JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
1, 40, 2002

ON CONTINUUM MODELLING OF THE DYNAMIC BEHAVIOUR OF PERIODIC LATTICE-TYPE PLATES WITH A COMPLEX STRUCTURE

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A new continuum model for studying the dynamic problems of periodic elastic lattice-type plates of an arbitrary lay-out is proposed. The general line of approach is partly based on the tolerance averaging techniques developed by Woźniak and Wierzbicki (2000) for the termomechanics of composite solids and applied by Cielecka et al. (1998, 2000) to the modelling of dense cellular structures. The proposed model describes the microstructure length-scale effect on the dynamic plate behaviour. The obtained equations are applied to the analysis of wave propagation in a special latticed plate. It is shown that the length-scale effect plays an important role and cannot be neglected in the above analysis.

Key words: periodic lattice-type plate, length-scale effect, tolerance averaging

Notations

Subscripts i, j, k, l run over 1,2 and are related to Cartesian orthogonal coordinates x_1, x_2 in the $0x_1x_2$-plane. Indices a and A run over $1, ..., n$ and $1, ..., N$, respectively; indices α, β take the values $1, ..., n-1$. Summation convention holds for all aforementioned indices unless otherwise stated. Points on the $0x_1x_2$-plane are denoted by $\boldsymbol{x} = (x_1, x_2)$ and t is the time coordinate.

1. Introduction

In the paper by Cielecka et al. (1998), a new approach to the analysis of the in-plane dynamic problems for periodic cellular media was proposed and applied. In this paper we deal with the formulation and application of a continuum model to study the linearised elastodynamics for lattice-type plates having an arbitrary complex periodic lay-out in $0x_1x_2$-plane; two examples of this lay-out are shown in Fig. 1. It is assumed that the length dimensions of a representative cell of the periodic structure are small compared to the minimum characteristic length dimension of the whole latticed plate, and that the mass distribution in this plate can be approximated by assigning concentrated masses and inertia moments to every nodal joint of the lattice. Hence the lattice-type plate under consideration is represented by a certain plane periodic system of mutually interacting rigid joints.

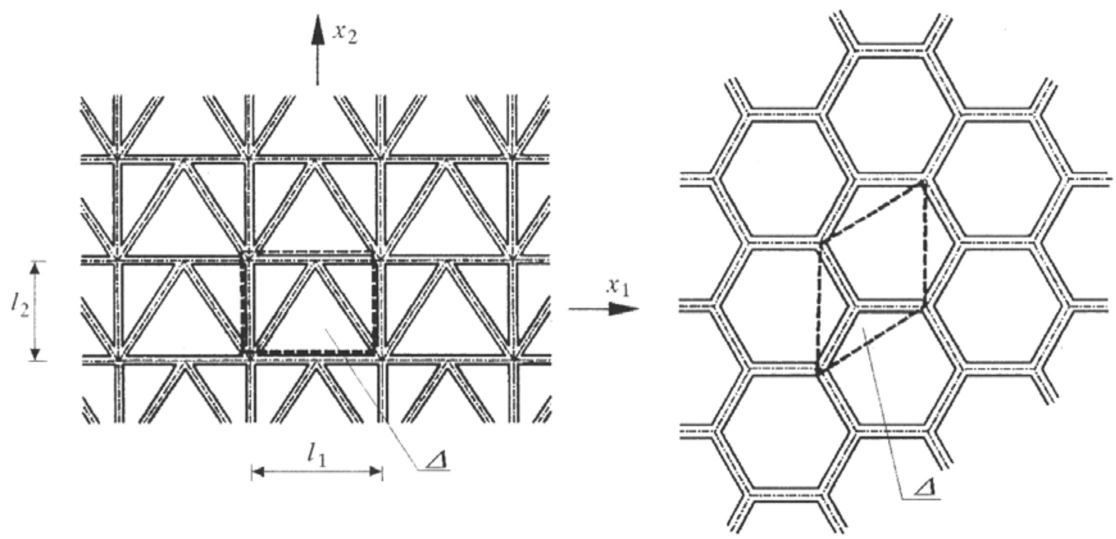

Fig. 1. Examples of lattice-type periodic plates

It is known that the direct approach to dynamics of periodic systems with a very large number of interacting rigid bodies leads to computational difficulties due to a large number of ordinary differential equations describing the problem under consideration. That is why different averaged continuum models have been proposed in order to reduce the number of basic unknowns and to simplify the analysis of particular problems. From many results obtained in this manner, let us mention those related to the frame-type lattice structures, summarised in Woźniak (1970) and the modelling procedures applied in Dow et al. (1985), Gibson et al. (1982), Horvay (1952). More sophisticated modelling approach, based on the asymptotic procedures of the homogenization theory in Bakhvalov and Panasenko (1984), Jikov et al. (1994), leads to the formulation of continuum models for periodic structures but neglects the effect of the unit cell size on the global behaviour of discrete system, because in the course of modelling all length dimensions of every unit cell tend to zero and the number of cells tends to infinity, Bakhvalov and Panasenko (1984), cf. also Cioranescu and Saint Paulin (1991). The continuum modelling of in-plane problems was considered in papers Lewiński (1984a,b, 1985, 1988), where hexagonal gridworks were investigated and Rogula-Kunin's approach was applied as a tool of modelling, cf. Kunin (1975).

Alternative continuum models of periodic trusses and plane cellular media were formulated in Cielecka (1995) and Cielecka et al. (1998). These models were applied to analyse honeycomb cellular media in Cielecka et al. (2000). The line of approach belongs to a new averaging technique in dynamics of periodic composites and structures, which introduces into the modelling procedure the concept of *tolerance* related to the accuracy of the performed manipulations. The aforesaid concept leads to the so-called *kinematic internal variables*, Woźniak and Wierzbicki (2000). These variables, being additional unknown functions, have described microdynamic phenomena and their effects on the global composite body behaviour. The characteristic feature of these functions is that they are governed by a system of ordinary differential equations (called the dynamic evolution equations) involving only their second order time derivatives, and hence they have not entered the boundary conditions. From a formal viewpoint, governing equations of the obtained models (cf. Cielecka, 1995; Cielecka et al., 1998, 2000) were similar to the Cosserat media equations of motion but, contrary to the Cosserat media, we have also dealt with the dynamic evolution equations. The averaging techniques based on the notion of *tolerance* were applied to analyse dynamic problems for different periodic structures, such as Reissner-type periodic plates (Baron and Woźniak, 1995), Kirchhoff-type periodic plates (Jędrysiak, 2000), wavy-type periodic plates (Michalak, 2000). These techniques applied to periodic composites is called *the tolerance averaging method* (see Woźniak and Wierzbicki, 2000).

The model proposed in this paper is also based on the concept of *tolerance*, which leads to kinematic internal variables, cf. Woźniak and Wierzbicki (2000), used to describe dynamics of lattice-type plates of an arbitrary complex lay-out and its foundations show a certain similarity to those explored in Cielecka et al. (1998). For the lattice-type plate under consideration, represented by a certain plane periodic system of mutually interacting rigid joints, it is assumed that the length dimension in $0x_1x_2$-plane of every rigid nodal joint are negligibly small as compared with the spans of interconnecting beams which can be bent and twisted. The obtained continuum model involves the effect of size of the representative periodicity cell on the global behaviour of plate under consideration. The proposed model is useful to the analysis of the long wave problems. In this paper the governing equations of the model are also applied to the analysis of free vibrations of the rectangular lattice structure.

2. Preliminaries

Let Δ be a parallelogram on the $0x_1x_2$-plane which constitutes a cell representative for a whole periodic lattice, cf. Fig. 1. It means that Δ contains the representative structural element for the lattice-type plate. In general the representative element can include one, two or several periodicity cells. The choice of this element is not unique and depends on the class of motions we are to investigate. It is assumed that the undeformed representative element is made of N prismatic linear-elastic beams B^A, $A = 1, ..., N$ axes of which are situated on the plane $0x_1x_2$. The beams B^A in the representative cell are interconnected by n rigid joints j^a, $a = 1, .., n$, with centers at points $\boldsymbol{x} = \boldsymbol{x}^a = (x_1^a, x_2^a)$ on the plane $0x_1x_2$. It is assumed that $0x_1x_2$ is a symmetry plane, both for every beam and every rigid joint treated as certain spatial (3-dimensional) elements. The beams are subjected to bending and torsion in the planes perpendicular to $0x_1x_2$-plane and the rigid joints rotate in the aforementioned planes, and their centers displace in the direction normal to $0x_1x_2$-plane.

