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ADHESIVE CONTACT PROBLEM FOR TRANSVERSELY ISOTROPIC ELASTIC HALF-SPACE

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The problem of adhesive contact for a transversely isotropic elastic half-space is considered. The problem is reduced to the solution of two coupled integral equations, and these are solved exactly. Explicit expressions are found for the contact compliance and for coefficients which characterise the singularities of contact stresses at the boundary of the contact region. The numerical results presented for some anisotropic materials show that the influence of anisotropy on the analysed mechanical quantities is significant.

Key words: adhesive contact, anisotropy, integral equations, compliance, stress concentration factors

1. Introduction

The problem of adhesive contact can be solved by the use of the Hankel transforms and the subsequent use of the Weiner-Hopf technique. The problem was first solved by Mossakovskiĭ (1954) and then was considered by Abramian et al. (1956) and Spence (1968a,b). The solutions of many adhesive contact problems can be found in Gladwell's book (1980). These solutions are related to isotropic materials.

The adhesive contact problem in the context of a transversely isotropic elastic stratum is considered in this paper. Many of fiber-reinforced, platelet and laminated systems, some soils and, of course, a number of crystalic and other real materials have transversely isotropic mechanical properties. The present paper clarifies the effect of anisotropy on the mechanical quantities under consideration in the adhesive contact problem.

2. Basic elasticity equations and their solutions

The axially symmetric problem of elasticity can be analysed by means of displacement functions, which are governed by differential equations

$$\nabla_i^2 \varphi_i(r, s_i z) = 0 \qquad i = 1, 2 \tag{2.1}$$

where ∇_i^2 is Laplace's operator referred to the cylindrical polar co-ordinate system (r, θ, z_i) with $z_i = s_i z$, where s_1 and s_2 are the parameters of a transversely isotropic medium. The displacement and stress can be uniquely expressed in terms of these displacement functions (Rogowski, 1975). The solution to equations (2.1) may be presented as a Hankel's (in terms of r) representation of the harmonic functions $\varphi_i(r, s_i z)$ in the domain $(r, s_i z)$, as follows

$$\varphi_i(r, s_i z) = \vartheta_i \int\limits_0^\infty \xi^{-1} H_i(\xi s_i z) J_0(\xi r)\, d\xi \tag{2.2}$$

where

$$\begin{aligned} &\vartheta_1 = -\frac{s_2}{G_z(k+1)(s_1 - s_2)} \qquad\qquad \vartheta_2 = \frac{s_1}{G_z(k+1)(s_1 - s_2)} \\ &H_i(\xi s_i z) = A_i(\xi) \mathrm{e}^{-\xi s_i z} \end{aligned} \tag{2.3}$$

and where G_z is the shear modulus along the axis of elastic symmetry of the material (z-axis) that has five components of the elastic stiffness c_{ij} or three equivalent parameters s_1, s_2 and k.

The corresponding displacement and stress components take the form

$$\begin{aligned} u_r(r, z) &= \frac{1}{G_z(k+1)(s_1 - s_2)} \int\limits_0^\infty \Big[k s_2 A_1(\xi) \mathrm{e}^{-\xi s_1 z} - s_1 A_2(\xi) \mathrm{e}^{-\xi s_2 z} \Big] J_1(\xi r)\, d\xi \\ u_z(r, z) &= \frac{s_1 s_2}{G_z(k+1)(s_1 - s_2)} \int\limits_0^\infty \Big[A_1(\xi) \mathrm{e}^{-\xi s_1 z} - k A_2(\xi) \mathrm{e}^{-\xi s_2 z} \Big] J_0(\xi r)\, d\xi \end{aligned} \tag{2.4}$$

$$\begin{aligned} \sigma_{zz}(r, z) &= -\frac{1}{s_1 - s_2} \int\limits_0^\infty \xi \Big[s_2 A_1(\xi) \mathrm{e}^{-\xi s_1 z} - s_1 A_2(\xi) \mathrm{e}^{-\xi s_2 z} \Big] J_0(\xi r)\, d\xi \\ \sigma_{rz}(r, z) &= -\frac{s_1 s_2}{s_1 - s_2} \int\limits_0^\infty \xi \Big[A_1(\xi) \mathrm{e}^{-\xi s_1 z} - A_2(\xi) \mathrm{e}^{-\xi s_2 z} \Big] J_1(\xi r)\, d\xi \end{aligned} \tag{2.5}$$

where $A_i(\xi)$ $(i=1,2)$ are arbitrary constants and $J_\nu(\xi r)$, $(\nu=0,1)$ are the Bessel functions.

By using the substitutions

$$
\begin{aligned}
A_1(\xi) &= s_1\widehat{t}(\xi) - \widehat{p}(\xi) \\
A_2(\xi) &= s_2\widehat{t}(\xi) - \widehat{p}(\xi)
\end{aligned}
\tag{2.6}
$$

we can transform these equations to give the displacements and stress on the plane $z=0$, and the resulting equations are then

$$
\begin{aligned}
u_r(r,0) &= \frac{1}{G_zC}\Big[\int\limits_0^\infty \widehat{t}(\xi)J_1(\xi r)\;d\xi - \vartheta_0\int\limits_0^\infty \widehat{p}(\xi)J_1(\xi r)\;d\xi\Big] \\
u_z(r,0) &= \frac{1}{G_zC}\Big[-\vartheta_0 s_1 s_2\int\limits_0^\infty \widehat{t}(\xi)J_0(\xi r)\;d\xi + \int\limits_0^\infty \widehat{p}(\xi)J_0(\xi r)\;d\xi\Big]
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
\sigma_{zz}(r,0) &= -\int\limits_0^\infty \xi\widehat{p}(\xi)J_0(\xi r)\;d\xi \\
\sigma_{zr}(r,0) &= -s_1s_2\int\limits_0^\infty \xi\widehat{t}(\xi)J_1(\xi r)\;d\xi
\end{aligned}
\tag{2.8}
$$

