JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
1, **40**, 2002

# A MACROSCOPIC MODEL FOR THE HEAT TRANSFER IN THE CHESSBOARD-TYPE NONHOMOGENEOUS MEDIA

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The aim of this paper is to formulate, discuss and apply a certain macroscopic model of the heat transfer in the rigid chessboard-type micro-inhomogeneous conductor. To this end the tolerance averaging approach is applied. Within the framework of this approach a certain approximate solution to the periodic cell problem is proposed. It leads to the initial-boundary value problem for the averaged temperature field coupled with the initial value problem for the so-called internal variable vector field. In contrast to homogenization, the obtained model describes the effect of microstructure size on the overall behaviour of the medium. It is shown that the proposed model has a physical sense provided that the inhomogeneity of the medium is not too large.

*Key words:* heat transfer, nonhomogeneous media, modelling

## 1. Introduction

The simplest mathematical models for the overall (macroscopic) behaviour of micro-periodic solids can be obtained by using results of the well-known asymptotic homogenization theory; we can mention here the monographs by Bakhvalov and Panasenko (1984), Bensoussan et al. (1978), Jikov et al. (1994), Sanchez-Palencia (1980). However, the coefficients of homogenized equations are independent of the microstructure size. Hence, these equations are incapable of describing the effect of microstructure size observed on the macroscopic level. To avoid this drawback, an alternative nonasymptotic modelling method was proposed and applied in a series of papers by Baron and Woźniak (1995), Baron and Jędrysiak (1998), Cielecka et al. (2001), Jędrysiak (1999, 2000), Matysiak (1991), Mazur-Śniady (1993), Michalak (2000) and others. The aforementioned method was referred to as the tolerance averaging approach and summarized in the book by Woźniak and Wierzbicki (2000).

The problem we are going to solve in this contribution is to formulate a tolerance-averaged model of heat transfer in the chessboard-type microperiodic medium, cf. Fig. 1. Exact form of the averaged equation for this medium is known within the framework of the homogenization theory. Thus, we compare both aforementioned models and we formulate certain conditions for the accuracy of the proposed tolerance averaging model. The discussion of the illustrative initial-boundary value problem concludes the paper. In the subsequent analysis, the Greek indices $\alpha, \beta, ...$ run over $1, 2$; summation convention over all twice repeated indices holds.

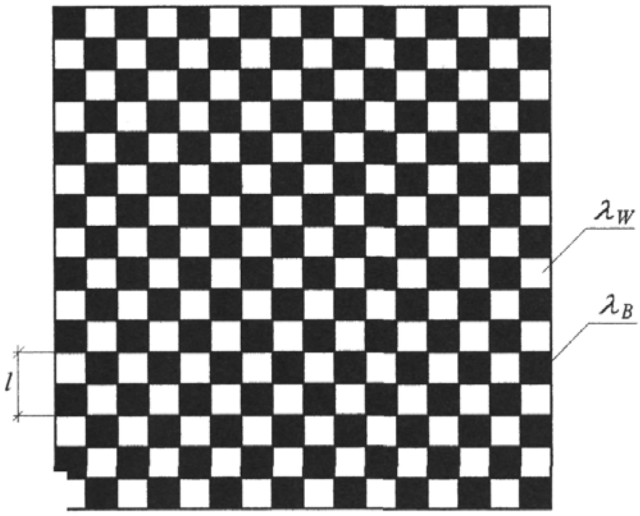


Fig. 1. The chessboard-type medium

## 2. Preliminaries

In this section, following Woźniak and Wierzbicki (2000), we recall some of the results concerning the tolerance averaging approach.

By $\Omega$ we denote the region in two-dimensional reference space parametrized with Carthesian orthogonal coordinates $x_1, x_2$ and occupied by certain periodic medium under consideration. Setting $\Delta = (-l_1/2, l_1/2) \times (-l_2/2, l_2/2)$ it is assumed that the medium is $\Delta$-periodic, i.e., $l_\alpha$-periodic in the direction of the $x_\alpha$-axis, $\alpha = 1, 2$. Moreover, the diameter $l$ of $\Delta$ is assumed to be very small compared with the smallest characteristic length dimension of $\Omega$. That is why $l$ will be referred to as the microstructure length. Denoting $\Delta(\boldsymbol{x}) := \boldsymbol{x} + \Delta$, $\Omega_\Delta := \{\boldsymbol{x} \in \Omega,\ \Delta(\boldsymbol{x}) \subset \Omega\}$, we shall use the known averaging formula

$$\langle f\rangle(\boldsymbol{x}) = \frac{1}{|\Delta|} \int\limits_{\Delta(\boldsymbol{x})} f(\boldsymbol{y})\, dy_1 dy_2 \qquad\qquad \boldsymbol{x} \in \Omega_\Delta$$

for an arbitrary integrable function $f$ defined in $\Omega$. Let $\mathcal{F}(\overline{\Omega})$ be a set of smooth enough, bounded functions defined in $\Omega$ and endowed with pertinent unit measures. Moreover, let $\varepsilon :\ \mathcal{F}(\overline{\Omega}) \ni \varphi \mapsto \varepsilon_\varphi \ \in\ R^+$ be a mapping which assigns to every $\varphi \in \mathcal{F}(\overline{\Omega})$ a positive real value $\varepsilon_\varphi$ related to the pertinent unit measure, which will be regarded as an admissible accuracy related to the computation of the values of $\varphi$ or to the measurements of a physical field represented by $\varphi$. For an arbitrary $\varphi \in \mathcal{F}(\overline{\Omega})$ we shall write $\varphi(\boldsymbol{x}) \cong \varphi(\boldsymbol{y})$ iff $|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})| \leqslant \varepsilon_\varphi$ and say that the values of $\varphi$ at $\boldsymbol{x}$ and $\boldsymbol{y}$ are in a tolerance. It means that the difference between the values of $\varphi$ at $\boldsymbol{x}$ and $\boldsymbol{y}$ can be neglected from the computational viewpoint. Every $\varepsilon_\varphi$ will be referred to as a tolerance parameter assigned to $\varphi \in \mathcal{F}(\overline{\Omega})$ and symbol $\cong$ will represent a certain tolerance relation (i.e. the binary relation being symmetric and reflexive) defined on a set $\mathbb{R}$ endowed with the known unit measure, cf. also Zeeman (1965).