By Ω we define a region on $0x_1x_2$-plane obtained as an interior of a union of all closures of repeated cells. It has to be remembered that the periodic structure of the whole lattice-type plate can be disturbed in the structural elements situated near the boundary $\partial\Omega$ of Ω. Denoting by L the smallest characteristic length dimension of Ω and by l the diameter of a cell Δ, it will be assumed that $l/L \ll 1$. This is why l will be referred to as the microstructure length parameter of the lattice-type plate.

Significant properties of a beam B^A will be given by the flexural stiffness E^AI^A, the torsional stiffness $G^AI_0^A$, the span l^A, the mass density ρ^A and the cross-section area F^A. The mass of a beam B^A will be represented by two equal concentrated masses assigned to the joints situated at the ends of a beam. The total concentrated mass assigned to a joint j^a will be denoted by M^a and given by

$$M^a = \frac{1}{2}\sum_{A=1}^{N_a} \rho^A F^A l^A \tag{2.1}$$

where N_a is the number of beams for which a joint j^a is an end. The rotational moment of inertia of a joint j^a will be represented by the second order tensor J^a_{ij}. To every beam B^A we shall assign unit vectors $\boldsymbol{t}^A$, $\boldsymbol{n}^A$ shown in Fig. 2. Describing the kinetic energy of a beam by velocities of displacements and rotations of its ends, the terms of tensor J^a_{ij} can be taken in the form

$$J^a_{ij} = \frac{1}{2}\sum_{A=1}^{N_a} \rho^A l^A (I^A n^A_i n^A_j + I^A_0 t^A_i t^A_j) \tag{2.2}$$

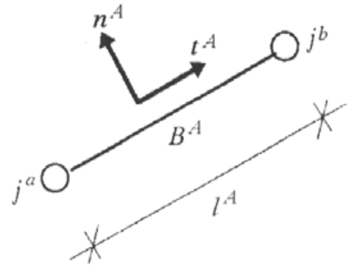

Fig. 2. Unit vectors of the beam

Let us denote by w^a a displacement (deflection) of the joint j^a in the direction of x_3-axis and by φ^a_n and φ^a_t – the rotations of j^a in the planes normal to $\boldsymbol{t}^A$, $\boldsymbol{n}^A$, respectively. Assuming that joints j^a and j^b are interconnected by a beam B^A, denote

$$\begin{aligned} \Delta_A w &:= \frac{w^b - w^a}{l^A} \qquad & \varphi_{An} &:= \frac{1}{2}(\varphi^a_n + \varphi^b_n) \\ \Delta_A \varphi_n &:= \varphi^b_n - \varphi^a_n \qquad & \Delta_A \varphi_t &:= \varphi^b_t - \varphi^a_t \end{aligned} \tag{2.3}$$

For the sake of simplicity let us also assume that every beam B^A can be considered in the framework of the Euler-Bernoulli beam theory. Then the strain components related to B^A can be taken in the form (no summation over A in formulae (2.4)-(2.6))

$$\widetilde{\varepsilon}^A := \Delta_A w + \varphi_{An} \qquad \kappa^A := \Delta_A \varphi_n \qquad \widetilde{\kappa}^A := \Delta_A \varphi_t \tag{2.4}$$

Hence, using additional notations

$$\widetilde{\Lambda}^A := 12E^A I^A (l^A)^{-1} \qquad K^A := E^A I^A (l^A)^{-1} \qquad \widetilde{K}^A := G^A I^A_0 (l^A)^{-1} \tag{2.5}$$

the strain energy σ^A assigned to a beam B^A is equal to

$$\sigma^A = \frac{1}{2}\widetilde{\Lambda}^A(\widetilde{\varepsilon}^A)^2 + \frac{1}{2}K^A(\kappa^A)^2 + \frac{1}{2}\widetilde{K}^A(\widetilde{\kappa}^A)^2 \tag{2.6}$$

It has to be remembered that all the aforementioned notations and formulae are related to an arbitrary but fixed repeated element of the periodic lattice-type plate under consideration (possibly except some elements situated near the boundary $\partial\Omega$ of Ω).

Let us denote by $\mathcal{L}$ the set of all periodically spaced points on the plane $0x_1x_2$ which are centers of all mutually disjoint cells constituting the region Ω. Then the deflection and rotation vector of the joint j^a belonging to a cell with the center $\boldsymbol{z}$, $\boldsymbol{z} \in \mathcal{L}$, at an arbitrary instant t, will be denoted by $w^a(\boldsymbol{z},t)$, $\varphi^a(\boldsymbol{z},t)$, respectively. All external loads acting on the medium are assumed to be applied exclusively to the centers of rigid joints. The resultant external force and external couples applied to the joint j^a in a cell with a center $\boldsymbol{z} \in \mathcal{L}$ will be denoted by $f^a(\boldsymbol{z},t)$ and $\boldsymbol{m}^a(\boldsymbol{z},t)$, respectively. Introducing the action functional $\mathcal{A} = \mathcal{E} - \mathcal{K} - \mathcal{W}$ where

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2}\sum_{\boldsymbol{z}\in\mathcal{L}}\sum_{A=1}^{n}\Big[\widetilde{\Lambda}^A\Big(\widetilde{\varepsilon}^A(\boldsymbol{z},t)\Big)^2 + K^A\Big(\kappa^A(\boldsymbol{z},t)\Big)^2 + \widetilde{K}^A\Big(\widetilde{\kappa}^A(\boldsymbol{z},t)\Big)^2\Big] \\
\mathcal{K} &= \frac{1}{2}\sum_{\boldsymbol{z}\in\mathcal{L}}\sum_{A=1}^{n}\Big[M^a\Big(\dot{w}^a(\boldsymbol{z},t)\Big)^2 + J^a_{ij}\dot{\varphi}^a_i(\boldsymbol{z},t)\dot{\varphi}^a_j(\boldsymbol{z},t)\Big] \\
\mathcal{W} &= \sum_{\boldsymbol{z}\in\mathcal{L}}\sum_{A=1}^{n}\Big[f^a(\boldsymbol{z},t)w^a(\boldsymbol{z},t) + m^a_i(\boldsymbol{z},t)\varphi^a_i(\boldsymbol{z},t)\Big]
\end{aligned} \tag{2.7}$$

and taking into account formulae (2.3), (2.4), from the principle of stationary action we derive equations of motion for $w^a(\boldsymbol{z},t)$, $\varphi^a_i(\boldsymbol{z},t)$, $\boldsymbol{z} \in \mathcal{L}$, $a = 1,...,n$, $i = 1,2$. These equations represent a discrete model of a periodic lattice-type plate but are not convenient in investigations of its global dynamic behaviour since the number of points $\mathcal{L}$ is very large. That is why relations (2.3), (2.4), (2.7) together with the assumptions formulated in Section 3 will be treated only as a basis for deriving a continuum model of the lattice-type plate under consideration.

3. Modelling assumptions

In order to pass from the discrete model of the periodic lattice-type plate under consideration to a certain refined continuum model, we have to recall the concept of a *slowly varying function*, Woźniak and Wierzbicki (2000). Let $F(\cdot, t)$ be a sufficiently regular real-valued function defined on Ω and depending on time t, and ε_F be a positive number determining the accuracy of calculations of the values of F. Assume that the values of $F(\cdot, t)$ for every t and every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ such that $\|\boldsymbol{x} - \boldsymbol{y}\| \leqslant l$ satisfy conditions $|F(\boldsymbol{x}, t) - F(\boldsymbol{y}, t)| \leqslant \varepsilon_F$. If similar conditions hold also for all derivatives of F (including time-derivatives) then $F(\cdot, t)$ will be called a *regular slowly varying function* (related to the microstructure length parameter l and to certain accuracy parameters $\varepsilon_F, \varepsilon_{\nabla F}, \varepsilon_{F}, \ldots$).