It is seen that $\widehat{p}(\xi)$ and $\widehat{t}(\xi)$ are the Hankel transforms of order zero and one, respectively, of the contact stress $\sigma_{zz}(r,0)=-p(r)$ and $\sigma_{zr}(r,0)=-s_1s_2t(r)$. In equations (2.7) the material constants C and ϑ_0 are defined by equations

$$
\begin{aligned}
C &= \frac{(k+1)(s_1-s_2)}{(k-1)s_1s_2} = 2\frac{G_r}{G_z}\frac{1}{(1-\nu_{r\theta})s_1s_2(s_1+s_2)} \\
\vartheta_0 &= \frac{ks_2-s_1}{(k-1)s_1s_2} = \frac{G_zC}{\sqrt{c_{11}c_{33}}+c_{13}}
\end{aligned}
\tag{2.9}
$$

where G_r and $\nu_{r\theta}$ are the shear modulus and Poissons ratio, respectively, in the isotropic plane. The constant C is real since s_1 and s_2 are real or complex conjugate; in consequence the parameter ϑ_0 is also real.

3. Boundary conditions and integral equations

Consider a rigid circular indenter loaded by the force P on a transversely isotropic half-space (Fig. 1). Assume that the friction at the interface is sufficient to prevent any slip between the indenter and the edge of the stratum. This states that the contact region $(r \leqslant a)$ has a constant displacement δ in the z-direction and zeroth displacement in the r-direction. The remainder of the plane $z = 0$ is stress-free. Thus

$$\begin{aligned} & u_z(r,0) = \delta && 0 \leqslant r \leqslant a \\ & u_r(r,0) = 0 && 0 \leqslant r \leqslant a \\ & \sigma_{zz}(r,0) = 0 && r > a \\ & \sigma_{zr}(r,0) = 0 && r > a \end{aligned} \tag{3.1}$$

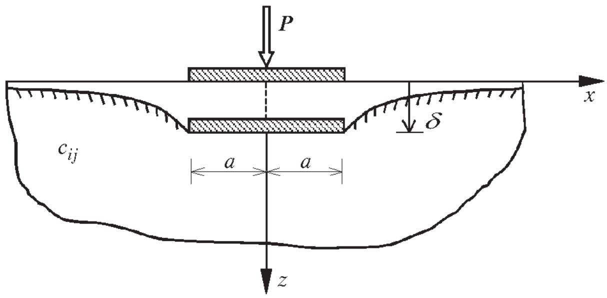

Fig. 1. Translation of a rigid indenter on a half-space

The boundary conditions (3.1) will be satisfied provided that

$$\begin{aligned} & -\vartheta_0 s_1 s_2 \int\limits_0^\infty \widehat{t}(\xi) J_0(\xi r)\, d\xi + \int\limits_0^\infty \widehat{p}(\xi) J_0(\xi r)\, d\xi = G_z C \delta && 0 \leqslant r \leqslant a \\ & \int\limits_0^\infty \widehat{t}(\xi) J_1(\xi r)\, d\xi - \vartheta_0 \int\limits_0^\infty \widehat{p}(\xi) J_1(\xi r)\, d\xi = 0 && 0 \leqslant r \leqslant a \end{aligned} \tag{3.2}$$

$$\begin{aligned} & \int\limits_0^\infty \xi \widehat{p}(\xi) J_0(\xi r)\, d\xi = 0 && r > a \\ & \int\limits_0^\infty \xi \widehat{t}(\xi) J_1(\xi r)\, d\xi = 0 && r > a \end{aligned} \tag{3.3}$$

Introducing the auxiliary functions $\varphi(t)$ and $\psi(t)$, on the assumption that $\psi(0)=0$, such that

$$\widehat{p}(\xi)=\int\limits_0^a \varphi(t)\cos(\xi t)\,dt \qquad\qquad \widehat{t}(\xi)=\int\limits_0^a \psi(t)\sin(\xi t)\,dt \tag{3.4}$$

we obtain from equations (2.8) the contact stresses

$$\begin{aligned}
p(r)&=-\frac{1}{r}\frac{d}{dr}\int\limits_r^a \frac{t\varphi(t)}{\sqrt{t^2-r^2}}\,dt \qquad && 0\leqslant r<a\\
t(r)&=-s_1s_2\frac{d}{dr}\int\limits_r^a \frac{\psi(t)}{\sqrt{t^2-r^2}}\,dt \qquad && 0\leqslant r<a
\end{aligned} \tag{3.5}$$

and $p(r)=0=t(r)$ for $r>a$, where the results (A.1)-(A.4) have been used (see Appendix).