Define $\mathcal{T} = (\mathcal{F}(\overline{\Omega}), \varepsilon(\cdot), l)$ and assume that in all subsequent considerations $\mathcal{T}$ is known. A sufficiently regular function $F \in \mathcal{F}(\overline{\Omega})$ will be called slowly varying (with respect to $\mathcal{T}$), $F(\cdot) \in SV(\mathcal{T})$, if for every $\boldsymbol{x}, \boldsymbol{y}$ from the domain of $F$, the condition $\|\boldsymbol{y} - \boldsymbol{x}\| \leqslant l$ implies $\|F(\boldsymbol{y}) - F(\boldsymbol{x})\| \leqslant \varepsilon_F$ and if similar conditions hold also for all derivatives of $F$; for the sake of simplicity we denote $\varepsilon_F = \varepsilon(F)$.

A continuous function $\psi \in \mathcal{F}(\overline{\Omega})$ is termed as *periodic-like*, $\psi \in PL(\mathcal{T})$, if for every $\boldsymbol{x} \in \Omega_\Delta$ there exists a $\Delta$-periodic function $\psi_{\boldsymbol{x}}(\cdot)$ such that for every $\boldsymbol{y} \in \Omega$, condition $\|\boldsymbol{y} - \boldsymbol{x}\| \leqslant l$ implies $\psi(\boldsymbol{y}) \cong \psi_{\boldsymbol{x}}(\boldsymbol{y})$. Function $\psi_{\boldsymbol{x}}$ is said to be a *periodic approximation* of $\psi$ in $\Delta(\boldsymbol{x})$. If $\psi \in PL(\mathcal{T})$ and $\langle \rho\psi\rangle(\boldsymbol{x}) = 0$ holds for every $\boldsymbol{x} \in \Omega_\Delta$ and for some integrable positive-valued function $\rho$ defined on $\Omega$, then we shall write $\psi \in PL^\rho(\mathcal{T})$ and call $\psi$ the *oscillating function.*

In the tolerance averaging approach we shall use the following assertion and lemmas (cf. Woźniak and Wierzbicki (2000)):

***Assertion***. *If $F \in SV(\mathcal{T})$, $\varphi \in PL(\mathcal{T})$ and $\varphi_{\boldsymbol{x}}$ is a $\Delta$-periodic approximation of $\varphi \in \Delta(\boldsymbol{x})$ then for every $f \in L^\infty_{per}(\Delta)$ and $h \in C^1_{per}(\overline{\Delta})$, such that* $\max\{h(\boldsymbol{y}) :\ \boldsymbol{y} \in \overline{\Delta}\} \leqslant l$, *the following propositions hold for every $\boldsymbol{x} \in \Omega_\Delta$:*

$$\begin{array}{lll}
(T1) & \langle fF\rangle(\boldsymbol{x}) \cong \langle f\rangle F(\boldsymbol{x}) & \textit{for} \quad \varepsilon = \langle|f|\rangle\varepsilon_F \\
(T2) & \langle f\varphi\rangle(\boldsymbol{x}) \cong \langle f\varphi_{\boldsymbol{x}}\rangle(\boldsymbol{x}) & \textit{for} \quad \varepsilon = \langle|f|\rangle\varepsilon_\varphi \\
(T3) & \langle f\partial_\alpha(hF)\rangle(\boldsymbol{x}) \cong \langle fF\partial_\alpha h\rangle(\boldsymbol{x}) & \textit{for} \quad \varepsilon = \langle|f|\rangle(\varepsilon_F + l\varepsilon_{\nabla F}) \\
(T4) & \langle h\partial_\alpha(f\varphi)\rangle(\boldsymbol{x}) \cong -\langle f\varphi\partial_\alpha h\rangle(\boldsymbol{x}) & \textit{for} \quad \varepsilon = \varepsilon_G + l\varepsilon_{\nabla G} \\
 & & \qquad \textit{and} \quad G = \langle hf\varphi\rangle l^{-1}
\end{array}$$

*where* $\varepsilon$ *is a tolerance parameter which defines the pertinent tolerance* $\cong$.

***Lemma.*** Functional spaces in the tolerance averaging technique have the following basic properties:

- ($L1$) *If* $g \in PL(\mathcal{T})$ *then for an arbitrary positive valued integrable* $\Delta$-*periodic function* $\rho$ *there exist functions* $g^\circ \in SV(\mathcal{T})$, $\widetilde{g} \in PL^\rho(\mathcal{T})$, *such that the decomposition* $g = g^\circ + \widetilde{g}$ *holds*
- ($L2$) *If* $g \in PL(\mathcal{T})$ *and* $f \in L^\infty_{per}(\Delta)$ *then* $\langle fg\rangle(\cdot) \in SV(\mathcal{T})$
- ($L3$) *If* $F \in SV(\mathcal{T})$ *and* $f \in C_{per}(\overline{\Delta})$ *then* $(fF)(\cdot) \in PL(\mathcal{T})$
- ($L4$) *If* $F \in SV(\mathcal{T})$, $G \in SV(\mathcal{T})$ *and* $kF + mG \in \mathcal{F}(\overline{\Omega})$ *for some reals* $k$, $m$, *then* $kF + mG \in SV(\mathcal{T})$.

