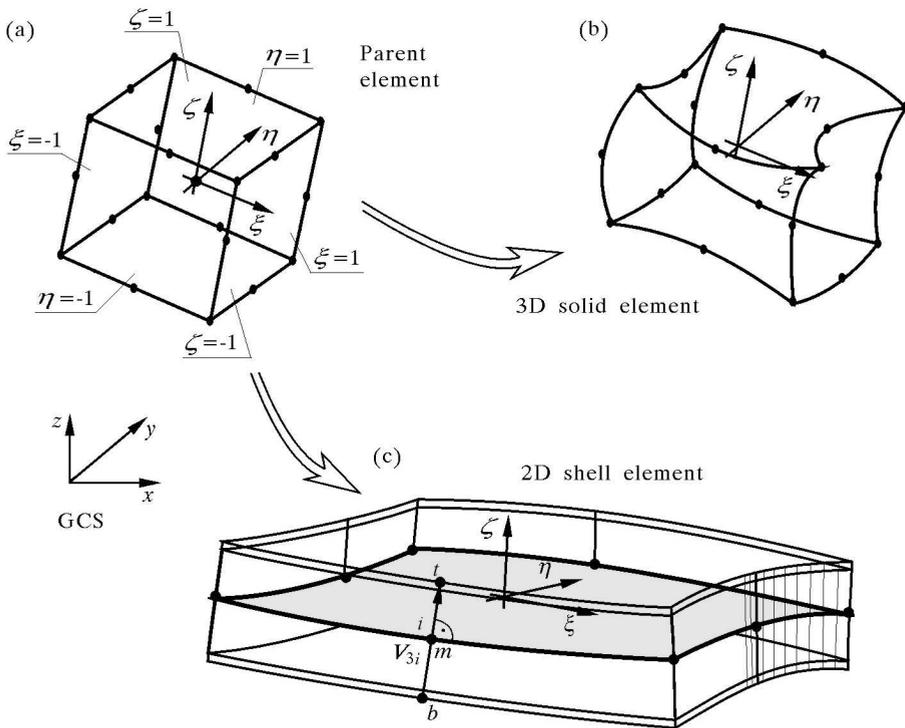


Fig. 3. Geometry approximation

The geometry approximation is defined as follows (see Fig. 3)

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_i N_i(\xi, \eta) \frac{1+\zeta}{2} \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_t + \sum_i N_i(\xi, \eta) \frac{1-\zeta}{2} \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_b \quad (3.1)$$

where: $\{x, y, z\}^\top$ are global coordinates of any point within the shell element volume, $\{x_i, y_i, z_i\}_{t(b)}^\top$ – global coordinates of the top (t) or bottom (b) node, ξ, η, ζ – curvilinear coordinates of any point within the shell element volume, $N_i(\xi, \eta)$ – the shape function corresponding to the node i , and summation holds over all nodes of the element.

Relation (3.1) can be rewritten in the following alternative form

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_i N_i(\xi, \eta) \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_m + \sum_i N_i(\xi, \eta) \begin{Bmatrix} (V_{3i})_x \\ (V_{3i})_y \\ (V_{3i})_z \end{Bmatrix} \quad (3.2)$$

where $\{x_i, y_i, z_i\}_m^\top$ are global coordinates of the node lying on the middle surface, $\{(V_{3i})_x, (V_{3i})_y, (V_{3i})_z\}$ – components of the nodal vector.

It is worthy to mention that the parent element for the defined one is the cube of side 2. It means that $\{\xi, \eta, \zeta\} \in \langle -1, 1 \rangle$. It is apparent from relations (3.1) and (3.2) that the defined approximation is linear with respect to the third coordinate ζ . The above definitions refer to the whole volume: the core together with two faces.