The equilibrium equation of the punch gives

$$P=2\pi\int\limits_0^a rp(r)\,dr \tag{3.6}$$

Substituting equation $(3.5)_1$ into (3.6) and integrating, we obtain

$$P=2\pi\int\limits_0^a \varphi(t)\,dt \tag{3.7}$$

Now substitute expressions (3.4) into equations (3.2), and use (A.5) and (A.6)

$$\begin{aligned}
&-\vartheta_0 s_1 s_2\int\limits_0^\infty t(\xi)J_0(\xi r)\,d\xi+\int\limits_0^r \frac{\varphi(t)}{\sqrt{r^2-t^2}}\,dt=G_zC\delta \qquad && 0\leqslant r\leqslant a\\
&\frac{1}{r}\int\limits_0^r \frac{t\psi(t)}{\sqrt{r^2-t^2}}\,dt-\vartheta_0\int\limits_0^\infty p(\xi)J_1(\xi r)\,d\xi=0 \qquad && 0\leqslant r\leqslant a
\end{aligned} \tag{3.8}$$

These equations are of Abel's type. Applying the inverse Abel's operator we obtain

$$\begin{aligned}
&\varphi(t)-\frac{\vartheta_0 s_1 s_2}{\pi}\frac{d}{dt}\int\limits_0^a \psi(x)\ln\frac{t+x}{|t-x|}\,dx=\frac{2}{\pi}G_zC\delta\\
&\psi(t)-\frac{\vartheta_0}{\pi}\frac{1}{t}\frac{d}{dt}\int\limits_0^a \varphi(x)\Big(2t-x\ln\frac{t+x}{|t-x|}\Big)\,dx=0
\end{aligned} \tag{3.9}$$

where results (A.7)-(A.10) have been used.

Multiplying both sides of these equations by dt and $t\,dt$, respectively, and integrating with respect to t from 0 to a and using result (3.7) we obtain the following equations

$$\begin{aligned} &\vartheta_0 s_1 s_2 \int\limits_0^a \psi(x) \ln \frac{a+x}{a-x}\; dx = \frac{1}{2}P - 2G_z C\delta a \\ &\pi \int\limits_0^a x\psi(x)\; dx + \vartheta_0 \int\limits_0^a x\varphi(x) \ln \frac{a+x}{a-x}\; dx = \frac{Pa\vartheta_0}{\pi} \end{aligned} \tag{3.10}$$

The problem is reduced to the solution of integral equations (3.10).

4. Solution to integral equations

By a suitable change of the variables

$$\begin{aligned} &x' = \frac{x}{a} \qquad && \Theta(x') = \frac{1}{2}\ln\frac{1+x'}{1-x'} \qquad && \tanh\Theta(x') = x' \\ &0 \leqslant x' < 1 && 0 \leqslant \Theta(x') < \infty && \end{aligned} \tag{4.1}$$

equations (3.10) become

$$\begin{aligned} &2\vartheta_0 s_1 s_2 \int\limits_0^1 \psi(x')\Theta(x')\; dx' = \\ &= 2\vartheta_0 s_1 s_2 \int\limits_0^\infty \psi(\tanh\Theta)\Theta\,\mathrm{sech}^2\Theta\; d\Theta = \frac{P}{2a} - 2G_z C\delta \\ &\pi \int\limits_0^1 x'\psi(x')\; dx' + 2\vartheta_0 \int\limits_0^1 x'\varphi(x')\Theta(x')\; dx' = \\ &= \pi \int\limits_0^\infty \tanh\Theta\psi(\tanh\Theta)\,\mathrm{sech}^2\Theta\; d\Theta + \\ &+2\vartheta_0 \int\limits_0^\infty \tanh\Theta\varphi(\tanh\Theta)\Theta\,\mathrm{sech}^2\Theta\; d\Theta = \frac{P\vartheta_0}{\pi a} \end{aligned} \tag{4.2}$$

Using the fact that $\psi(x')$ is an odd function and $\varphi(x')$ is an even function, we assume the solution to these equations having the forms

$$\begin{aligned}\psi(x') &= B(\lambda)\sin(\lambda\Theta)\\ \varphi(x') &= A(\lambda)\cos(\lambda\Theta)\end{aligned} \tag{4.3}$$

where $A(\lambda)$ and $B(\lambda)$ are constants and λ plays the role of an eigenvalue.

Substituting equations (4.3) into equations (4.2), and using integrals (A.11), (A.12) and (A.13), see Appendix, we obtain the following algebraic equations

$$\begin{aligned}&B(\lambda)\Big[1-\frac{\pi}{2}\lambda\coth\Big(\frac{\pi}{2}\lambda\Big)\Big]\mathrm{cosech}\,\Big(\frac{\pi}{2}\lambda\Big) = \frac{2G_zC\delta}{\pi\vartheta_0 s_1 s_2} - \frac{P}{2\pi\vartheta_0 s_1 s_2 a}\\ &B(\lambda)\lambda^2\mathrm{cosech}\,\Big(\frac{\pi}{2}\lambda\Big) + \frac{4\lambda\vartheta_0}{\pi}A(\lambda)\Big[1-\frac{\pi}{4}\lambda\coth\Big(\frac{\pi}{2}\lambda\Big)\Big]\mathrm{cosech}\,\Big(\frac{\pi}{2}\lambda\Big) = \frac{4P\vartheta_0}{\pi^3 a}\end{aligned} \tag{4.4}$$

The third equation is obtained from condition (3.7), which gives

$$A(\lambda)\mathrm{cosech}\,\Big(\frac{\pi}{2}\lambda\Big) = \frac{P}{\pi^2 a\lambda} \tag{4.5}$$

where the integral (A.14) is used (see Appendix).