In the linear approximation, the heat conduction properties of a medium are uniquely described by the second order heat conduction tensor $A_{\alpha\beta}$ and by the specific heat scalar $c$. For every $\Delta$-periodic medium under consideration, the functions $A_{\alpha\beta} = A_{\alpha\beta}(\cdot)$, $c = c(\cdot)$ are $\Delta$-periodic where $\Delta$ is assumed to be known. Let $\theta = \theta(\cdot, t)$ be a temperature field in $\Omega$ at time $t$, and let $f = f(\cdot, t)$ be the known intensity of heat sources at $t$. Under the aforementioned notations, a temperature field has to satisfy in $\Omega$ the well-known heat transfer equation

$$\partial_\alpha(A_{\alpha\beta}\partial_\beta\theta) - c\dot{\theta} = f \tag{2.1}$$

The tolerance averaging of the heat transfer equations in micro-periodic media will be based on two assumptions. First, it is the heuristic assumption that in the problem under consideration, the temperature field conforms to the periodic structure of the medium. The above heuristic statement can be written in the following mathematical form.

***Conformability Assumption*** ($CA$). *In the modelling of the heat transfer problems in microperiodic media, every temperature field* $\theta(\cdot, t)$ *has to satisfy the condition*

$$\theta(\cdot, t) \in PL(\mathcal{T})$$

This condition may be violated only in the certain near-boundary layer of $\Omega$. The second assumption is related to formulas ($T1$)-($T4$).

***Tolerance Averaging Assumption*** $(TA)$. *In averaging the equations involving slowly varying and periodic-like functions, the left-hand sides of formulae* $(T1)$*-*$(T4)$ *will be approximated respectively by their right-hand sides.*

From $(CA)$ and $(L1)$ it follows that there exists the decomposition $\theta(\cdot,t) = \theta^0(\cdot,t) + \vartheta(\cdot,t)$, with $\theta^0(\cdot,t) \in SV(\mathcal{T})$ and $\vartheta(\cdot,t) \in PL^c(\mathcal{T})$. It was shown in Woźniak and Wierzbicki (2000) that, under $(CA)$ and $(T1)$ the tolerance averaging of (2.1) yields

$$\begin{aligned} &\partial_\alpha[\langle A_{\alpha\beta}\rangle\partial_\beta\theta^0(\boldsymbol{x},t) + \langle A_{\alpha\beta}\partial_\beta\vartheta\rangle(\boldsymbol{x},t)] - \langle c\rangle\dot{\theta}^0(\boldsymbol{x},t) - \\ &-\langle c\dot{\vartheta}\rangle(\boldsymbol{x},t) = \langle f\rangle(\boldsymbol{x},t) \end{aligned} \tag{2.2}$$

for $\boldsymbol{x} \in \Omega_\Delta$. At the same time, using $(TA)$, we can prove that the following periodic variational equation for the $\Delta$-periodic function $\vartheta_{\boldsymbol{x}}(\boldsymbol{y},t)$, $\boldsymbol{y} \in \Delta(\boldsymbol{x})$, holds

$$\begin{aligned} &\langle\partial_\alpha\vartheta^* A_{\alpha\beta}\partial_\beta\vartheta_{\boldsymbol{x}}\rangle(\boldsymbol{x},t) + \langle\vartheta^*\dot{\vartheta}_{\boldsymbol{x}}c\rangle(\boldsymbol{x},t) = \\ &= -\langle\vartheta^* f\rangle(\boldsymbol{x},t) - \langle\vartheta^* c\rangle\dot{\theta}^0(\boldsymbol{x},t) - \langle\partial_\alpha\vartheta^* A_{\alpha\beta}\rangle\partial_\beta\theta^0(\boldsymbol{x},t) \end{aligned} \tag{2.3}$$

where $\boldsymbol{x} \in \Omega_\Delta$ and $\vartheta^*(\cdot)$ is a $\Delta$-periodic test function, $\vartheta^* \in H^1_{per}(\Delta)$; here either $\langle\vartheta^*\rangle = 0$, $\langle\vartheta_{\boldsymbol{x}}\rangle = 0$ or $\langle c\vartheta^*\rangle = 0$, $\langle c\vartheta_{\boldsymbol{x}}\rangle = 0$. Equations (2.2), (2.3) constitute the fundamentals of the tolerance averaging approach to the modelling of heat transfer problems in micro-periodic media on the macroscopic level. In order to obtain the model equations we shall look for an approximate solutions to the periodic cell problem (2.3) in the form $\vartheta_{\boldsymbol{x}}(\boldsymbol{y},t) = h^A(\boldsymbol{y})V^A(\boldsymbol{x},t)$ (summation convention over $A = 1,...,N$ holds), $\boldsymbol{y} \in \Delta(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega_\Delta$, where $h^A(\cdot)$, $A = 1,...,N$, are postulated $\Delta$-periodic mode shape functions and $V^A(\cdot,t) \in SV(\mathcal{T})$ are extra unknowns. The aforementioned mode shape functions have to satisfy condition $\langle h^A\rangle = 0$ or $\langle ch^A\rangle = 0$, and can be derived as solutions to a certain eigenvalue problem related to (2.3), or are resulting from a periodic discretization of the cell $\Delta$. In this way, setting $\vartheta^* = h^A$ and applying $(TA)$, $(L2)$, $(L4)$, after many manipulations, we obtain for $\theta^0$ and $V^A$, $A = 1,...,N$, the following system of equations

$$\begin{aligned} &\partial_\alpha[\langle A_{\alpha\beta}\rangle\partial_\beta\theta^0(\boldsymbol{x},t) + \langle A_{\alpha\beta}\partial_\beta h^A\rangle V^A(\boldsymbol{x},t)] - \langle c\rangle\dot{\theta}^0(\boldsymbol{x},t) = \langle f\rangle(\boldsymbol{x},t) \\ &\langle ch^A h^B\rangle\dot{V}^B(\boldsymbol{x},t) + \langle\partial_\alpha h^A A_{\alpha\beta}\partial_\beta h^B\rangle V^B(\boldsymbol{x},t) + \\ &+\langle\partial_\alpha h^A A_{\alpha\beta}\rangle\partial_\beta\theta^0(\boldsymbol{x},t) = -\langle h^A f\rangle(\boldsymbol{x},t) \qquad A = 1,...,N \end{aligned} \tag{2.4}$$