In the course of modelling we shall assume two hypotheses. To this end define $\nu := n - 1$ and let $h^{a\alpha}$, $g_{ij}^{a\alpha}$, $\alpha = 1, \ldots, \nu$, $i, j = 1, 2$, be the systems of real numbers satisfying conditions

$$\sum_{a=1}^{n} M^a h^{a\alpha} = 0 \qquad \sum_{a=1}^{n} J_{kl}^a g_{ki}^{a\alpha} = 0 \qquad \begin{array}{l} \alpha = 1, \ldots, \nu \\ i, k, l = 1, 2 \end{array} \tag{3.1}$$

It has to be emphasised that the systems $h^{a\alpha}$, $g_{ij}^{a\alpha}$, $a = 1, \ldots, n$, $\alpha = 1, \ldots, \nu$, are not uniquely determined but their choice will be irrelevant.

The first modelling hypothesis interrelates the deflection $w^a(\boldsymbol{z}, t)$ and the rotations $\varphi_i^a(\boldsymbol{z}, t)$ of the joint j^a in a cell with the center $\boldsymbol{z}$, $\boldsymbol{z} \in \mathcal{L}$, with certain regular slowly varying functions $W(\cdot, t)$, $Q^\alpha(\cdot, t)$, $\Phi_i(\cdot, t)$, $R_i^\alpha(\cdot, t)$ which will be treated as basic kinematic unknowns. This statement is called the kinematic hypothesis which will be assumed in the form

$$\begin{aligned} w^a(\boldsymbol{z}, t) &= W(\boldsymbol{x}^a, t) + l h^{a\alpha} Q^\alpha(\boldsymbol{x}^a, t) \\ \varphi_i^a(\boldsymbol{z}, t) &= \Phi_i(\boldsymbol{x}^a, t) + l g_{ij}^{a\alpha} R_j^\alpha(\boldsymbol{x}^a, t) \qquad \boldsymbol{z} \in \mathcal{L} \end{aligned} \tag{3.2}$$

where $\boldsymbol{x}^a$ is the position vector of the joint j^a, $a = 1, \ldots, n$. Because of $|W(\boldsymbol{x}^a, t) - W(\boldsymbol{z}, t)| \leqslant \varepsilon_W$, $|\Phi_i(\boldsymbol{x}^a, t) - \Phi_i(\boldsymbol{z}, t)| \leqslant \varepsilon_{\Phi_i}$ etc. and bearing in mind conditions (3.1) imposed on $h^{a\alpha}$, $g_{ij}^{a\alpha}$, under notations $M = \sum_{a=1}^n M^a$, $J_{ik} = \sum_{a=1}^n J_{ik}^a$ and neglecting terms involving ε_W, ε_{Φ_i}, we obtain

$$\begin{aligned} W(\boldsymbol{z}, t) &= M^{-1} \sum_{a=1}^{n} M^a w^a(\boldsymbol{z}, t) \\ \Phi_i(\boldsymbol{z}, t) &= J_{ik}^{-1} \sum_{a=1}^{n} J_{kl}^a \varphi_l^a(\boldsymbol{z}, t) \qquad \boldsymbol{z} \in \mathcal{L} \end{aligned} \tag{3.3}$$

where $\mathbf{J}^{-1}$ is the inverse of the matrix $\mathbf{J}$ having components J_{ik}. Thus, we conclude that the fields $W(\boldsymbol{z}, t)$, $\Phi_i(\boldsymbol{z}, t)$ represent respectively weighted averaged deflections and rotations of repeated elements of the structure.

Hence $Q^{\alpha}(\boldsymbol{z},t)$, $R_i^{\alpha}(\boldsymbol{z},t)$ describe respectively the disturbances in deflections and rotations at a time t within these elements. Fields Q^{α}, R_i^{α} will be called *kinematic internal variables*; the meaning of this term will be explained below.

The second modelling hypothesis is related to the concept of slowly varying function. On this basis we shall approximate finite differences of these functions by the values of their appropriate derivatives, we shall neglect increments of the introduced functions inside an arbitrary cell in calculation of averages over this cell, and after that we shall approximate finite sums over $\mathcal{L}$ in (2.7) by the integrals over Ω. This statement will be called *the smoothness hypothesis* and it leads to the continuum representation of strain components in an arbitrary beam B^A belonging to a cell with the center $\boldsymbol{z}$. To this end, under assumption that joints j^a, j^b are interconnected by a beam B^A, define $h^{A\alpha} := h^{b\alpha} - h^{a\alpha}$, $g^{\alpha}_{Aij} := 0.5(g^{a\alpha}_{ij} + g^{b\alpha}_{ij})$, $g^{A\alpha}_{ij} := g^{b\alpha}_{ij} - g^{a\alpha}_{ij}$ and $\lambda^A := l/l^A$. Also define

$$\Psi_i(\boldsymbol{x},t) := W_{,i}(\boldsymbol{x},t) + \varepsilon_{ij}\Phi_j(\boldsymbol{x},t) \qquad \boldsymbol{x} \in \Omega \tag{3.4}$$

where ε_{ij} stand for the Ricci symbol. After simple calculations, from (2.4), (3.2) and (3.4) we obtain (no summation over A!)

$$\begin{aligned}
\widetilde{\varepsilon}^A(\boldsymbol{z},t) &= t_i^A\Psi_i(\boldsymbol{z},t) + \lambda^A h^{A\alpha}Q^{\alpha}(\boldsymbol{z},t) + l n_i^A g^{\alpha}_{Aij}R^{\alpha}_j(\boldsymbol{z},t) \\
\kappa^A(\boldsymbol{z},t) &= l^A n_i^A t_j^A \Phi_{i,j}(\boldsymbol{z},t) + l n_i^A g^{A\alpha}_{ij} R^{\alpha}_j(\boldsymbol{z},t) \\
\widetilde{\kappa}^A(\boldsymbol{z},t) &= l^A t_i^A t_j^A \Phi_{i,j}(\boldsymbol{z},t) + l t_i^A g^{A\alpha}_{ij} R^{\alpha}_j(\boldsymbol{z},t) \qquad \boldsymbol{z} \in \mathcal{L}
\end{aligned} \tag{3.5}$$

It has to be emphasised that restrictions imposed on the class of motions under consideration reduce to the requirement that the basic unknown fields $W(\cdot,t)$, $Q^{\alpha}(\cdot,t)$, $\Phi_i(\cdot,t)$, $R_i^{\alpha}(\cdot,t)$ have to be regular, slowly varying functions for every t.

The above two modelling hypotheses have to be supplement by the condition imposed on the external loading on the lattice plate under consideration. Namely, we shall assume that there exist continuous slowly varying functions $f(\cdot,t)$, $f^{\alpha}(\cdot,t)$, $m_i(\cdot,t)$, $m_i^{\alpha}(\cdot,t)$ defined on Ω for every t, such that the conditions

$$\begin{aligned}
f(\boldsymbol{z},t) &= |\Delta|^{-1}\sum_{a=1}^{n} f^a(\boldsymbol{z},t) \qquad & m_i(\boldsymbol{z},t) &= h^{-1}|\Delta|^{-1}\sum_{a=1}^{n} m_i^a(\boldsymbol{z},t) \\
f^{\alpha}(\boldsymbol{z},t) &= |\Delta|^{-1}\sum_{a=1}^{n} f^a(\boldsymbol{z},t)h^{a\alpha} \qquad & m_i^{\alpha}(\boldsymbol{z},t) &= h^{-1}|\Delta|^{-1}\sum_{a=1}^{n} m_i^a(\boldsymbol{z},t)g^{a\alpha}_{ij}
\end{aligned} \tag{3.6}$$

hold for every $\boldsymbol{z} \in \mathcal{L}$; where the cell area is equal to $|\Delta|$.