Eliminating $A(\lambda)$ from equations $(4.4)_2$ and (4.5), we obtain

$$B(\lambda)\mathrm{cosech}\,\Big(\frac{\pi}{2}\lambda\Big) = \frac{P\vartheta_0}{\pi^2 a\lambda}\coth\Big(\frac{\pi}{2}\lambda\Big) \tag{4.6}$$

Substituting (4.6) into $(4.4)_1$, we have

$$\frac{P\vartheta_0}{\pi^2 a\lambda}\coth\Big(\frac{\pi}{2}\lambda\Big)\Big[1-\frac{\pi}{2}\lambda\coth\Big(\frac{\pi}{2}\lambda\Big)\Big] = \frac{2G_zC\delta}{\pi\vartheta_0 s_1 s_2} - \frac{P}{2\pi\vartheta_0 s_1 s_2 a} \tag{4.7}$$

If we define the eigenvalue λ by equation

$$\tanh\Big(\frac{\pi}{2}\lambda\Big) = \vartheta_0 \tag{4.8}$$

which has the solution

$$\lambda = \frac{1}{\pi}\ln\frac{1+\vartheta_0}{1-\vartheta_0} \tag{4.9}$$

then

$$\delta = \frac{Ps_1s_2}{2G_zCa}\Big[\frac{\vartheta_0}{\pi\lambda} - \frac{1}{2}\Big(1-\frac{1}{s_1s_2}\Big)\Big] \tag{4.10}$$

This solution determines the compliance of a transversely isotropic half-space in the adhesive contact problem. For real materials the quantity of ϑ_0 is real, positive and $0 \leqslant \vartheta_0 < 1$. For example, ϑ_0 takes the values: 0.1833; 0.2474; 0.4020 for cadmium, laminated composite consisting of alternating layers of two isotropic materials with $\mu/\overline{\mu} = 0.5$, $\overline{h}/h = 0.5$, $\mu = 10^4$ MPa (shear modulus) and for E-glass-epoxy composite, respectively.

The constants $A(\lambda)$ and $B(\lambda)$ are equal, and are given by equation

$$A(\lambda) = B(\lambda) = \frac{2}{\pi} G_z C\delta \frac{1}{\sqrt{1-\vartheta_0^2}} \tag{4.11}$$

and the functions $\psi(x)$ and $\varphi(x)$ defined by equations (4.3) are as follows

$$\begin{aligned} \psi(x) &= \frac{2}{\pi} G_z C\delta \frac{1}{\sqrt{1-\vartheta_0^2}} \sin(\lambda\Theta) \\ \varphi(x) &= \frac{2}{\pi} G_z C\delta \frac{1}{\sqrt{1-\vartheta_0^2}} \cos(\lambda\Theta) \qquad \Theta = \frac{1}{2}\ln\frac{a+x}{a-x} \end{aligned} \tag{4.12}$$

For an isotropic material the following hold

$$\vartheta_0 = \frac{1-2\nu}{2(1-\nu)} \qquad C = \frac{1}{1-\nu} \qquad s_1 = s_2 = 1 \tag{4.13}$$

and

$$\delta = \frac{P}{4G_z a} \frac{1-2\nu}{\ln(3-4\nu)} \tag{4.14}$$

Result (4.14) agrees with Spence's solution (Spence, 1968a). For an incompressible material ($\nu = 1/2$ for isotropy or $\vartheta_0 = 0$ for transverse isotropy) we have the limiting values

$$\lim_{\nu\to\frac{1}{2}} \frac{1-2\nu}{\ln(3-4\nu)} = \frac{1}{2} \qquad \text{or} \qquad \lim_{\vartheta_0\to 0} \frac{\vartheta_0}{\ln\frac{1+\vartheta_0}{1-\vartheta_0}} = \frac{1}{2} \tag{4.15}$$

so that

$$\delta = \frac{P}{8G_z a} \qquad \text{or} \qquad \delta = \frac{P}{4G_z a}\frac{s_1+s_2}{s_1 s_2} \qquad \text{for } ks_2 = s_1 \tag{4.16}$$

and

$$\psi(x) = 0 \qquad \varphi(x) = \frac{2}{\pi} G_z C\delta \qquad \text{for} \quad \vartheta_0 = 0 \qquad \lambda = 0 \tag{4.17}$$

Equations (3.9) show that for $\vartheta_0 = 0$ the solutions are given by (4.17). This is a confirmation of the correctness of the obtained results and the proper definition of the eigenvalue λ by equation (4.9). Equations (4.16) agree with the result of the frictionless contact related to the incompressible material of a half-space.

5. The stress in the contact region and displacements outside of one

The contact stresses are given by equations (3.5) or, alternatively, by the following integrals

$$\begin{aligned} p(r) &= \frac{1}{\vartheta_0}\frac{1}{r}\frac{d}{dr}\int\limits_0^r \frac{x\psi(x)}{\sqrt{r^2-x^2}}\,dx = \frac{1}{\vartheta_0}\int\limits_0^r \frac{d\psi(x)}{dx}\frac{dx}{\sqrt{r^2-x^2}} \\ t(r) &= -\frac{s_1s_2}{\vartheta_0}\frac{d}{dr}\int\limits_0^r \frac{\varphi(x)}{\sqrt{r^2-x^2}}\,dx = -\frac{s_1s_2}{\vartheta_0}\frac{1}{r}\int\limits_0^r \frac{d\varphi(x)}{dx}\frac{x\,dx}{\sqrt{r^2-x^2}} \end{aligned} \tag{5.1}$$

where the second representations are obtained with the use of formula (A.17). Note, that the equivalence in equation $(5.1)_1$ holds since $\psi(x)$ is an odd function, while in equation $(5.1)_2$ it does since $\varphi(x)$ is an even function.