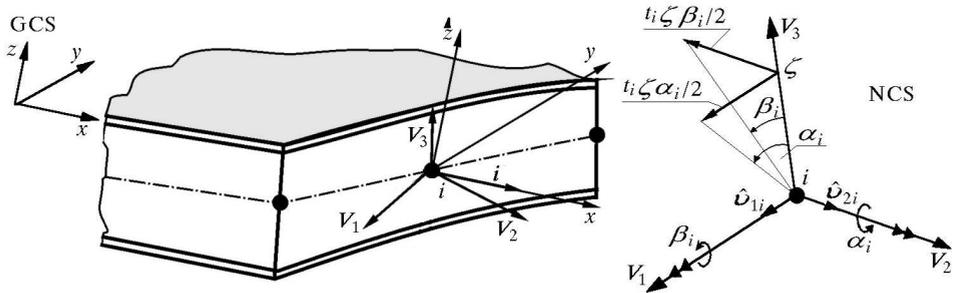


Fig. 4. Displacement approximation

The displacement approximation is more general, and is defined as follows

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_i N_i(\xi, \eta) \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix}_m + \sum_i N_i(\xi, \eta) \zeta \frac{t_i}{2} \begin{bmatrix} (\hat{v}_{1i})_x & -(\hat{v}_{2i})_x \\ (\hat{v}_{1i})_y & -(\hat{v}_{2i})_y \\ (\hat{v}_{1i})_z & -(\hat{v}_{2i})_z \end{bmatrix} \begin{Bmatrix} \alpha_i \\ \beta_i \end{Bmatrix} \quad (3.3)$$

where $\{u_i, v_i, w_i\}^\top$ are components of nodal displacements in the global coordinate system, $\{u, v, w\}^\top$ are components of displacements at a given point $\{\xi, \eta, \zeta\}$, $[\hat{v}_{1i}, -\hat{v}_{2i}]$ – matrix composed of unit vectors defining axes of the nodal coordinate system (cf. Fig. 4), $\{\alpha_i, \beta_i\}^\top$ – two independent rotations defined appropriately in the nodal coordinate system.

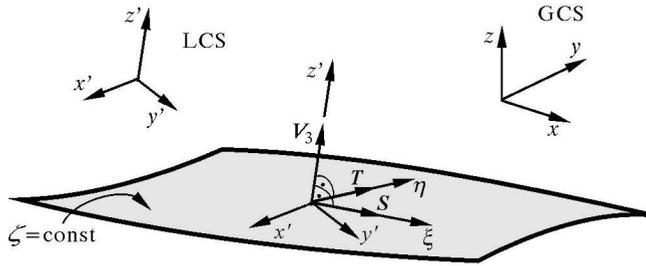


Fig. 5. Local coordinate system

In this definition formula, the linear approximation of displacements with respect to the third coordinate is visible. It is the guarantee that the plane section will remain plane after deformation. Two independent rotation parameters make possible free rotation of the straight normal to the middle surface, according to the adopted assumptions.

In this manner, the displacement field within the whole sandwich element is described by means of five nodal parameters: three translations and two independent rotations. It is exactly the same as it was done in original Ahmad's element.

To define strains, a local coordinate system (LCS) is introduced in such a way that the axes x' and y' occur in the plane parallel to the middle surface, and the axis z' is perpendicular to it. Strain-displacements relations ensue from the full Green-de Saint Venant strain tensor and take the following form