4. Governing equations of internal variable models

The governing equations for the deflection W, rotations Φ_i and the extra unknowns Q^α, R_i^α will be obtained from the principle of stationary action under two modelling hypotheses mentioned in Section 3. Introduce the notations

$$
\begin{aligned}
A_{ij} &:= |\Delta|^{-1} \sum_{A=1}^{N} \widetilde{\Lambda}^A t_i^A t_j^A \\
C_{ijkl} &:= |\Delta|^{-1} \sum_{A=1}^{N} (l^A)^2 (K^A n_i^A n_k^A + \widetilde{K}^A t_i^A t_k^A) t_j^A t_l^A \\
A^{\alpha\beta} &:= |\Delta|^{-1} \sum_{A=1}^{N} (\lambda^A)^2 \widetilde{\Lambda}^A h^{A\alpha} h^{A\beta} \\
A_{ij}^{\alpha\beta} &:= |\Delta|^{-1} \sum_{A=1}^{N} \Big[\widetilde{\Lambda}^A n_k^A n_l^A g_{Aki}^{\alpha} g_{Alj}^{\beta} + (K^A n_k^A n_l^A + \widetilde{K}^A t_k^A t_l^A) g_{ki}^{A\alpha} g_{lj}^{A\beta} \Big] \\
B_{ijk}^{\alpha} &:= |\Delta|^{-1} \sum_{A=1}^{N} (\lambda^A)^{-1} (K^A n_l^A n_j^A + \widetilde{K}^A t_l^A t_j^A) t_k^A g_{li}^{A\alpha} \\
D_i^{\alpha} &:= |\Delta|^{-1} \sum_{A=1}^{N} \lambda^A \widetilde{\Lambda}^A t_i^A h^{A\alpha} \\
D_{ij}^{\alpha} &:= |\Delta|^{-1} \sum_{A=1}^{N} \widetilde{\Lambda}^A t_i^A n_k^A g_{Akj}^{\alpha} \\
D_i^{\alpha\beta} &:= |\Delta|^{-1} \sum_{A=1}^{N} \lambda^A \widetilde{\Lambda}^A n_k^A h^{A\alpha} g_{Aki}^{\beta} \\
\mu &:= |\Delta|^{-1} \sum_{a=1}^{n} M^a \\
\chi_{ij} &:= h^{-2} |\Delta|^{-1} \sum_{a=1}^{n} J_{ij}^a \\
\mu^{\alpha\beta} &:= |\Delta|^{-1} \sum_{a=1}^{n} M^a h^{a\alpha} h^{a\beta} \\
\chi_{ij}^{\alpha\beta} &:= h^{-2} |\Delta|^{-1} \sum_{a=1}^{n} J_{kl}^a g_{ki}^{a\alpha} g_{lj}^{a\beta}
\end{aligned}
\tag{4.1}
$$

where h stands for mean height of the beams in the direction normal to

$0x_1x_2$-plane. After substituting to (2.7) the right-hand sides of equations (3.2), (3.5), (3.6) and taking into account the smoothness hypothesis (related to calculations of averages), as well as the conditions (3.1) and the notations (4.1), we arrive at the integral form of the action functional $\mathcal{A} = \mathcal{E} - \mathcal{K} - \mathcal{W}$, where now

$$\begin{aligned}
\mathcal{E} &= \int\limits_{\Omega} \Big(\frac{1}{2} A_{ij} \Psi_i \Psi_j + \frac{1}{2} C_{ijkl} \Phi_{i,j} \Phi_{k,l} + \frac{1}{2} A^{\alpha\beta} Q^\alpha Q^\beta + \frac{1}{2} l^2 A_{ij}^{\alpha\beta} R_i^\alpha R_j^\beta \\
&\quad + l^2 B_{ijk}^\alpha R_i^\alpha \Phi_{j,k} + D_i^\alpha \Psi_i Q^\alpha + l D_{ij}^\alpha \Psi_i R_j^\alpha + l D_i^{\alpha\beta} Q^\alpha R_i^\beta \Big) \, dx_1 dx_2 \\
\mathcal{K} &= \int\limits_{\Omega} \Big(\frac{1}{2} \dot{W} \dot{W} + \frac{1}{2} l^2 \mu^{\alpha\beta} \dot{Q}^\alpha \dot{Q}^\beta + \frac{1}{2} h^2 \chi_{ij} \dot{\Phi}_i \dot{\Phi}_j + \frac{1}{2} h^2 l^2 \chi_{ij}^{\alpha\beta} \dot{R}_i^\alpha \dot{R}_j^\beta \Big) \, dx_1 dx_2 \\
\mathcal{W} &= \int\limits_{\Omega} \Big(f W + l f^\alpha Q^\alpha + h m_i \Phi_i + h l m_i R_i^\alpha \Big) \, dx_1 dx_2
\end{aligned} \tag{4.2}$$

From the principle of stationary action we obtain the following equations for deflection W and rotations Φ_i

$$\begin{aligned}
&(A_{ij} \Psi_j + D_i^\alpha Q^\alpha + l D_{ij}^\alpha R_j^\alpha)_{,i} - \mu \ddot{W} + f = 0 \\
&(C_{kijl} \Phi_{j,l} + l^2 B_{jki}^\alpha R_j^\alpha)_{,i} + \varepsilon_{ki} (A_{ij} \Psi_j + D_i^\alpha Q^\alpha + l D_{ij}^\alpha R_j^\alpha) - h^2 \chi_{ki} \ddot{\Phi}_i + h m_k = 0
\end{aligned} \tag{4.3}$$

which are coupled with the following equations for the extra unknowns Q^α, R_i^α

$$\begin{aligned}
&l^2 \mu^{\alpha\beta} \ddot{Q}^\beta + A^{\alpha\beta} Q^\beta + D_i^\alpha \Psi_i + l D_i^{\alpha\beta} R_i^\beta = l f^\alpha \\
&h^2 l^2 \chi_{ij}^{\alpha\beta} \ddot{R}_j^\beta + l^2 A_{ij}^{\alpha\beta} R_j^\beta + l D_{ji}^\alpha \Psi_j + l^2 B_{ijl}^\alpha \Phi_{j,l} + l D_i^{\beta\alpha} Q^\beta = h l m_i^\alpha
\end{aligned} \tag{4.4}$$

here Ψ_j is defined by Eq. (3.4). The obtained equations have to be satisfied for every t in the region Ω of $0x_1x_2$ and represent a continuum model of the periodic lattice-type plate under consideration.

It can be seen that the extra unknowns Q^α, R_i^α are governed by the ordinary differential equations (4.4). Hence in general, Q^α, R_i^α do not enter the boundary conditions and that is why they have been called *kinematic internal variables*. Similarly, the obtained continuum model will be referred to as *the internal variable model* (*IV*-model).

The governing equations (4.3)-(4.4) can be also written in the equivalent form:

— constitutive equations

$$\begin{bmatrix} P_i \\ M_{ki} \\ S^\alpha \\ H_i^\alpha \end{bmatrix} = \begin{bmatrix} A_{ij} & 0 & D_i^\beta & lD_{ij}^\beta \\ 0 & C_{kijl} & 0 & l^2 B_{jki}^\beta \\ D_i^\alpha & 0 & A^{\alpha\beta} & lD_j^{\alpha\beta} \\ lD_{ji}^\alpha & l^2 B_{ijl}^\alpha & lD_i^{\beta\alpha} & l^2 A_{ij}^{\alpha\beta} \end{bmatrix} \begin{bmatrix} \Psi_j \\ \Phi_{j,l} \\ Q^\beta \\ R_j^\beta \end{bmatrix} \tag{4.5}$$

$$\Psi_i = W_{,i} + \varepsilon_{ij}\Phi_j$$

— equations of motion

$$\begin{aligned} &P_{i,i} - \mu\ddot{W} + f = 0 \\ &M_{ki,i} + \varepsilon_{ki}P_i - h^2\chi_{ki}\ddot{\Phi}_i + hm_k = 0 \end{aligned} \tag{4.6}$$

— dynamic evolution equations

$$\begin{aligned} &l^2\mu^{\alpha\beta}\ddot{Q}^\beta + S^\alpha = lf^\alpha \\ &h^2 l^2 \chi_{ij}^{\alpha\beta}\ddot{R}_j^\beta + H_i^\alpha = hlm_i^\alpha \end{aligned} \tag{4.7}$$

From a formal viewpoint, equations (4.6) are similar to the known plate-type Cosserat continuum equations, Woźniak (1970). However, contrary to the Cosserat media, we also deal here with dynamic evolution equations (4.7) which are coupled with equations of motion (4.6) via the constitutive equations (4.5).