The displacements outside the contact region are defined by equations (2.7), in which the functions $\widehat{t}(\xi)$ and $\widehat{p}(\xi)$ are given by integrals (3.4). The substitution with the use of equations (A.3) and (A.4) (for displacements) yields

$$\begin{aligned} p(\rho) &= \frac{P}{\pi^2a^2\lambda}\frac{\vartheta_0}{\sqrt{1-\vartheta_0^2}}S_1(\rho) && 0 \leqslant \rho < 1 \\ t(\rho) &= \frac{Ps_1s_2}{\pi^2a^2\lambda}\frac{\vartheta_0}{\sqrt{1-\vartheta_0^2}}S_2(\rho) && 0 \leqslant \rho < 1 \\ u_r(\rho) &= \frac{\delta}{\rho}\Big[\frac{2}{\pi}\frac{1}{\sqrt{1-\vartheta_0^2}}U_2(\rho) - \lambda\Big] && \rho \geqslant 1 \\ u_z(\rho) &= \frac{2}{\pi}\delta\frac{1}{\sqrt{1-\vartheta_0^2}}U_1(\rho) && \rho \geqslant 1 \end{aligned} \tag{5.2}$$

where

$$
\begin{aligned}
S_1(\rho) &= -\frac{1}{\rho}\frac{d}{d\rho}\int\limits_{\rho}^{1}\frac{x\cos(\lambda\Theta)}{\sqrt{x^2-\rho^2}}\,dx = \frac{1}{\vartheta_0}\frac{1}{\rho}\frac{d}{d\rho}\int\limits_{0}^{\rho}\frac{x\sin(\lambda\Theta)}{\sqrt{\rho^2-x^2}}\,dx = \\
&= \frac{1}{\vartheta_0}\frac{1}{\rho^2}\int\limits_{0}^{\rho}\frac{d[x\sin(\lambda\Theta)]}{dx}\frac{xdx}{\sqrt{\rho^2-x^2}} \\
S_2(\rho) &= -\frac{d}{d\rho}\int\limits_{\rho}^{1}\frac{\sin(\lambda\Theta)}{\sqrt{x^2-\rho^2}}\,dx = -\frac{1}{\vartheta_0}\frac{d}{d\rho}\int\limits_{0}^{\rho}\frac{\cos(\lambda\Theta)}{\sqrt{\rho^2-x^2}}\,dx = \\
&= -\frac{1}{\vartheta_0}\frac{1}{\rho}\int\limits_{0}^{\rho}\frac{d[\cos(\lambda\Theta)]}{dx}\frac{xdx}{\sqrt{\rho^2-x^2}}
\end{aligned}
\tag{5.3}
$$

$$
\begin{aligned}
&U_1(\rho) = \int\limits_{0}^{1}\frac{\cos(\lambda\Theta)}{\sqrt{\rho^2-x^2}}\,dx \qquad && U_1(1) = \frac{\pi}{2}\sqrt{1-\vartheta_0^2} \\
&U_2(\rho) = \int\limits_{0}^{1}\frac{x\sin(\lambda\Theta)}{\sqrt{\rho^2-x^2}}\,dx \qquad && U_2(1) = \frac{\pi}{2}\lambda\sqrt{1-\vartheta_0^2} \\
&\Theta = \frac{1}{2}\ln\frac{1+x}{1-x} \qquad && 0 \leqslant x < 1
\end{aligned}
\tag{5.4}
$$

In deriving the third representations in (5.3) the integral (A.17) is used.

The stress distribution on the contact surface can be characterised by the load-transfer factor, $P(\rho)$, which is defined as

$$
P(\rho) = 2\pi a^2\int\limits_{\rho}^{1}\rho p(\rho)\,d\rho \tag{5.5}
$$

From equations $(5.3)_1$ and (5.5), we have

$$
\begin{aligned}
P(\rho) &= \frac{2}{\pi}P\frac{1}{\lambda}\frac{\vartheta_0}{\sqrt{1-\vartheta_0^2}}\int\limits_{\rho}^{1}\frac{x\cos(\lambda\Theta)}{\sqrt{x^2-\rho^2}}\,dx = \\
&= P - \frac{2}{\pi}P\frac{1}{\sqrt{1-\vartheta_0^2}}\int\limits_{0}^{\rho}\frac{\sqrt{\rho^2-x^2}\cos(\lambda\Theta)}{1-x^2}\,dx \\
P(0) &= P \qquad\qquad P(1) = 0
\end{aligned}
\tag{5.6}
$$

In deriving $P(0)$ and $P(1)$ the integrals (A.14) and (A.15) were employed.