$$\left\{ \begin{matrix} \varepsilon_{x'} \\ \varepsilon_{y'} \\ \gamma_{x'y'} \\ \gamma_{x'z'} \\ \gamma_{y'z'} \end{matrix} \right\} = \left\{ \begin{matrix} \frac{\partial u'}{\partial x'} + \frac{1}{2} \left[\left(\frac{\partial u'}{\partial x'} \right)^2 + \left(\frac{\partial v'}{\partial x'} \right)^2 + \left(\frac{\partial w'}{\partial x'} \right)^2 \right] \\ \frac{\partial v'}{\partial y'} + \frac{1}{2} \left[\left(\frac{\partial u'}{\partial y'} \right)^2 + \left(\frac{\partial v'}{\partial y'} \right)^2 + \left(\frac{\partial w'}{\partial y'} \right)^2 \right] \\ \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial x'} \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'} \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial x'} \frac{\partial w'}{\partial y'} \\ \frac{\partial u'}{\partial z'} + \frac{\partial w'}{\partial x'} + \frac{\partial u'}{\partial x'} \frac{\partial u'}{\partial z'} + \frac{\partial v'}{\partial x'} \frac{\partial v'}{\partial z'} + \frac{\partial w'}{\partial x'} \frac{\partial w'}{\partial z'} \\ \frac{\partial v'}{\partial z'} + \frac{\partial w'}{\partial y'} + \frac{\partial u'}{\partial y'} \frac{\partial u'}{\partial z'} + \frac{\partial v'}{\partial y'} \frac{\partial v'}{\partial z'} + \frac{\partial w'}{\partial y'} \frac{\partial w'}{\partial z'} \end{matrix} \right\} \quad (3.4)$$

The component $\varepsilon_{z'}$ was omitted due to the assumption that the influence of $\sigma_{z'}$ on the strain energy is ignored.

Only first three components will appear within the faces of the sandwich element.

The constitutive relations are formulated in LCS as well, and due to the assumed orthotropy properties, take the form

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{E_x}{1 - \nu_{xy}\nu_{yx}} & \frac{E_x\nu_{xy}}{1 - \nu_{xy}\nu_{yx}} & 0 \\ \frac{E_y\nu_{yx}}{1 - \nu_{xy}\nu_{yx}} & \frac{E_y}{1 - \nu_{xy}\nu_{yx}} & 0 \\ 0 & 0 & G_{xy} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (3.5)$$

within the faces, and

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} \frac{E_x}{1 - \nu_{xy}\nu_{yx}} & \frac{E_x\nu_{xy}}{1 - \nu_{xy}\nu_{yx}} & 0 & 0 & 0 \\ \frac{E_y\nu_{yx}}{1 - \nu_{xy}\nu_{yx}} & \frac{E_y}{1 - \nu_{xy}\nu_{yx}} & 0 & 0 & 0 \\ 0 & 0 & G_{xy} & 0 & 0 \\ 0 & 0 & 0 & kG_{xz} & 0 \\ 0 & 0 & 0 & 0 & kG_{yz} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \quad (3.6)$$

inside the core.

In these relations the following denote: E_x, E_y – Young’s moduli in the x and y directions, ν_{xy} – ratio of the lateral strain $|\varepsilon_x|$ to the strain ε_y when loaded in the y direction, G_{xy}, G_{xz}, G_{yz} – shear moduli in the planes xy, xz and yz respectively; the condition $E_x\nu_{xy} = E_y\nu_{yx}$ must hold. The correction factor $k = 5/6$ was introduced here due to the fact that the uniform shear stress distribution follows from relation (3.3) while it is parabolic actually.

The governing equations of the problem are obtained from the principle of virtual work. If \mathbf{p} denotes the vector of external load this principle written for the whole sandwich shell takes the following form

$$\int_{V^C} (\boldsymbol{\sigma}^C)^\top \delta \boldsymbol{\varepsilon}^C dv + \int_{V^{TF}} (\boldsymbol{\sigma}^{TF})^\top \delta \boldsymbol{\varepsilon}^{TF} dv + \int_{V^{BF}} (\boldsymbol{\sigma}^{BF})^\top \delta \boldsymbol{\varepsilon}^{BF} dv - \int_A \mathbf{p}^\top \delta \mathbf{u} dA = 0 \quad (3.7)$$

where V^C, V^{TF}, V^{BF} are volumes of the core, of the top face and the bottom face respectively, A is the area where the external load is applied, $\delta \boldsymbol{\varepsilon}$ denotes strain variations due to virtual displacements, $\delta \mathbf{u}$ are virtual displacements.