At the same time, using ($L3$), we can prove that the temperature field can be approximated by means of the formula

$$\theta(\boldsymbol{x},t) \simeq \theta^0(\boldsymbol{x},t) + h^A(\boldsymbol{x})V^A(\boldsymbol{x},t) \qquad \boldsymbol{x} \in \Omega_\Delta \tag{2.5}$$

where the accuracy of approximation $\simeq$ depends on the number $N$ of terms in the formula $\vartheta_{\boldsymbol{x}}(\boldsymbol{y},t) = h^A(\boldsymbol{y})V^A(\boldsymbol{x},t)$, $y \in \Delta(\boldsymbol{x})$. Solutions $\theta^0$, $V^A$ to problems described by Eqs (2.4) can be physically reliable only if

$$\theta^0(\cdot,t) \in SV(\mathcal{T}) \qquad V^A(\cdot,t) \in SV(\mathcal{T}) \tag{2.6}$$

for every $t$ and $A = 1,...,N$. For a more detailed discussion of this model the reader is referred to Woźniak and Wierzbicki (2000).

Equations (2.4), (2.5) together with conditions (2.6) represent the tolerance model of nonstationary heat transfer problems in a periodic microheterogeneous solid.

## 3. Formulation of the problem

Let us assume that the chessboard-type medium under consideration is made of two isotropic materials. In this case $A_{\alpha\beta} = \delta_{\alpha\beta}\lambda$ and (2.1) takes the form

$$\partial_\alpha(\lambda\partial_\alpha\theta) - c\dot{\theta} = f \tag{3.1}$$

where $\lambda = \lambda(\cdot)$ and $c = c(\cdot)$ denote the heat conduction and the specific heat coefficients, respectively. These coefficients take the values $\lambda_B$, $c_B$ and $\lambda_W$, $c_W$ in the sets $\Omega_B$, $\Omega_W$ occupied by the "black" and "white" constituents, respectively. Remember that $\langle\lambda\rangle = (\lambda_B + \lambda_W)/2$, and denote $[\![\lambda]\!] = (\lambda_B - \lambda_W)/2$. It is known that, using the homogenization approach, from (3.1) we obtain

$$\lambda^h\partial_\alpha\partial_\alpha\theta^0 - \langle c\rangle\dot{\theta}^0 = f \tag{3.2}$$

where

$$\lambda^h = \sqrt{\lambda_B\lambda_W} = \langle\lambda\rangle\sqrt{1 - \frac{[\![\lambda]\!]^2}{\langle\lambda\rangle^2}} \tag{3.3}$$

is the homogenized heat conduction modulus, Jikov et al. (1994). However, this approach neglects the effect of the microstructure size on the overall medium behaviour. The aim of this contribution is to detect this effect by using the tolerance model described in Section 2. Main points of this contribution can be stated as follows:

- To propose a certain approximate solution $\vartheta_{\boldsymbol{x}}(\boldsymbol{y},t) \cong h^A(\boldsymbol{y})V^A(\boldsymbol{x},t)$, $\boldsymbol{y} \in \Delta(\boldsymbol{x})$, to the periodic cell problem (2.3) by a specification of the shape functions $h^A$.
- To compare the obtained tolerance model with the homogenized one.
- To illustrate the effect of microstructure size $l$ by an example of a certain special initial-boundary value problem.

## 4. Tolerance averaged model

Equation (2.2) for the chessboard-type medium takes the form

$$\partial_\alpha[\langle\lambda\rangle\partial_\alpha\theta^0(\boldsymbol{x},t) + \langle\lambda\partial_\alpha\vartheta_{\boldsymbol{x}}\rangle(\boldsymbol{x},t)] - \langle c\rangle\dot{\theta}^0(\boldsymbol{x},t) = f(\boldsymbol{x},t) \tag{4.1}$$

where $\lambda = \lambda_W$, $c = c_W$ in $\Omega_W$ and $\lambda = \lambda_B$, $c = c_B$ in $\Omega_B$. The periodic variational cell problem (2.3) is given by:

*Find* $\vartheta_{\boldsymbol{x}}(\cdot,t) \in H^1_{per}(\Delta)$ *such that* $\langle c\vartheta_{\boldsymbol{x}}\rangle(\boldsymbol{x},t) = 0$ *and*

$$\langle\lambda\partial_\alpha\widetilde{\vartheta}\partial_\alpha\vartheta_{\boldsymbol{x}}\rangle(\boldsymbol{x},t) + \langle c\widetilde{\vartheta}\dot{\vartheta}_{\boldsymbol{x}}\rangle(\boldsymbol{x},t) + \langle\lambda\partial_\alpha\widetilde{\vartheta}\rangle\partial_\alpha\theta^0(\boldsymbol{x},t) = -\langle\widetilde{\vartheta}f\rangle(\boldsymbol{x},t) \tag{4.2}$$

*holds for every* $\Delta-$*periodic field* $\widetilde{\vartheta}(\cdot) \in H^1_{per}(\Delta)$ *satisfying* $\langle c\widetilde{\vartheta}\rangle = 0$.