Applying the differentiation rule of the integrand (equation (A.16), see Appendix), we derive the following relations from equations (5.3)

$$\begin{aligned} S_1(\rho) &= \frac{\cos(\lambda\Theta)}{\sqrt{1-\rho^2}} + \lambda\int\limits_{\rho}^{1} \frac{\sin(\lambda\Theta)}{(1-x^2)\sqrt{x^2-\rho^2}}\,dx \\ S_2(\rho) &= \rho\Big[\frac{\sin(\lambda\Theta)}{\sqrt{1-\rho^2}} + \int\limits_{\rho}^{1} \frac{\sin(\lambda\Theta)}{x^2\sqrt{x^2-\rho^2}}\,dx - \lambda\int\limits_{\rho}^{1} \frac{\cos(\lambda\Theta)}{x(1-x^2)\sqrt{x^2-\rho^2}}\,dx\Big] \end{aligned} \tag{5.7}$$

The integrals $S_1(\rho)$ and $S_2(\rho)$ show singularities at the boundary of the contact region, i.e. as $\rho \to 1$, which results in the relevant stresses singularities, too. Such behaviour is well known in the analysis of contact and interface crack problems (Ting, 1990; Ni and Nemat Nasser, 1991, 1992).

In the case of an incompressible material, i.e. when $\lambda = 0$, we obtain the square root of the singularity for the normal stress, while the shear stress vanishes in this case. This corresponds to the frictionless contact problem of an incompressible half-space.

The oscillations occur in the regions defined by

$$\begin{aligned} &\frac{1+\rho}{1-\rho} > \mathrm{e}^{\pi/\lambda} \qquad \text{or} \qquad \varepsilon_0 < \frac{2}{1+\mathrm{e}^{\pi/\lambda}} \qquad \rho = 1-\varepsilon_0 \\ &\frac{1+\rho}{1-\rho} > \mathrm{e}^{2\pi/\lambda} \qquad \text{or} \qquad \varepsilon_1 < \frac{2}{1+\mathrm{e}^{2\pi/\lambda}} \qquad \rho = 1-\varepsilon_1 \end{aligned} \tag{5.8}$$

for the normal and shear stresses, respectively. For example, for an isotropic and extreme case when $\nu = 0$ we have $\vartheta_0 = 0.5$, $\lambda = 0.3497$, $\varepsilon_0 = 0.00025$. The calculation of the local extremum of the first term on the right hand side of equations (5.7) results in the following relationships, respectively

$$\begin{aligned} &\tan[\lambda\Theta(\rho_0)] = \frac{\rho_0}{\lambda} \qquad \mathrm{extr}\, S_1(\rho_0) = \frac{\lambda}{\sqrt{\lambda^2+\rho_0^2}\sqrt{1-\rho_0^2}} \\ &\tan[\lambda\Theta(\rho_1)] = -\lambda\rho_1 \qquad \mathrm{extr}\, S_2(\rho_1) = \frac{\lambda\rho_1^2}{\sqrt{1+\lambda^2\rho_1^2}\sqrt{1-\rho_1^2}} \end{aligned} \tag{5.9}$$

There are many roots of ρ_0 and ρ_1 in the small intervals $(1-\varepsilon_0, 1)$ and $(1-\varepsilon_1, 1)$ which can be obtained from the foregoing equation. The one which yields the first extremum of the contact stresses at the end (ρ_0 or ρ_1) is chosen for numerical computation. The values of ρ_0 and ρ_1 for different materials are presented in Table 1.

The stress concentration factors defined by equations

$$
\begin{aligned}
K_z &= \sqrt{2a(1-\rho_0)}p(\rho_0) \\
K_{zr} &= \sqrt{2a(1-\rho_1)}t(\rho_1)
\end{aligned}
\tag{5.10}
$$

are obtained as follows

$$
\begin{aligned}
K_z &= \frac{P\sqrt{2}}{\pi^2 a\sqrt{a}} \frac{\vartheta_0}{\sqrt{\lambda^2+\rho_0^2}\sqrt{1+\rho_0}} \\
K_{zr} &= \frac{P\sqrt{2}}{\pi^2 a\sqrt{a}} \frac{\vartheta_0 s_1 s_2 \rho_1^2}{\sqrt{1+\lambda^2\rho_1^2}\sqrt{1+\rho_1}}
\end{aligned}
\tag{5.11}
$$

For $\vartheta_0/\lambda \to \pi/2$ the obtained results are reduced to the following formulae

$$
\begin{aligned}
p(\rho) &= \frac{P}{2\pi a^2}\frac{1}{\sqrt{1-\rho^2}} && 0 \leqslant \rho < 1 \\
t(\rho) &= 0 && 0 \leqslant \rho < 1 \\
u_r(\rho) &= \begin{cases} 0 & \text{for the incompressible half–space, } \rho \geqslant 1 \\ -\dfrac{2}{\pi}\delta\vartheta_0\dfrac{1}{\rho} & \text{for the frictionless contact, } \rho \geqslant 1 \end{cases} \\
u_z(\rho) &= \frac{2}{\pi}\delta \arcsin\frac{1}{\rho} && \rho \geqslant 1 \\
\delta &= \frac{P}{4G_z C a}
\end{aligned}
\tag{5.12}
$$

Equations (5.12) are well known, and therefore tend to confirm the present analysis.

6. Numerical results

Table 1 shows the values of compliance $(\delta a\mu/P)$, the stress concentration factors and the parameters ε_0, ε_1, ρ_0, ρ_1 obtained from equations (4.10), (5.11), (5.8), respectively, for six different materials such as cadmium (denoted symbolically as C), magnesium (M) cristals, E-glass-epoxy (EG-E), graphite epoxy (G-E) composite materials and comparative layered (L) and isotropic (ISO) materials. For the layered material it is assumed $\mu/\overline{\mu} = 0.5$ and $\overline{h}/h = 0.5$; $\mu = 10^4$ MPa.