Passing to finite elements, this principle takes the following form

$$\sum_{(e)} \left(\int_{V^{Ce}} \sigma^i \delta \varepsilon^i dv + \int_{A^{TF}} \sigma^i \delta \varepsilon^i dA + \int_{A^{BF}} \sigma^i \delta \varepsilon^i dA - \lambda F_i \delta d_i \right) = 0 \quad (3.8)$$

where: t_t, t_b are thicknesses of the top and bottom faces respectively, λ – load intensity factor, F_i – nodal forces due to the load on the reference level, V^{Ce} – volume of the core, A^{TF}, A^{BF} – area of the top and bottom face respectively, $\sum_{(e)}$ – denotes aggregation over all elements, d_i are nodal displacements.

Two terms corresponding to the strain energy generated in the faces appear in this relation. Because the strains are defined by the same nodal parameters, the above relation can be rewritten, after some further derivations, as follows

$$\sum_{(e)} [(\mathbf{K}^C + \mathbf{K}^{TF} + \mathbf{K}^{BF})\mathbf{d}^e - \lambda\mathbf{F}^e] = \mathbf{0} \quad (3.9)$$

where: $\mathbf{K}^C, \mathbf{K}^{TF}, \mathbf{K}^{BF}$ are stiffness matrices of the core, top face and bottom face, respectively, \mathbf{F}^e – vector of the nodal forces due to the load on the reference level.

All stiffness matrices are dependent on the nodal displacements.

The resulting set of nonlinear algebraic equations assumes the following form

$$\Psi(\mathbf{d}, \lambda) = \mathbf{K}_N \mathbf{d} - \lambda \mathbf{F} = \mathbf{0} \quad (3.10)$$

where \mathbf{K}_N is the global stiffness matrix dependent on \mathbf{d} , \mathbf{d} – global vector of the nodal displacements, \mathbf{F} – global vector of the nodal forces.

It is worthy to mention that \mathbf{K}^C is calculated using the reduced integration scheme $2 \times 2 \times 2$ Gauss points, while $\mathbf{K}^{TF}, \mathbf{K}^{BF}$ are calculated using 2×2 integration scheme (2D integrals).

To obtain a load-displacement curve, set (3.10) have to be solved for the whole range of the load intensity factor λ . The adopted procedures are exactly the same like these described in the work by Marcinowski (1999) and need not to be discussed here. These procedures are versatile enough to trace even the most complicated equilibrium paths with bifurcation points, limit points, secondary paths, etc.

4. Verification problems

To verify the correctness of the proposed approach and the computer program itself, a few problems were solved. The first one was taken from the library of verification problems of the COSMOS/M system (cf. [2]). It refers

to deflections of a square sandwich plate clamped along all edges and loaded by a uniform pressure. The geometry and material data are given in Fig. 6, on which the load intensity factor versus central deflection curve was shown. The presented solution was compared with that obtained by COSMOS/M and the solution taken from the paper of Schmit and Monforton (1970). Quite good correspondence can be observed. The core in this problem exhibits only shear rigidity, while the faces are rigid in the plane and in the lateral direction. As a matter of fact G_{xz} and G_{yz} of the faces could not be incorporated in the present model because only the membrane effect has been taken into account within the faces.