In this paper an approximate solution $\vartheta_{\boldsymbol{x}}(\cdot,t)$ to the cell problem (4.2) will be assumed in the form

$$\vartheta_{\boldsymbol{x}}(\boldsymbol{y},t) = \psi_\alpha(\boldsymbol{y})v_\alpha(\boldsymbol{x},t) \qquad \boldsymbol{y} \in \Delta(\boldsymbol{x}) \tag{4.3}$$

with $v_\alpha(\boldsymbol{x},t)$ as the unknown amplitudes, and $\Delta$-periodic shape functions $\psi_\alpha$ given in $\overline{\Delta}$ by

$$\psi_1(\boldsymbol{y}) = \Big(\frac{l}{4} - |y_1|\Big)\sin\frac{2\pi y_2}{l} \qquad\qquad \psi_2(\boldsymbol{y}) = \Big(\frac{l}{4} - |y_2|\Big)\sin\frac{2\pi y_1}{l}$$

where $\boldsymbol{y} = (y_1,y_2) \in \langle -l/2, l/2\rangle^2$. The diagrams of these functions are presented in Fig. 2. It means that $v_\alpha = \delta^A_\alpha V^A$, $\psi_\alpha = \delta^A_\alpha h^A$, $A = 1,2$, and hence

$$\theta(\boldsymbol{x},t) \simeq \theta^0(\boldsymbol{x},t) + \psi_\alpha(\boldsymbol{x})v_\alpha(\boldsymbol{x},t) \tag{4.4}$$

Substituting (4.3) into (4.1), (4.2), denoting

$$a = \frac{1}{96} \qquad\qquad b = \frac{2}{\pi} \qquad\qquad d = \frac{1}{2} + \frac{\pi^2}{24}$$

and applying theorems of the tolerance averaging technique, we arrive at the following form of equations (2.4)

$$\begin{aligned} &\langle\lambda\rangle\partial_\alpha\partial_\alpha\theta^0 + b[\![\lambda]\!]\partial_\alpha v_\alpha - \langle c\rangle\dot{\theta}^0 = \langle f\rangle \\ &al^2\langle c\rangle\dot{v}_\gamma + d\langle\lambda\rangle v_\gamma + b[\![\lambda]\!]\partial_\gamma\theta^0 = -\langle\psi_\gamma f\rangle \end{aligned} \tag{4.5}$$

with $\theta^0(\cdot,t)$ and $v_\alpha(\cdot,t)$ as basic unknowns. Hence we arrived at the system of equations (with constant coefficients) for $\theta^0$ and $v_\alpha$. It has to be remembered (cf. Section 2) that these equations have the physical sense only if basic unknowns $\theta^0(\cdot,t)$ and $v_\alpha(\cdot,t)$ are slowly varying, i.e.

$$\theta^0(\cdot,t) \in SV(\mathcal{T}) \qquad\qquad v_\alpha(\cdot,t) \in SV(\mathcal{T}) \tag{4.6}$$

Formulas (4.5) and (4.6) represent the proposed tolerance-averaged model of the chessboard-type medium under consideration.

It has to be mentioned that the chessboard-type medium can be treated as a special case of a medium investigated by Matysiak (1991), where the piecewise linear saw-like periodic shape functions were used. However, the approach in the aforementioned paper applied to the chessboard-type medium leads to uncoupling of the equations for $\theta^0$ and $v_\alpha$, and hence does not describe the effect of microstructure size on the distribution of the averaged temperature field $\theta^0$.

## 5. Applicability of the model

For the asymptotic approach $l \to 0$ equations (4.5) yield the single equation

$$\lambda^0\partial_\alpha\partial_\alpha\theta^0 - \langle c\rangle\dot{\theta}^0 = \langle f\rangle \tag{5.1}$$

where

$$\lambda^0 = \langle\lambda\rangle - \frac{b^2}{d}\frac{[\![\lambda]\!]^2}{\langle\lambda\rangle} \tag{5.2}$$

and $b^2/d \cong 0.45$. Formulae (5.1), (5.2) represent the asymptotic approximation of the tolerance-averaged model proposed in this contribution. By means of the approximate form (4.3) of the solution to the periodic cell problem (4.2), the modulus $\lambda^0$ represents a certain approximation of the known homogenized (effective) modulus $\lambda^h$ given by (3.3). Now we shall compare $\lambda^0$ with $\lambda^h$. To this end we introduce the parameter

$$\xi = \frac{[\![\lambda]\!]}{\langle\lambda\rangle} = \frac{\lambda_W - \lambda_B}{\lambda_W + \lambda_B} \qquad -1 < \xi < 1 \tag{5.3}$$

which can be treated as a certain measure of inhomogeneity of the chessboard-type medium under consideration. Under the above notation, the exact value of the effective modulus $\lambda^h$, derived from homogenization, and the approximate value $\lambda^0$, resulting from the asymptotic approximation of the tolerance-averaged model, can be represented by

$$\lambda^h = \langle\lambda\rangle\sqrt{1-\xi^2} \qquad \lambda^0 = \langle\lambda\rangle(1 - 0.44\xi^2) \tag{5.4}$$

respectively. The diagrams of functions $\lambda^h/\langle\lambda\rangle$ and $\lambda^0/\langle\lambda\rangle$ are shown in Fig. 2.