Table 1. Compliance, stress concentration factors, parameters ε_0, ε_1, ρ_0, ρ_1 for different materials

	C	M	EG-E	G-E	L	ISO ($\nu = 0.3$)
$\frac{\delta a \mu}{P}$	0.1216	0.09639	0.1894	0.1455	0.1605	0.1701
$\frac{K_z \pi^2 a\sqrt{a}}{P\sqrt{2}}$	0.1287	0.2114	0.2743	0.3647	0.1727	0.1986
$\frac{K_{rz} \pi^2 a\sqrt{a}}{P\sqrt{2}}$	0.1972	0.2080	0.1558	0.1121	0.1797	0.1986
ε_0	$0.5532 \cdot 10^{-11}$	$0.3141 \cdot 10^{-6}$	$0.1865 \cdot 10^{-4}$	$0.7531 \cdot 10^{-3}$	$0.6536 \cdot 10^{-8}$	$0.1020 \cdot 10^{-6}$
ε_1	$0.1530 \cdot 10^{-22}$	$0.4932 \cdot 10^{-13}$	$0.1738 \cdot 10^{-9}$	$0.2838 \cdot 10^{-6}$	$0.2136 \cdot 10^{-16}$	$0.5205 \cdot 10^{-14}$
$1 - \rho_0$	$0.4050 \cdot 10^{-10}$	$0.2261 \cdot 10^{-5}$	$0.1315 \cdot 10^{-3}$	$0.5085 \cdot 10^{-2}$	$0.4748 \cdot 10^{-7}$	$0.7369 \cdot 10^{-6}$
$1 - \rho_1$	$0.1120 \cdot 10^{-21}$	$0.3550 \cdot 10^{-12}$	$0.1225 \cdot 10^{-8}$	$0.1904 \cdot 10^{-5}$	$0.1552 \cdot 10^{-15}$	$0.3759 \cdot 10^{-13}$

Figure 2 shows the load-transfer curves obtained from equation (5.6) for different materials.

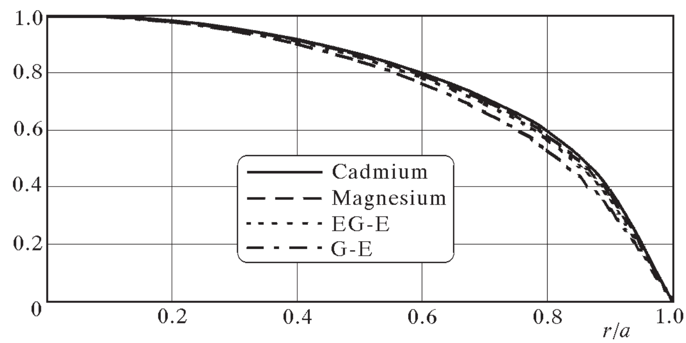

Fig. 2. Load transfer characteristics $P(\rho)/P$ (equation (5.6)) for different materials shown in Table 1

The distributions of contact stresses: normal $p = p(\rho)a^2/P$ and tangential $t = t(\rho)a^2/P$ for cadmium are shown in Fig. 3 (see equations $(5.2)_{1,2}$).

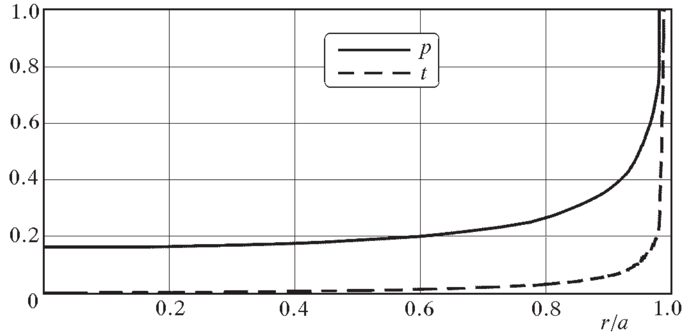

Fig. 3. Contact stresses $p = p(\rho)a^2/P$ and $t = t(\rho)a^2/P$, $\rho = r/a$ for cadmium (equations $(5.2)_{1,2}$)

7. Conclusions

The equations derived in the paper make it possible to completely describe the compliance of the elastic transversely isotropic half-space loaded by a rigid indenter in the adhesive contact problem. As it could be expected, this compliance appears to be strongly dependent on mechanical properties of the presented materials.

The contact pressures (normal and tangential), regadless of their closed mathematical structures, contain integrals which can only be determined numerically. Those integrals exhibit however singular behaviour, which results in the oscillations of contact stresses near the contact region edge. The oscillation regions are characterised by two parameters: width $(\varepsilon_0$ or $\varepsilon_1)$ and location of the first extremum $(\rho_0$ or $\rho_1)$. These parameters are defined by the closed form equations. They are closely related to the material anisotropy.

The contact stress distribution is illustrated by the load-transfer curves, and it is visible that these curves are almost material independent.