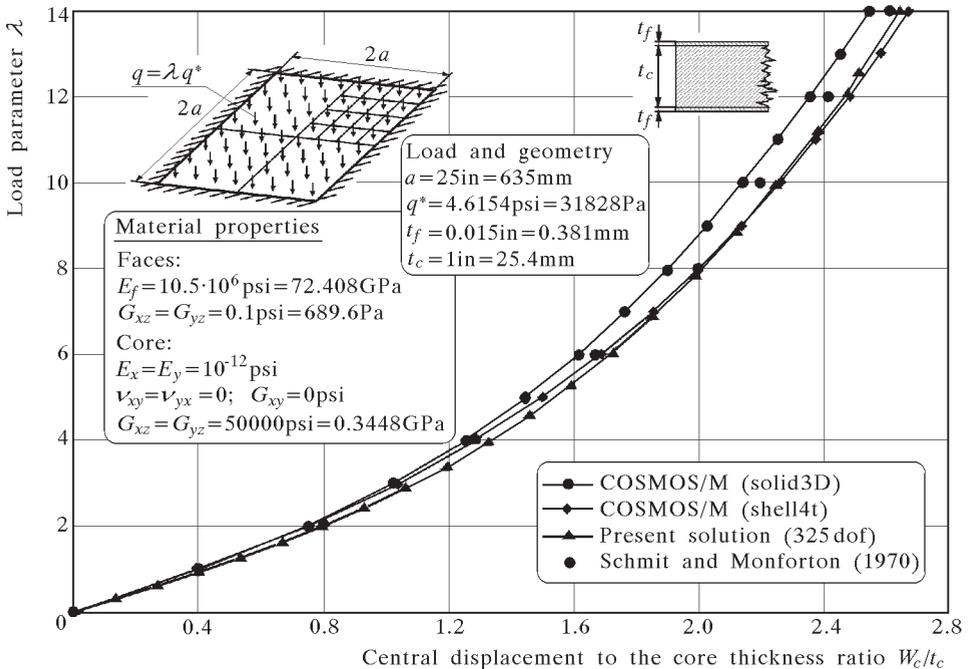


Fig. 6. Nonlinear equilibrium path for the square clamped plate

As a second example the well known benchmark of Sabir and Lock (1972) will be considered. It is a cylindrical shell simply supported along their rectilinear edges and loaded laterally by a concentrated force applied to its center. Details of boundary conditions, geometry and material data are shown in Fig. 7. In this figure FE mesh is also shown. This division is chosen after checking that the shell does not exhibit bifurcation phenomenon.