As it was stated above, for the internal variables Q^α, R_i^α we have obtained ordinary differential equations (4.4) involving exclusively time-derivatives of Q^α, R_i^α while the deflection W and rotations Φ_i are governed by the partial differential equations (4.3). Thus, in formulation of the initial-boundary value problems, equations (4.3)-(4.4) have to be considered together with two boundary and initial conditions for W, Φ_i and with two initial conditions for Q^α, R_i^α. Besides, in the framework of the derived continuum model of latticed plates only boundary conditions for W and Φ_i have a physical motivation, being independent of possible disturbances of the periodic structure of the medium near the boundary of a region Ω. It has to be emphasised that solutions W, Φ_i, Q^α, R_i^α to any initial-boundary value problem have a physical meaning only if they are represented by slowly varying functions.

In the governing equations of the IV-model there exist two length parameters h and l. The parameter h treated as a certain mean height of beams in the direction normal to $0x_1x_2$-plane is much smaller than the microstructure length parameter l. From the IV-model proposed we can pass to two models, the first one will be called the model without rotational inertia terms while the second one can be called the local model and it will be discussed in Section 5.

The model without rotational inertia terms will be obtained under assumptions that the terms involving h in equations (4.3), (4.4) will be neglected. In this situation the internal variables R_j^β can be eliminated from the governing equations of the IV-model. Denoting by $E_{ij}^{\alpha\beta}$ elements of the inverse matrices of matrices $A_{ij}^{\alpha\beta}$ (which are non-singular because the strain energy $(4.2)_1$ is positive definite), from $(4.4)_2$ we obtain

$$lR_j^\beta = -E_{ij}^{\alpha\beta}(D_{ki}^\alpha \Psi_k + lB_{ikl}^\alpha \Phi_{k,l} + D_i^{\gamma\alpha} Q^\gamma) \tag{4.8}$$

where $E_{ij}^{\alpha\beta} A_{jk}^{\alpha\beta} = \delta_{ik}$ (no summation over α, β). After substituting to equations (4.3) and $(4.4)_1$ the right-hand side of (4.8), we arrive at the following equations

$$\begin{aligned}
&\Big[(A_{ij} - E_{kl}^{\alpha\beta} D_{jk}^\alpha D_{il}^\beta)\Psi_j + (D_i^\gamma - E_{kl}^{\alpha\beta} D_k^{\gamma\alpha} D_{il}^\beta) Q^\gamma - lE_{kl}^{\alpha\beta} B_{kjp}^\alpha D_{il}^\beta \Phi_{j,p}\Big]_{,i} - \\
&-\mu \ddot{W} + f = 0 \\[2ex]
&\Big[(C_{kijl} - l^2 E_{rp}^{\alpha\beta} B_{rjl}^\alpha B_{pki}^\beta)\Phi_{j,l} - lE_{rp}^{\alpha\beta} D_{jr}^\alpha B_{pki}^\beta \Psi_j - lE_{rp}^{\alpha\beta} D_r^{\gamma\alpha} B_{pki}^\beta Q^\gamma\Big]_{,i} + \\
&+\varepsilon_{ki}\Big[(A_{ij} - E_{ml}^{\alpha\beta} D_{jm}^\alpha D_{il}^\beta)\Psi_j + (D_i^\gamma - E_{ml}^{\alpha\beta} D_m^{\gamma\alpha} D_{il}^\beta) Q^\gamma - \\
&-lE_{ml}^{\alpha\beta} B_{mjp}^\alpha D_{il}^\beta \Phi_{j,p}\Big] = 0 \\[2ex]
&l^2 \mu^{\alpha\beta} \ddot{Q}^\beta + (A^{\alpha\beta} - E_{kl}^{\delta\gamma} D_k^{\beta\delta} D_l^{\alpha\gamma}) Q^\beta + (D_j^\alpha - E_{kl}^{\delta\gamma} D_l^{\alpha\gamma} D_{jk}^\delta)\Psi_j - \\
&-lE_{kl}^{\delta\gamma} D_l^{\alpha\gamma} B_{kjp}^\delta \Phi_{j,p} = lf^\alpha
\end{aligned} \tag{4.9}$$

where $\Psi_i := W_{,i} + \varepsilon_{ij}\Phi_j$. Now basic unknowns are functions W, Φ_j, Q^γ. The obtained equations have to be satisfied for every t in the region Ω of $0x_1x_2$-plane and their solutions have a physical sense only if W, Φ_j, Q^γ are slowly varying functions. This continuum model will be referred to as *the model without rotational inertia terms* of the periodic lattice-type plates under consideration.

Both the model without rotational inertia terms and the IV-model can be used to describe the length-scale effect on the dynamic behaviour of latticed plates under consideration, on account of the presence of the microstructure length parameter l in the governing equations of these models.

The governing equations of proposed models describe dynamics of a lattice-type plate of an arbitrary complex periodic lay-out in the $0x_1x_2$-plane. If we deal with the latticed plate of a simple lay-out, i.e., having only one rigid joint in every repeated element (in this case $n = 1$) then Q^α, R_i^α drop out from all equations and we pass to the Cosserat model of lattice-type plate which coincides with that discussed in Woźniak (1970).

At the end of this section let us observe that in the quasi-stationary problems the internal kinematic variables can be eliminated from the governing equations of the proposed models by means of equations (4.4) in the framework of the IV-model or by means of equations $(4.9)_3$ within the model without rotational inertia terms.

5. Passage to the local model

The continuum model called the local model will be derived from equations (4.3)-(4.4) by the asymptotic procedure in which the microstructure length parameter l is scaled down. At the same time, it is assumed that the length parameter h tends towards zero much faster than the parameter l, i.e., $h = o(l)$.

Taking into account definitions (4.1) it can be seen that all coefficients except C_{kijl} will be constant under the above rescaling. Neglecting the terms involving h and setting $l \to 0$ in governing equations (4.3) and $(4.4)_1$, we arrive at the following equations

$$
\begin{gathered}
(A_{ij}\Psi_j + D_i^\alpha Q^\alpha)_{,i} - \mu\ddot{W} + f = 0 \\
C_{kijl}\Phi_{j,li} + \varepsilon_{ki}(A_{ij}\Psi_j + D_i^\alpha Q^\alpha) = 0
\end{gathered}
\tag{5.1}
$$

and

$$
A^{\alpha\beta}Q^\beta + D_i^\alpha\Psi_i = 0 \qquad\qquad \Psi_i = W_{,i} + \varepsilon_{ij}\Phi_j \tag{5.2}
$$

Equations (5.2) represent the system of linear algebraic equations for Q^β. Because the strain energy given by $(4.2)_1$ is positive definite, hence the matrix $A^{\alpha\beta}$ has to be non-singular. Denoting by $E^{\alpha\beta}$ the elements of the matrix inverse to $A^{\alpha\beta}$, $E^{\alpha\beta}A^{\beta\gamma} = \delta^{\alpha\gamma}$, we obtain for the internal kinematic variables Q^β the formulae

$$
Q^\beta = -E^{\alpha\beta}D_i^\alpha\Psi_i \tag{5.3}
$$

and hence from equations (5.1) we can eliminate the internal kinematic variables by means of (5.3). After some manipulations we obtain

$$\begin{aligned} &\varepsilon_{kp} C_{kijl} \Phi_{j,lip} + \mu \ddot{W} - f = 0 \\ &C_{kijl} \Phi_{j,li} + \varepsilon_{ki} (A_{ij} - E^{\alpha\beta} D_i^\alpha D_j^\beta) \Psi_j = 0 \end{aligned} \tag{5.4}$$

From notations (4.1) it follows that coefficients C_{kijl} are of the order of l^2 and hence they can be written down in the form $C_{kijl} = l^2 \widehat{C}_{kijl}$. The coefficients $\widehat{C}_{kijl}$, similarly to the coefficients $A_{ij} - E^{\alpha\beta} D_i^\alpha D_j^\beta$ are constant under the limit passage $l \to 0$. Hence equation $(5.4)_2$ yields

$$l^2 \widehat{C}_{kijl} \Phi_{j,li} + \varepsilon_{ki} (A_{ij} - E^{\alpha\beta} D_i^\alpha D_j^\beta) \Psi_j = 0 \tag{5.5}$$

Under the limit passage $l \to 0$ we obtain

$$\Psi_j = 0 \tag{5.6}$$

and from the definition (3.9) we have

$$\Phi_j = -\varepsilon_{mj} W_{,m} \tag{5.7}$$

Define

$$D_{piml} := \varepsilon_{kp} \varepsilon_{jm} C_{kijl} \tag{5.8}$$

where C_{kijl} is given by $(4.1)_2$. Substituting to $(5.4)_1$ the right-hand side of (5.7) and using (5.8) we obtain finally

$$D_{piml} W_{,piml} + \mu \ddot{W} - f = 0 \tag{5.9}$$

The obtained equation (5.9) represents an asymptotic model of the periodic lattice-type plate under consideration which can be called the local model. The only unknown in this model is the deflection W which has to be a slowly varying function for every t in the region Ω of $0x_1x_2$.