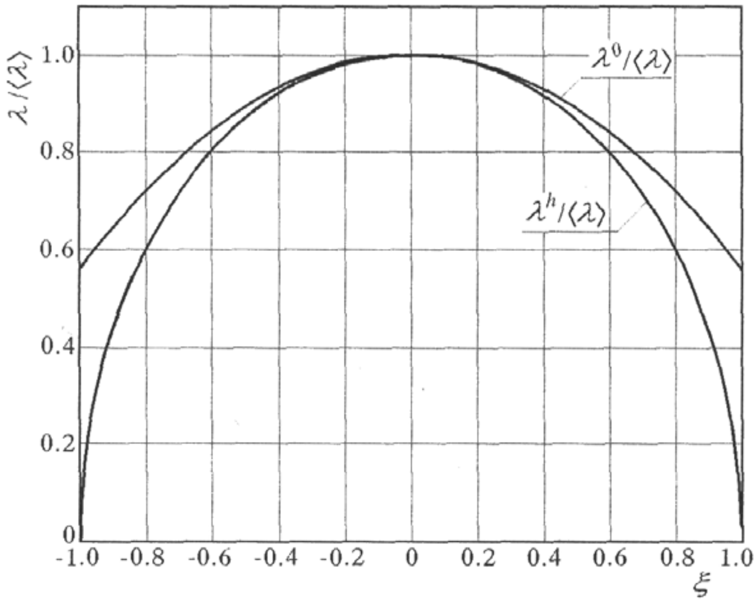


Fig. 2. Diagrams of functions $\lambda^0/\langle\lambda\rangle = 1 - 0.44\xi^2$, $\lambda^h/\langle\lambda\rangle = \sqrt{1-\xi^2}$

To compare the exact effective modulae $\lambda^h$ and its approximation $\lambda^0$ for different values of $\xi$, we introduce the relative error parameter $\delta$ and the inhomogeneity ratio $\eta$ defined by

$$\delta = \frac{\lambda^0 - \lambda^h}{\lambda^h} \qquad \eta = \frac{\lambda_B}{\lambda_W} \qquad 0 < \eta < \infty \tag{5.5}$$

It can be shown that

$$\delta = \frac{(1+\eta)^2 - 0.44(1-\eta)^2}{\sqrt{4\eta}(1+\eta)} - 1 \qquad \delta(\eta) = \delta\Big(\frac{1}{\eta}\Big) \tag{5.6}$$

where the diagram of $\delta(\cdot)$ is presented in Fig. 3.

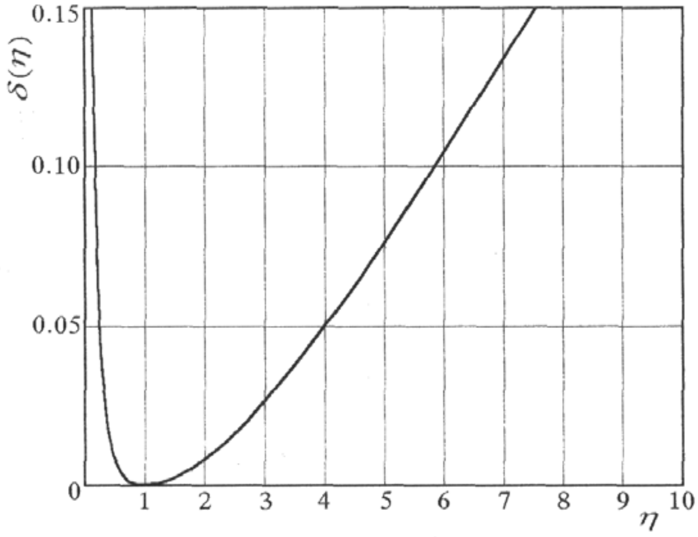


Fig. 3. Diagram of function $\delta(\eta)$

Thus, by means of (5.6) we finally conclude that the proposed model can be applied only if the ratio $\eta = \lambda_B/\lambda_W$ is not too large (if $0.25 \leqslant \eta \leqslant 4$ then $\delta < 5\%$). This is the necessary condition for applicability of the model equations (4.5) proposed in this contribution.

## 6. Illustrative example

Let us consider a one-dimensional initial-boundary problem for (4.5) and neglect the heat sources in all the subsequent analysis. Setting $\theta^0 = \theta^0(x,t)$, $v_\gamma = v_\gamma(x,t)$, and denoting $\partial = \partial/\partial_1$, where $x = x_1$, from (4.5) we obtain the system of partial differential equations for $\theta^0 = \theta^0(x,t)$ and $v_1 = v_1(x,t)$

$$\begin{gathered}\langle\lambda\rangle\partial^2\theta^0 + b[\![\lambda]\!]\partial v_1 - \langle c\rangle\dot{\theta}^0 = 0 \\ al^2\langle c\rangle\dot{v}_1 + d\langle\lambda\rangle v_1 + b[\![\lambda]\!]\partial\theta^0 = 0\end{gathered} \tag{6.1}$$

and the first order ordinary differential equation for $v_2 = v_2(x,t)$

$$al^2\langle c\rangle\dot{v}_2 + d\langle\lambda\rangle v_2 = 0 \tag{6.2}$$

Let $x \in < -L, L >$ and $t > 0$. Let us also assume that $\psi_1(\pm L, x_2) = 0$ for every $x_2$. We postulate the following initial conditions

$$\begin{aligned} &\theta^0(x,0) = A_0 \cos\frac{\pi x}{2L} \\ &v_1(x,0) = 0 \qquad\qquad v_2(x,0) = 0 \end{aligned} \tag{6.3}$$

and the boundary conditions

$$\theta^0(-L,t) = \theta^0(L,t) = 0 \qquad\qquad t > 0 \tag{6.4}$$

Let us observe that from $(6.3)_3$ and (6.2) we obtain $v_2(x,t) = 0$ for every $x \in \langle -L, L\rangle$ and $t > 0$. Hence, formula (4.4) yields

$$\theta(x_1,x_2,t) \simeq \theta^0(x,t) + \psi_1(x_1,x_2)v_1(x,t) \qquad\qquad x = x_1 \tag{6.5}$$

From (6.4) it follows that the boundary conditions for $\theta$ have the form $\theta(\pm L,t) \simeq 0$ and the initial condition for $\theta$, by means of (6.3), is given by $\theta(x_1,x_2,,0) \simeq A_0\cos[\pi x_1/(2L)]$.