A. Appendix

A.1. Integrals involving Bessel functions

The following relations are used

$$\frac{1}{r}\frac{d}{dr}[rJ_1(\xi r)] = \xi J_0(\xi r) \tag{A.1}$$

$$\frac{d}{dr}[J_0(\xi r)] = -\xi J_1(\xi r) \tag{A.2}$$

The following integrals are used

$$\int_0^\infty J_1(\xi r)\cos(\xi t)\,d\xi = \begin{cases} \dfrac{1}{r} & 0 < t < r \\ \dfrac{1}{r}\Big[1 - \dfrac{t}{\sqrt{t^2 - r^2}}\Big] & t > r \end{cases} \tag{A.3}$$

$$\int_0^\infty J_0(\xi r)\sin(\xi t)\,d\xi = \begin{cases} 0 & 0 < t < r \\ \dfrac{1}{\sqrt{t^2 - r^2}} & t > r \end{cases} \tag{A.4}$$

$$\int_0^\infty J_0(\xi r)\cos(\xi t)\,d\xi = \begin{cases} \dfrac{1}{\sqrt{r^2 - t^2}} & 0 < t < r \\ 0 & t > r \end{cases} \tag{A.5}$$

$$\int_0^\infty J_1(\xi r)\sin(\xi t)\,d\xi = \begin{cases} \dfrac{t}{r\sqrt{r^2 - t^2}} & 0 < t < r \\ 0 & t > r \end{cases} \tag{A.6}$$

$$\int_0^t \frac{rJ_0(\xi r)}{\sqrt{t^2 - r^2}}\,dr = \frac{\sin\xi t}{\xi} \tag{A.7}$$

$$\int_0^t \frac{r^2 J_1(\xi r)}{\sqrt{t^2 - r^2}}\,dr = \frac{t}{\xi}\Big(\frac{\sin\xi t}{\xi t} - \cos\xi t\Big) = -\frac{d}{d\xi}\frac{\sin\xi t}{\xi} \tag{A.8}$$

A.2. Integrals involving trigonometric and hyperbolic functions

$$\int_0^\infty \frac{\sin\xi t \sin\xi x}{\xi}\,d\xi = \frac{1}{2}\ln\frac{t + x}{|t - x|} \tag{A.9}$$

$$\int_0^\infty \cos\xi x\frac{d}{d\xi}\Big(\frac{\sin\xi t}{\xi}\Big)\,d\xi = -t + \frac{x}{2}\ln\frac{t + x}{|t - x|} \tag{A.10}$$

$$\int_0^\infty \sin(\lambda\Theta)\Theta\operatorname{sech}^2\Theta\,d\Theta = -\frac{\pi}{2}\Big[1 - \frac{\pi}{2}\lambda\coth\Big(\frac{\pi}{2}\lambda\Big)\Big]\operatorname{cosech}\Big(\frac{\pi}{2}\lambda\Big) \tag{A.11}$$

$$\int_0^\infty \sin(\lambda\Theta)\tanh\Theta\operatorname{sech}^2\Theta\,d\Theta = \frac{\pi}{4}\lambda^2\operatorname{cosech}\Big(\frac{\pi}{2}\lambda\Big) \tag{A.12}$$

$$\begin{aligned}
&\int\limits_0^\infty \Theta\cos(\lambda\Theta)\tanh\Theta\mathrm{sech}^2\Theta\; d\Theta = \\
&\qquad = \frac{\pi}{2}\lambda\Big[1-\frac{\pi}{4}\lambda\coth\Big(\frac{\pi}{2}\lambda\Big)\Big]\mathrm{cosech}\,\Big(\frac{\pi}{2}\lambda\Big)
\end{aligned} \tag{A.13}$$

$$\int\limits_0^\infty \cos(\lambda\Theta)\mathrm{sech}^2\Theta\; d\Theta = \frac{\pi}{2}\lambda\mathrm{cosech}\,\Big(\frac{\pi}{2}\lambda\Big) \tag{A.14}$$

$$\int\limits_0^\infty \cos(\lambda\Theta)\mathrm{sech}\,\Theta\; d\Theta = \frac{\pi}{2}\mathrm{sech}\,\Big(\frac{\pi}{2}\lambda\Big) \tag{A.15}$$

Results (A.12) and (A.13) have been deducted from the results given by Erdelyi (page 30 and 88 of Vol. I book by Erdelyi (1954)). The following rule of differentiation of the integrand was employed in deriving equations (5.7) and (5.3)

$$\frac{d}{dr}\int\limits_r^a \frac{h(t)dt}{\sqrt{t^2-r^2}} = -\frac{rh(a)}{a\sqrt{a^2-r^2}} + r\int\limits_r^a \frac{d}{dt}\Big(\frac{h(t)}{t}\Big)\frac{dt}{\sqrt{t^2-r^2}} \tag{A.16}$$

$$r\frac{d}{dr}\int\limits_0^r \frac{f(t)dt}{\sqrt{r^2-t^2}} = \int\limits_0^r \frac{df(t)}{dt}\frac{tdt}{\sqrt{r^2-t^2}} \tag{A.17}$$

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Zagadnienie kontaktowe z adhezją dla poprzecznie izotropowej sprężystej półprzestrzeni

Streszczenie

Rozpatrzono adhezyjne zagadnienie kontaktowe dla poprzecznie izotropowej sprężystej półprzestrzeni. Zagadnienie zredukowano do rozwiązania dwóch sprzężonych równań całkowych, które rozwiązano dokładnie. Znaleziono w postaci jawnej wzory na podatność oraz współczynniki określające osobliwości naprężeń kontaktowych na brzegu obszaru kontaktu. Wyniki liczbowe przedstawione dla różnych materiałów pokazują wpływ anizotropii na analizowane wielkości mechaniczne.

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