6. Applications

The governing equations of models proposed in Sections 4 and 5 will be now applied to the analysis of free vibrations of the lattice-type plate strip simply supported on the opposite edges $x_1 = \pm 0.5L$; the lay-out of this

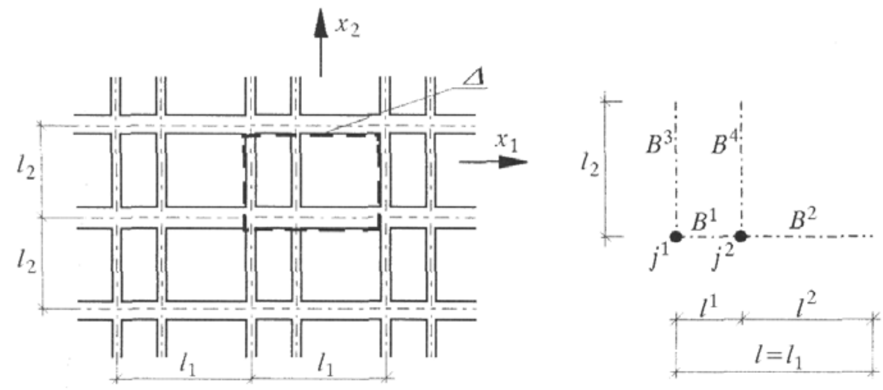

Fig. 3. The lattice-type periodic plate and its periodicity cell

latticed plate and a cell Δ are shown in Fig. 3. The cell has two rigid joints; in this case $n = 2$ and $\nu = n - 1 = 1$, i.e., the systems of numbers $h^{a\alpha}$, $g_{ij}^{a\alpha}$, $a = 1, ..., n$, $\alpha = 1, ..., \nu$, reduce to vectors with components h^{11}, h^{21} and g^{11}, g^{21}. Moreover, let the axes of all beams be parallel to the pertinent coordinate axes x_1, x_2. Let us consider the case in which the material of every beam is characterised by the Young modulus E, the Kirchhoff modulus G and the mass density ρ ($E^1 = E^2 = E^3 = E^4 = E$, $G^1 = G^2 = G^3 = G^4 = G$, $\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho$); the inertia moments of beams I^A, I_0^A, $A = 1, ..., 4$, are assumed to be interrelated by $I^3 = I^4$, $I_0^3 = I_0^4$ and $I^1 = I^2$, $I_0^1 = I_0^2$; moreover, the cross-sectional area of beams is assumed to be equal to F ($F^1 = F^2 = F^3 = F^4 = F$), and the lengths of beams l^A are defined as $l^3 = l^4 = l_2$, $l^1 \neq l^2$. In this special case, the masses assigned to all nodal joints are equal and the rotational inertia moments assigned to all joints satisfy conditions $J_{12}^a = J_{21}^a = 0$, $a = 1, 2$. Let $l = l_1$ be the microstructure length parameter shown in Fig. 3. Assuming that all unknown functions depend only on x_1 and time t, bearing in mind definitions (3.4), (4.1) and neglecting external loadings, we obtain from (4.3)-(4.4) a system of governing equations for the plate strip. Calculate from (4.1) the coefficients and denote

$$
\begin{aligned}
&C_{2121} = EI^1(l_2)^{-1} \\
&C_{1111} = GI_0^1(l_2)^{-1} \\
&B_{111}^1 = B_{211}^1 = B_{221}^1 = B_{121}^1 = 0 \\
&\mu = \mu^{11} = M(l_1 l_2)^{-1} \\
&A_{11} = 12EI^1[(l^1)^{-1} + (l^2)^{-1}](l_1 l_2)^{-1} \\
&A_{22} = 24EI^3(l^1)^{-1}(l^2)^{-2} \\
&A^{11} = 48EI^1 l_1[(l^1)^{-3} + (l^2)^{-3}](l_2)^{-1}
\end{aligned}
\tag{6.1}
$$

$$
\begin{aligned}
&D_1^1 = 24EI^1[(l^2)^{-2} - (l^1)^{-2}](l_2)^{-1} \\
&A_{11}^{11} = A_{22}^{11} = A_{12}^{11} = A_{21}^{11} = \\
&\quad = 4(l_1 l_2)^{-1}\Big\{6EI^3(l_2)^{-1} + (EI^1 + GI_0^1)[(l^1)^{-1} + (l^2)^{-1}]\Big\} \\
&\chi_{11} = h^{-2}J_{11}(l_1 l_2)^{-1} \\
&\chi_{22} = h^{-2}J_{22}(l_1 l_2)^{-1} \\
&\chi_{11}^{11} = \chi_{22}^{11} = \chi_{12}^{11} = \chi_{21}^{11} = h^{-2}(l_1 l_2)^{-1}(J_{11} + J_{22})
\end{aligned}
$$

where $M = \sum_{a=1}^{2} M^a$, $J_{ik} = \sum_{a=1}^{2} J_{ik}^a$ and M^a, J_{ik}^a are defined by (2.1), (2.2), and denote

$$
\begin{aligned}
&\widehat{A} \equiv A_{11}^{11} = A_{22}^{11} = A_{12}^{11} = A_{21}^{11} \qquad \widehat{D} \equiv D_1^1 \\
&\chi \equiv \chi_{11}^{11} = \chi_{22}^{11} = \chi_{12}^{11} = \chi_{21}^{11}
\end{aligned}
\tag{6.2}
$$

Introducing a new unknown defined as

$$
R^1 \equiv R_1^1 + R_2^1 \tag{6.3}
$$

we obtain the system of three equations for W, Q, Φ_2

$$
\begin{aligned}
&A_{11}(W_{,1} + \Phi_2)_{,1} + \widehat{D}Q_{,1} - \mu\ddot{W} = 0 \\
&C_{2121}\Phi_{2,11} - A_{11}(W_{,1} + \Phi_2) - \widehat{D}Q^1 - h^2\chi_{22}\ddot{\Phi}_2 = 0 \\
&l^2\mu\ddot{Q}^1 + A^{11}Q^1 + \widehat{D}(W_{,1} + \Phi_2) = 0
\end{aligned}
\tag{6.4}
$$

one equation for Φ_1

$$
C_{1111}\Phi_{1,11} - A_{22}\Phi_1 - h^2\chi_{11}\ddot{\Phi}_1 = 0 \tag{6.5}
$$

and one equation for R^1

$$
h^2\chi\ddot{R}^1 + \widehat{A}R^1 = 0 \tag{6.6}
$$

The equations (6.4) describe bending of the plate strip under consideration in the planes parallel to $0x_1x_3$-plane while equations (6.5) describe torsion in the planes parallel to $0x_2x_3$-plane. In the aforementioned equations there exist two length parameters h and l, where $h \ll l$. Parameter h have been introduced in a formal way in Sections 3 and 4 (cf. formulae $(3.6)_{3,4}$ and $(4.1)_{9,10}$ and that is why in the further calculations we shall put

$$
\widehat{\chi}_{ij} \equiv h^2\chi_{ij} \qquad \widehat{\chi} \equiv h^2\chi \tag{6.7}
$$

Looking for the free vibration frequencies of the lattice-type plate strip, solutions of the above equations can be assumed in the following forms

$$
\begin{aligned}
W &= A_W \sin(kx_1)\cos(\omega t)\\
\Phi_2 &= A_{\Phi_2} \cos(kx_1)\cos(\omega t)\\
Q^1 &= A_{Q^1} \cos(kx_1)\cos(\omega t)\\
\Phi_1 &= A_{\Phi_1} \cos(kx_1)\cos(\omega t)\\
R^1 &= A_{R^1} \sin(kx_1)\cos(\omega t)
\end{aligned}
\tag{6.8}
$$