We shall look for the solution to the equations (6.1) in the form

$$\theta^0(x,t) = \overline{\theta}(x)\mathrm{e}^{-\gamma t} \qquad\qquad v_1(x,t) = \overline{v}(x)\mathrm{e}^{-\gamma t} \tag{6.6}$$

where $\gamma$ is a certain positive number. Substituting (6.6) to (6.1) we obtain

$$\begin{aligned} &\langle\lambda\rangle\partial^2\overline{\theta} + b[\![\lambda]\!]\partial\overline{v} + \gamma\langle c\rangle\overline{\theta} = 0 \\ &(d\langle\lambda\rangle - \gamma a l^2\langle c\rangle)\overline{v} + b[\![\lambda]\!]\partial\overline{\theta} = 0 \end{aligned} \tag{6.7}$$

Hence

$$\overline{v} = -\frac{b[\![\lambda]\!]}{d\langle\lambda\rangle - \gamma a l^2\langle c\rangle}\partial\overline{\theta} \tag{6.8}$$

Eliminating $\overline{v}$ from (6.7) we obtain the second order ordinary differential equation for $\overline{\theta}$

$$\frac{d\lambda^0 - \gamma a l^2\langle c\rangle}{d\langle\lambda\rangle - \gamma a l^2\langle c\rangle}\partial^2\overline{\theta} + \gamma\frac{\langle c\rangle}{\langle\lambda\rangle}\overline{\theta} = 0 \tag{6.9}$$

In order to satisfy (6.4) we assume $\overline{\theta}(x) = \cos[\pi x/(2L)]$. Let us define $\varepsilon = l/L$. Then by means of (6.8) and (6.9) we conclude that

$$\begin{aligned} &\overline{\theta}(x) = \cos\frac{\pi x}{2L} \\ &\overline{v}(x) = \frac{\pi}{2L}\frac{b[\![\lambda]\!]}{d\langle\lambda\rangle - \gamma a l^2\langle c\rangle}\sin\frac{\pi x}{2L} \end{aligned} \tag{6.10}$$

is a certain solution to the system (6.7) provided that the following condition for $\gamma$ holds

$$a\varepsilon^2\gamma^2 - \frac{4d+\pi^2 a\varepsilon^2}{4L^2}\frac{\langle\lambda\rangle}{\langle c\rangle}\gamma + \frac{\pi^2 d}{aL^4}\frac{\langle\lambda\rangle\lambda^0}{\langle c\rangle^2} = 0 \tag{6.11}$$

Taking into account the conditions (4.6) we conclude that functions $\theta^0(\cdot,t)$, $v_1(\cdot,t)$ are slowly varying only if $l \ll L$. Thus Eqs (6.6) with $\gamma$ given by (6.11) have a physical sense only under the condition $\varepsilon \ll 1$. Hence, bearing in mind that $\varepsilon$ is a small parameter, we obtain from (6.11) the following asymptotic formulae for $\gamma$

$$\begin{aligned}
\gamma_1 &= \frac{\pi^2}{4L^2}\frac{\lambda^0}{\langle c\rangle}\Big(1 - \frac{\pi^2\varepsilon^2}{4}\frac{ab^2}{d^2}\frac{[\![\lambda]\!]^2}{\langle\lambda\rangle^2}\Big) + O(\varepsilon^4) \\
\gamma_2 &= \frac{d}{aL^2\varepsilon^2}\frac{\langle\lambda\rangle}{\langle c\rangle} + \frac{\pi^2 b^2}{4L^2 d}\frac{[\![\lambda]\!]^2}{\langle c\rangle\langle\lambda\rangle} + \frac{ad\pi^2\varepsilon^2}{d^2}\Big(1 - \frac{b^2[\![\lambda]\!]^2}{d\langle\lambda\rangle^2}\Big) + O(\varepsilon^4)
\end{aligned} \tag{6.12}$$

Combining (6.6), (6.10) and bearing in mind (6.12), we obtain the following general solution to (6.1) satisfying boundary conditions (6.4)

$$\begin{aligned}
\theta^0(x,t) &= \Big(A\mathrm{e}^{-\gamma_1 t} + \overline{A}\mathrm{e}^{-\gamma_2 t}\Big)\cos\frac{\pi x}{2L} \\
v_1(x,t) &= \frac{b[\![\lambda]\!]\pi}{2L}\Big(\frac{A}{d\langle\lambda\rangle - \gamma_1 a l^2\langle c\rangle}\mathrm{e}^{-\gamma_1 t} + \frac{\overline{A}}{d\langle\lambda\rangle - \gamma_2 a l^2\langle c\rangle}\mathrm{e}^{-\gamma_2 t}\Big)\sin\frac{\pi x}{2L}
\end{aligned} \tag{6.13}$$

with $A$ and $\overline{A}$ as arbitrary constants. Bearing in mind the initial conditions $(6.3)_{1,2}$ we obtain from (6.13)

$$\begin{aligned}
\theta^0(x,t) &= \frac{A_0}{al^2\langle c\rangle}\Big(\frac{d\langle\lambda\rangle - \gamma_1 a l^2\langle c\rangle}{\gamma_2-\gamma_1}\mathrm{e}^{-\gamma_1 t} + \frac{d\langle\lambda\rangle - \gamma_2 a l^2\langle c\rangle}{\gamma_1-\gamma_2}\mathrm{e}^{-\gamma_2 t}\Big)\cos\frac{\pi x}{2L} \\
v_1(x,t) &= \frac{A_0\pi}{al^2\langle c\rangle(\gamma_2-\gamma_1)}\Big(\mathrm{e}^{-\gamma_1 t} - \mathrm{e}^{-\gamma_2 t}\Big)\frac{b[\![\lambda]\!]}{2L}\sin\frac{\pi x}{2L}
\end{aligned} \tag{6.14}$$