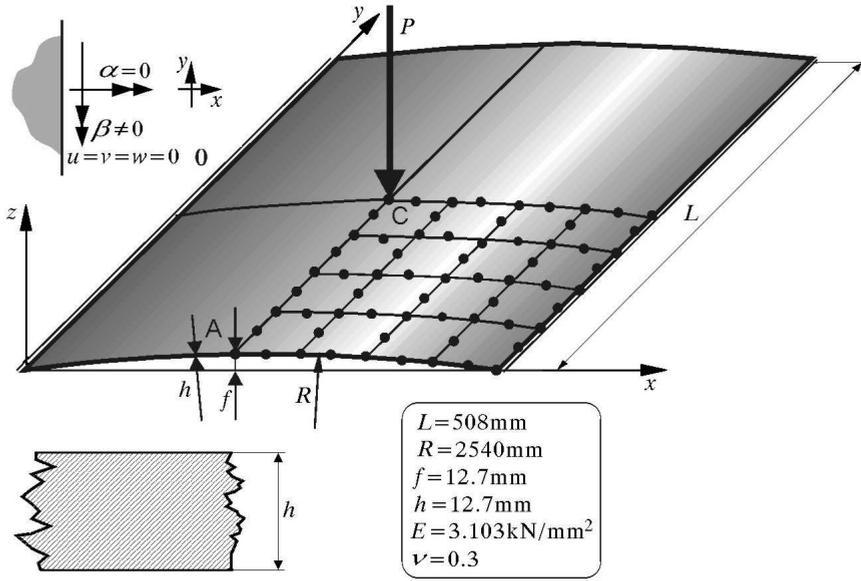


Fig. 7. Cylindrical shell loaded by a concentrated force

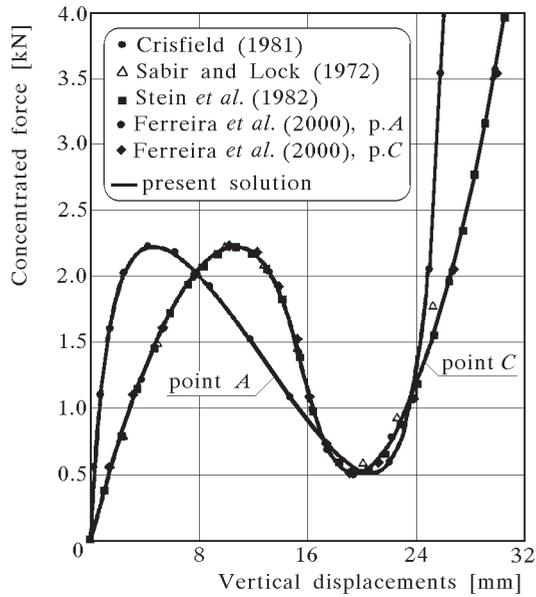


Fig. 8. Equilibrium paths for the homogeneous shell

The homogeneous case of such a shell was solved by many authors. A comparison of the present solution with those obtained by other authors is shown in Fig. 8. Very good correspondence can be observed.

Using the same geometry, a new problem is created following the idea of Ferreira *et al.* (2000). The thickness of the shell is now divided into three zones. Two thin faces ($t_f = 0.635$ mm) are separated from the homogeneous shell and, in this way, a sandwich shell is created. To analyse the influence of the core rigidity on the overall stiffness of the sandwich shell the Young modulus of the core (and the bulk modulus which is expressed by E , and $G = E/[2(1 + \nu)]$) was reduced 10, 100 and 1000 times, while the material parameters of the faces are kept constant. In this manner, three cases of a more and more weak core are considered. The results of analysis are presented in Fig. 9 (load versus vertical displacements of the point C) and in Fig. 10 (load versus vertical displacements of the point A). They are shown together with the solutions obtained by means of the system COSMOS/M for the same data.

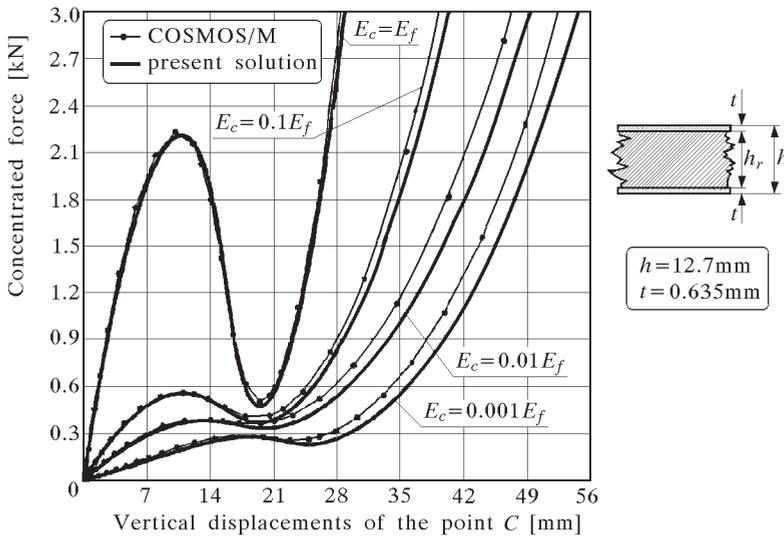


Fig. 9. Equilibrium path for the sandwich shell

Discrepancies can be observed for very large displacements. These displacements are accompanied by finite rotations which were not taken into account