Substituting the right-hand sides of $(6.8)_{1,2,3}$ into equations (6.4) we obtain the system of three linear algebraic equations for A_W, A_{Φ_2}, A_Q, which has nontrivial solutions provided its determinant is equal to zero. This way we arrive at the characteristic equation for the free vibration frequencies related to equations (6.4). This equation, being the dispersion relation for $(6.8)_{1,2,3}$, has the form

$$
\begin{aligned}
&l^2\mu\widehat{\chi}_{22}\omega^6 - \mu\Big[\mu(l^2A_{11} + l^2k^2C_{2121}) + \widehat{\chi}_{22}(l^2k^2A_{11} + A^{11})\Big]\omega^4 + \\
&+\Big[(A^{11}A_{11} - \widehat{D}^2)(\mu + k^2\widehat{\chi}_{22}) + \mu C_{2121}(l^2k^2A_{11} + A^{11})k^2\Big]\omega^2 - \\
&-C_{2121}(A^{11}A_{11} - \widehat{D}^2)k^4 = 0
\end{aligned}
\tag{6.9}
$$

Let us define the nondimensional parameter

$$
\varepsilon := kl = \frac{2\pi l}{L}
\tag{6.10}
$$

where L is the wavelength. Because W, Φ_2, Q have to be regular slowly varying functions, hence $l/L \ll 1$ and parameter ε is sufficiently small compared to 1. Bearing in mind this fact, from (6.9) we obtain the free vibration frequencies which can be represented by the following formulae

$$
\begin{aligned}
(\omega_1)^2 &= \frac{C_{2121}}{\mu}k^4 - \frac{C_{2121}}{\mu}\Big(\frac{A^{11}C_{2121}}{A_{11}A^{11} - \widehat{D}^2} + \frac{\widehat{\chi}_{22}}{\mu}\Big)k^6 + \mathcal{O}(\varepsilon^8)\\
(\omega_2)^2 &= \frac{A^{11}}{l^2\mu} + \frac{\widehat{D}^2}{\widehat{\chi}_{22}A^{11}} + \mathcal{O}(\varepsilon^2)\\
(\omega_3)^2 &= \frac{A_{11}A^{11} - \widehat{D}^2}{\widehat{\chi}_{22}A^{11}} + \mathcal{O}(\varepsilon^2)
\end{aligned}
\tag{6.11}
$$

Taking into account the equation (6.5) and its nontrivial solution as $(6.8)_4$, we arrive at the following free vibration frequency

$$
(\omega_4)^2 = \frac{A_{22} + C_{1111}k^2}{\widehat{\chi}_{11}}
\tag{6.12}
$$

Substituting the nontrivial solution $(6.8)_5$ to equation (6.6) we obtain the following free vibration frequency

$$(\omega_5)^2 = \frac{\widehat{A}}{\widehat{\chi}} \tag{6.13}$$

Thus, in the framework of the IV-model for the lattice-type plate strip under consideration shown in Fig. 3, we have obtained free vibration frequencies defined by (6.11), (6.12), (6.13).

Now we shall consider the model without rotational inertia terms which is governed by equations (4.9). For the identical plate strip and representative cell, as previously we obtain the following system of equations

$$\begin{aligned}
&A_{11}(W_{,1} + \Phi_2)_{,1} + \widehat{D}Q_{,1} - \mu\ddot{W} = 0\\
&C_{2121}\Phi_{2,11} - A_{11}(W_{,1} + \Phi_2) - \widehat{D}Q = 0\\
&l^2\mu\ddot{Q} + A^{11}Q + \widehat{D}(W_{,1} + \Phi_2) = 0
\end{aligned} \tag{6.14}$$

Coefficients in the above equations are defined by formulae (6.1). Assuming solutions to the above system in the form given by $(6.8)_{1,2,3}$ we arrive at the dispersion relation

$$\begin{aligned}
&\mu^2(l^2A_{11} + l^2k^2C_{2121})\omega^4 - \mu\Big[A^{11}A_{11} - \widehat{D}^2 + (l^2k^2A_{11} + A^{11})C_{2121}k^2\Big]\omega^2 + \\
&+(A^{11}A_{11} - \widehat{D}^2)C_{2121}k^4 = 0
\end{aligned} \tag{6.15}$$

Under similar assumptions as above we obtain the asymptotic formulae for the free vibration frequencies

$$\begin{aligned}
(\omega_L)^2 &= \frac{C_{2121}}{\mu}k^4 - \frac{A^{11}(C_{2121})^2}{\mu(A_{11}A^{11} - \widehat{D}^2)}k^6 + \mathcal{O}(\varepsilon^8)\\
(\omega_H)^2 &= \frac{A_{11}A^{11} - \widehat{D}^2}{l^2\mu(A_{11} + C_{2121}k^2)} + \frac{A^{11}C_{2121}}{l^2\mu(A_{11} + C_{2121}k^2)}k^2 + \mathcal{O}(\varepsilon^2)
\end{aligned} \tag{6.16}$$

Thus, in the framework of the model without rotational inertia terms we have obtained two free vibration frequencies defined by (6.16). The frequency ω_H is the higher one.

Using the local model for the plate under consideration, from (5.9) we obtain

$$D_{1111}W_{,1111} + \mu\ddot{W} = 0 \tag{6.17}$$

where assuming the definition (5.8) we have $D_{1111} = C_{2121}$, and hence we arrive at the following free vibration frequency

$$\omega^2 = \frac{C_{2121}}{\mu} k^4 \tag{6.18}$$

Thus, in the framework of the local model we can obtain only one vibration frequency given by (6.18).

Comparing the results obtained from three models derived above we conclude that from the local model, only one free vibration frequency can be obtained. This frequency does not depend on the microstructure parameter l characterising a size of representative cell of the lattice-type plate under consideration. Hence in the framework of the local model, dispersion phenomena cannot be described. From the governing equations of the IV-model and the model without rotational inertia terms we have obtained also higher vibration frequencies because these equations contain the microstructure parameter. Higher frequencies are defined by $(6.11)_{2,3}$ and by (6.13) in the IV-model, and by $(6.16)_2$ in the model without rotational inertia terms. The influence of rotational inertia on dynamic behaviour of the lattice-type plate under consideration is visible both in the formulae for lower and higher frequencies (i.e. ω_1 and ω_2). Moreover, the frequencies ω_4-ω_5 have been caused by the presence of terms χ_{ij}, $\chi_{ij}^{\alpha\beta}$ (related to inertia moments) in governing equations of the IV-model.

7. Numerical example

Now, an application of formulae for free vibration frequencies obtained in the previous section will be illustrated by a numerical example. We introduce the following dimensionless coefficients

$$\begin{array}{lll} \alpha \equiv I_0^1 (I^1)^{-1} & \beta \equiv I^3 (I^1)^{-1} & \gamma \equiv I_0^3 (I^3)^{-1} \\ \varphi \equiv I^1 [F(l_1)^2]^{-1} & \delta \equiv G E^{-1} & \xi \equiv l_2 (l_1)^{-1} \\ \lambda \equiv l^1 (l_1)^{-1} & & \end{array} \tag{7.1}$$

where E is the Young modulus; G is the Kirchhoff modulus; I^1, I_0^1, I^3, I_0^3 are the inertia moments; l_1, l_2 are the lengths of the cell; l^1 is the length of the beam B^1 (see Fig. 3); F is the cross-sectional area of beams.

Free vibration frequencies for the IV-model, (6.11), (6.12), (6.13), for the model without rotational inertia terms, (6.16), and for the local model, (6.18), we will write in the dimensionless form

$$(\Omega)^2 \equiv \frac{\rho(l_1)^2}{E}(\varpi)^2 \tag{7.2}$$

where ρ is the mass density of the material of the beam; ϖ is the frequency described by (6.11), (6.12), (6.13), (6.16) or (6.18).