The above formulae together with $v_2(x,t) = 0$ represent the solution to the initial-boundary value problem represented by (6.1)-(6.4). Neglecting all higher order terms in (6.12), we obtain

$$\gamma_1 \cong \frac{\pi^2}{4L^2}\frac{\lambda^0}{\langle c\rangle} \qquad \gamma_2 \cong \frac{d\langle\lambda\rangle}{al^2\langle c\rangle}$$

and hence (6.14) can be also rewritten in the form

$$
\begin{aligned}
\theta^0(x,t) &\cong A_0 \exp\Big(-\frac{\pi^2\lambda^0}{4L^2\langle c\rangle}t\Big)\cos\frac{\pi x}{2L} \\
v_1(x,t) &\cong A_0 \frac{\pi b[\![\lambda]\!]}{2Ld\langle\lambda\rangle}\Big[\exp\Big(-\frac{\pi^2\lambda^0}{4L^2\langle c\rangle}t\Big) - \exp\Big(-\frac{d\langle\lambda\rangle}{al^2\langle c\rangle}t\Big)\Big]\sin\frac{\pi x}{2L}
\end{aligned}
\tag{6.15}
$$

Substituting the right-hand sides of (6.15) into (6.5) we obtain the distribution of temperature $\theta(x_1, x_2, t)$ in the problem under consideration.

In order to investigate the length size effect on the heat transfer, we shall pass to the analysis of the initial-boundary value problem under consideration in the asymptotic approximation. Applying the limit passage $l \to 0$ in Eqs (6.1), we obtain

$$
\begin{aligned}
&\langle\lambda\rangle\partial^2\theta^0 + b[\![\lambda]\!]\partial v_1 - \langle c\rangle\dot{\theta}^0 = 0 \\
&d\langle\lambda\rangle v_1 + b[\![\lambda]\!]\partial\theta^0 = 0
\end{aligned}
\tag{6.16}
$$

The same limit passage applied to (6.2) yields $v_2 = 0$. From (6.16) it follows that

$$
v_1 = -\frac{b[\![\lambda]\!]}{d\langle\lambda\rangle}\partial\theta^0 \tag{6.17}
$$

and eliminating $v_1$ from the system (6.16) one can obtain

$$
\lambda^0\partial^2\theta^0 - \langle c\rangle\dot{\theta}^0 = 0 \tag{6.18}
$$

where $\lambda^0$ is given by (5.2). Similar equation

$$
\lambda^h\partial^2\theta^0 - \langle c\rangle\dot{\theta}^0 = 0
$$

we also obtain in the framework of homogenization. We look for the solution to (6.18) in the form

$$
\theta^0(x,t) = \overline{\theta}\mathrm{e}^{-\gamma t} \tag{6.19}
$$

Substituting (6.19) into (6.18) we obtain

$$
\lambda^0\partial^2\overline{\theta} + \gamma\langle c\rangle\overline{\theta} = 0 \tag{6.20}
$$

This equations has the solution $\overline{\theta}(x) = A_0\cos[\pi x/(2L)]$ for which the averaged temperature $\theta^0(x,t) = A_0\exp(-\gamma t)\cos[\pi x/(2L)]$ satisfies the initial condition $(6.3)_1$ as well as the boundary conditions (6.4), provided that

$$
\gamma = \frac{\pi^2\lambda^0}{4L^2\langle c\rangle}
$$

At the same time (6.20) yields

$$v_1(x,t) = -\frac{\pi b[\![\lambda]\!]A_0}{2Ld\langle\lambda\rangle}\mathrm{e}^{-\gamma t}\sin\frac{\pi x}{2L}$$

Thus we conclude that after neglecting the length-scale effect ($l \to 0$) the initial condition $v_1(x,0) = 0$ cannot be satisfied.

## 7. Conclusions

Summarizing the results obtained in this contribution it is possible to formulate the following conclusions.

- By means of (4.5) the proposed averaged model of the heat transfer in chessboard-type media is isotropic and depends on the microstructure size $l$.
- The proposed model can be applied only if the ratio $\lambda_B/\lambda_W$ is not too large (if $0.25 \leqslant \lambda_B/\lambda_W \leqslant 4$ then $\delta < 5\%$).
- The proposed model can be used to the analysis of initial boundary value problems for the temperature $\theta$, in contrast to the homogenized (asymptotic) model, where only initial conditions for the averaged temperature $\theta^0$ can be satisfied.

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## Makroskopowy model przewodnictwa ciepła w niejednorodnym ośrodku typu szachownicy

### Streszczenie

Celem pracy jest opracowanie, dyskusja i zastosowanie makroskopowego modelu przewodnictwa ciepła w przewodniku posiadającym periodyczną mikroniejednorodną strukturę typu szachownicy. Stosując technikę tolerancyjnego uśredniania zaproponowano pewne przybliżone rozwiązanie zagadnienia na komórce. Rozwiązanie to prowadzi do zagadnienia początkowo-brzegowego dla uśrednionego pola temperatury i sprzężonego z tym problemem zagadnienia początkowego dla pola wektorowego zmiennych

wewnętrznych. W przeciwieństwie do homogenizacji otrzymany model opisuje wpływ wymiaru charakterystycznego komórki periodyczności na makroskopowe właściwości ciała. Wykazano, że zaproponowany model ma sens fizyczny, jeżeli niejednorodność ośrodka typu szachownicy jest niezbyt duża.

*Manuscript received July 11, 2001; accepted for print October 8, 2001*