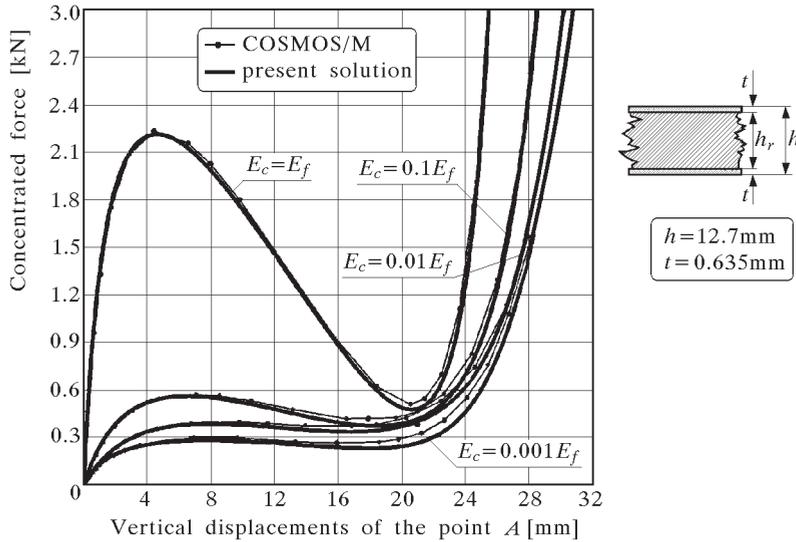


Fig. 10. Equilibrium path for the sandwich shell

in the presented version of program. It is the reason of these discrepancies. The coincidence is pretty good within the initial portions of the curves, and particularly in the vicinity of the critical points.

5. Final remarks

The conception of degeneration usually adopted to homogeneous shells has turned out to be effective also in the case of sandwich shells. The performed test confirmed that the proposed approach is correct within the adopted assumptions. The limit ratio t_f/t_c (the face thickness over the core thickness) was established, and it seems that the adopted assumptions are valid for $t_f/t_c < 1/15$. There is possibility of extending the proposed approach to a case of thicker layers and a multilayer case, but it would require integration along the lateral direction within every layer.

There is no possibility of taking into account local deformations of the faces (cf. Fig. 1c), and this is the obvious drawback of the presented approach. All numerical procedures used before for nonlinear static analysis of homogeneous plates and shells (cf. Marcinowski, 1999) turned out to be effective also in the case of sandwich shells.

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Geometrycznie nieliniowa analiza statyczna płyt i powłok warstwowych

Streszczenie

W pracy rozważa się płyty i powłoki warstwowe złożone z trzech warstw: dwóch twardych okładek i stosunkowo miękkiego rdzenia, który jest znacznie grubszy od przykrywających go okładek. Zaprezentowano element skończony, oparty na koncepcji degeneracji ośrodka trójwymiarowego, dobrze opisujący własności mechaniczne takiej powłoki. W opracowanym elemencie wykorzystano oryginalną koncepcję Ahmada, Ironsa i Zienkiewicza rozszerzoną przez autora na zakres geometrycznie nieliniowy. Zakładając błonowy stan odkształceń i naprężeń w okładkach, uzupełniono macierz sztywności rdzenia o macierze sztywności okładek wykorzystując przy tym te same stopnie swobody, które wprowadził Ahmad. Założono, że materiały rdzenia i okładek są liniowo sprężyste i wykazują własności ortotropii. Przedstawiona analiza obejmuje zagadnienia quasistatyczne w zakresie dużych przemieszczeń i umiarkowanych obrotów ze szczególnym uwzględnieniem utraty stateczności równowagi. Do wyznaczania geometrycznie nieliniowych ścieżek równowagi wykorzystano procedury autorskie stosowane wcześniej z powodzeniem w przypadku powłok pełnych, jednorodnych. W pracy przedstawiono kilka przykładów potwierdzających poprawność proponowanej koncepcji. Przeprowadzone testy wykazały, że sformułowane założenia i oparta na nich koncepcja opisu są prawdziwe dla powłok warstwowych, w których stosunek grubości okładki do grubości rdzenia jest mniejszy od $1/15$.

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