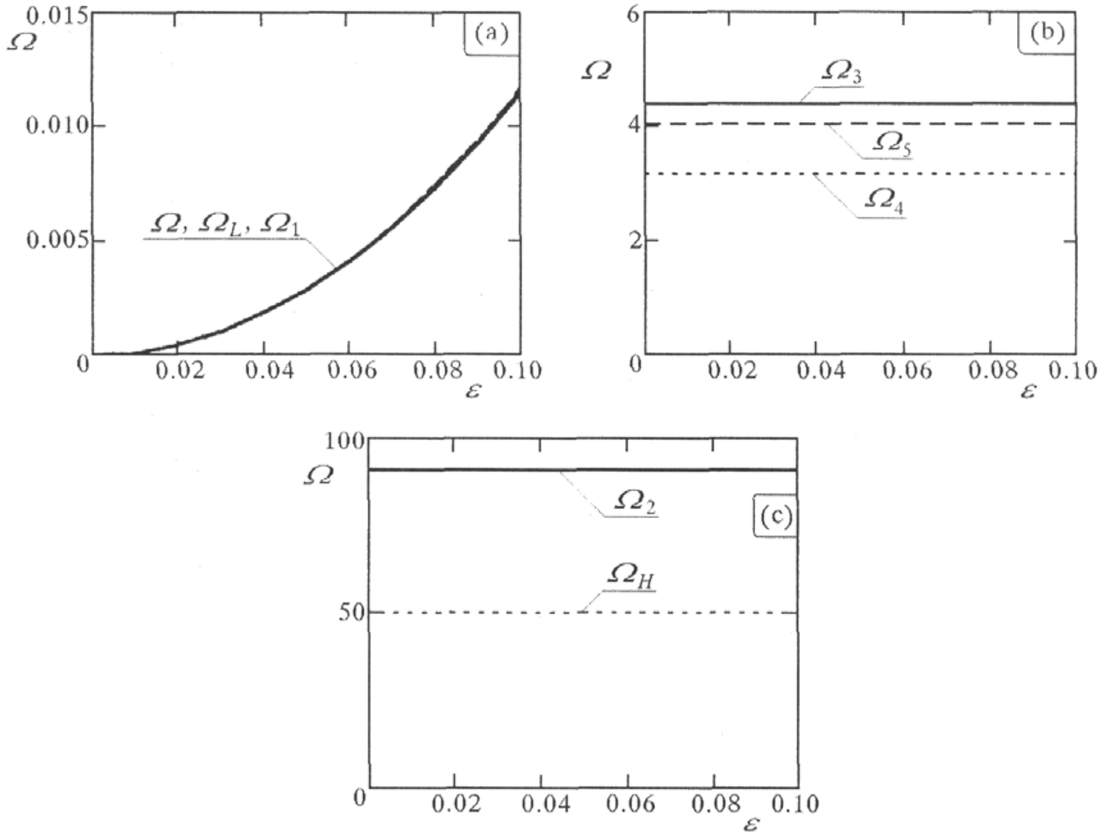

Fig. 4. Diagrams of spectral lines: (a) for lower frequencies: $\Omega_1 - \varepsilon$ (the IV-model), $\Omega_L - \varepsilon$ (the model without rotational inertia terms), $\Omega - \varepsilon$ (the local model); (b) for higher frequencies: $\Omega_3 - \varepsilon$, $\Omega_4 - \varepsilon$, $\Omega_5 - \varepsilon$ (the IV-model); (c) for higher frequencies: $\Omega_2 - \varepsilon$ (the IV-model), $\Omega_H - \varepsilon$ (the model without rotational inertia terms); values of parameters defined by (7.1): $\alpha = 0.2$, $\beta = 0.5$, $\gamma = 0.2$, $\varphi = 4$, $\delta = 0.4$, $\xi = 1$, $\lambda = 0.2$

Numerical results obtained by using the aforesaid formulae for the frequencies and formula (7.2) are presented as spectral lines in Fig. 4. These plots are made for the following parameters (7.1): $\alpha = 0.2$, $\beta = 0.5$, $\gamma = 0.2$, $\varphi = 4$, $\delta = 0.4$, $\xi = 1$, $\lambda = 0.2$. In Fig. 4a we have plots of the relations for the lower frequncies: $\Omega_1 - \varepsilon$, $\Omega_L - \varepsilon$, $\Omega - \varepsilon$. Diagrams of higher frequencies are presented in Fig. 4b: the relations $\Omega_3 - \varepsilon$, $\Omega_4 - \varepsilon$, $\Omega_5 - \varepsilon$; and in Fig. 4c: the relations $\Omega_2 - \varepsilon$, $\Omega_H - \varepsilon$. The dimensionless wave number ε defined by (6.10) is from the interval [0,0.1].

Analysing the diagrams of spectral lines we can observe that:

- Differences between values of lower free vibration frequencies calculated within the IV- model, the model without rotational inertia terms and the local model are negligible (Fig. 4a).

- Taking into account the rotational inertia terms in the IV-model we can investigate three additional higher frequencies (Fig. 4b).

- Higher free vibration frequencies related to the periodic structure of the lattice-type plate, which plots are shown in Fig. 4c, can be obtained only within models based on the tolerance averaging method – the IV-model and the model without rotational inertia terms; but values of these frequencies are much smaller than those resulting from the IV-model after neglecting the inertia terms.

8. Conclusions

In this paper, the general averaged continuum model (the IV-model) of periodic lattice-type plates for the analysis of dynamic problems has been derived. The obtained IV-model equations constitute a certain generalisation of the Cosserat continuum equations. The model has been derived by using the approach which is similar to *the tolerance averaging method* proposed by Woźniak and Wierzbicki (2000) for periodic composites. In what follows the main features of the presented model will be listed:

- The proposed models can be applied to the analysis of periodic lattice-type plates which have an arbitrary complex lay-out.

- The IV-model describes the cell size effect on the global dynamic behaviour of latticed plates, because in governing equations of the model there exists a microstructure parameter l.

- Contrary to the homogenized model, by using the IV-model it is possible to investigate dispersion phenomena and calculate higher free vibration frequencies (and also higher wave propagation speeds).

- The IV-models can be formulated on different levels of accuracy; on every level the system of real numbers $h^{a\alpha}$, $g_{ij}^{a\alpha}$, $\alpha = 1, ..., \nu$, $i, j = 1, 2$, have to be assumed *a priori* as describing the class of motions we are going to investigate. The most simple IV-model is based on the smallest repetitive cell and consists of the minimum number of internal kinematic variables. More accurate IV-models are based on cells composed of two or more adjacent repetitive elements and hence, they may involve even a large number of kinematic internal variables.

- The form of governing equations of the IV-model is relatively simple since the extra unknowns called internal kinematic variables are governed by ordinary differential equations involving only time-derivatives of these variables, and they do not enter the boundary conditions. Moreover, the boundary conditions with extra unknowns may be not well motivated from the physical viewpoint.

- The IV-model is mainly restricted to the analysis of long wave problems (compared with the microstructure length parameter) because it was obtained under the assumption that all unknown functions (deflections, rotations and internal variables) are slowly varying.

- The governing equations of the general IV-model involve inertia terms as a result of assigning inertia moments to every nodal joint of the lattice under consideration. From this model, the model without rotational inertia terms has been derived which also describes dispersion phenomena and makes it possible to calculate higher order motions. Both models can be approximated by the local model which neglects the effect of the microstructure size of latticed plates on their global dynamic behaviour.

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Ciagły model zagadnień dynamicznych periodycznych płyt siatkowych o złożonej strukturze

Streszczenie

W pracy zaproponowano nowy ciągły model do analizowania zagadnień dynamicznych sprężystych, periodycznych płyt siatkowych. W przedstawionym podejściu częściowo wykorzystano technikę tolerancyjnego uśredniania, opracowaną przez Woźniaka i Wierzbickiego (2000) dla termomechaniki kompozytów. Podejście to zastosowano w pracach Cieleckiej i in. (1998, 2000) w modelowaniu gęstych struktur komórkowych. Zaproponowany model opisuje wpływ wielkości mikrostruktury na dynamikę płyty siatkowej. Otrzymane równania zastosowano do analizy propagacji fal w pewnym szczególnym przypadku płyty. Pokazano, że efekt skali odgrywa ważną rolę i nie może być pominięty w powyższej analizie.

Manuscript received August 1, 2001; accepted for print September 18, 2001