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ASSESSMENT OF THE STRUCTURE STABILITY ACCORDING TO THE THEORY OF FUZZY SETS

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There are two different concepts that are often used incorrectly – randomness and imprecision. The randomness is described on the basis of the theory of probability, whereas the analysis of the imprecision is a subject of research of the fuzzy sets theory. The theory of reliability uses the Hasofer-Lind's index for the assessment of probability of failure of structures which are risked by a stability loss. The limit state function in distinction of the classical one was described by applying the fuzzy sets theory. Three fuzzy areas were defined: desirable states, limit states and undesirable states. The suitable probabilities were calculated using the approximation method FORM. It enabled taking into account in the reliability analysis the weakening of a structure before appearing of the limit point on the equilibrium path.

Key words: reliability, stability, fuzzy sets theory

1. Introduction

The question of structure reliability is associated with two kinds of uncertainty:

- randomness
- imprecision.

The randomness is the lack of certainty relative to the likelihood of a given event. Objectively, it is described on the basis of the frequency of events. Modelling this kind of uncertainty is the subject of the theory of probability.

The imprecision is the lack of certainty in defining a given event. Modelling and processing imprecise data are the subject of the theory of fuzzy sets.

The essence of the theory, created by Zadeh (1965), is the generalisation of two-value logic into multi-value logic. As a result, the generalisation of the classical notion of the set – otherwise called the crisp set – is the fuzzy set (Kacprzyk, 1986; Yager and Filer, 1995; Piegat, 1999). Each element x of examined space R may not only belong or not belong to the fuzzy set $\underline{A}$ but it may also belong to it to a certain degree. The above set is described by the so-called membership function $\chi_{\underline{A}}(x)$, which assigns to every $x \in R$ the number $\alpha \in <0, 1>$, in the following way

$$\chi_{\underline{A}}(x) = \begin{cases} 1 & \text{for} \quad x \in \underline{A} \\ \alpha \in (0,1) & \text{for} \quad x \in \underline{A} \ \text{ to a certain degree} \\ 0 & \text{for} \quad x \notin \underline{A} \end{cases} \tag{1.1}$$

As a result, the generalisation of the notion of an event – otherwise called the crisp event – is a fuzzy event. Individual elementary events may not only favour the fuzzy event $\underline{A}$ or not but they also may favour it to a certain degree. Its probability according to Zadeh (see Kacprzyk, 1986) is defined as follows

$$P(\underline{A}) = \int\limits_{\underline{A}} f(x)\ dx = \int\limits_{R} \chi_{\underline{A}} f(x)\ dx \tag{1.2}$$

where $f(x)$ is the probability density function of the variable X.

When it concerns the structural reliability of building constructions, all uncertainties are treated as random and the probability of failure is treated as a measure of the reliability (Blockley, 1980; Doliński, 1983; Ditlevsen and Bjerager, 1986; Melchers, 1987). This probability is defined as the integral of the joint probability density function $f_{1...n}(x_1, ..., x_n)$ of n random basic variables $(X_1, ..., X_n)$ over the so-called undesirable states domain F, the region where the so-called limit state function $g(X_1, ..., X_n)$ is negative

$$P_f = P[g(X_1, ..., X_n) < 0] = \int\limits_{g(x)<0} ... \int f_{1...n}(x_1, ..., x_n)\ dx_1 ... dx_2 \tag{1.3}$$

Usually, the so-called First Order Reliability Method (FORM) is used to determine it. The n-dimensional space of the basic random variables $\boldsymbol{X}$ is transformed into the space of uncorrelated random variables of normal standardised distributions $\boldsymbol{U}$ (Fig. 1). Meanwhile, the limit state surface $g(x_1, ..., x_n) = 0$ is approximated by the linear surface $h(u_1, ..., u_n) = 0$, tangent in the so-called design point $\boldsymbol{u}^*$ in the system $\boldsymbol{U}$. It is a point on the limit state surface closest to the origin of the system $\boldsymbol{U}$ – the most likely point of failure. Its distance from the origin of the system β – the so-called Hasofer-Lind index – is accepted as a measure of structural reliability

$$P_f \approx \Phi(-\beta) \tag{1.4}$$

where $\Phi(\cdot)$ is the normal cumulative distribution function.

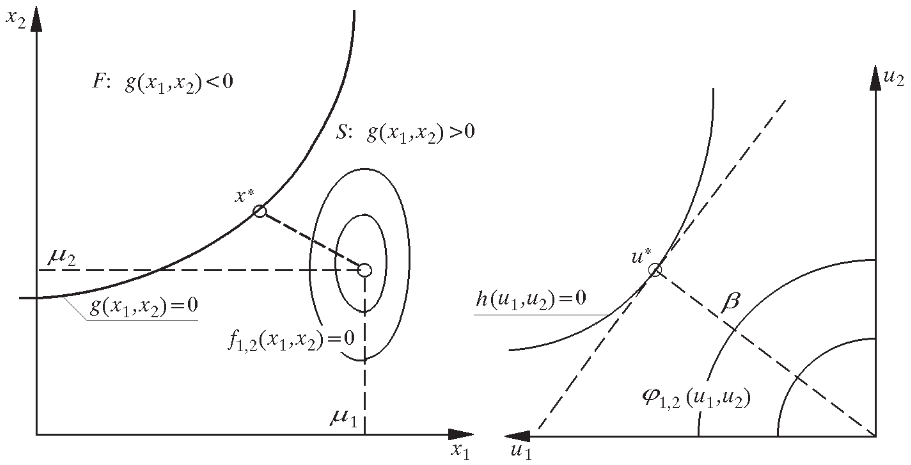

Fig. 1. The area of undesirable states F and desirable states S in the system of basic random variables X_1, X_2 and the system of standardised random variables U_1, U_2

Modelling of the imprecision as applied to the problems of structural reliability, was the subject Ph.D. thesis by Szeliga (see Szeliga and Witkowski, 1997; Szeliga, 2000). In this approach, the limit state function divides the sample space $\boldsymbol{X}$ into fuzzy areas (Fig. 2): a fuzzy area of the desirable states $\underline{S}$, fuzzy area of the limit states $\underline{L}$ and fuzzy area of the undesirable states $\underline{F}$. The structural survival or failure are treated as fuzzy events, when the variable $\boldsymbol{X}$ hits the areas mentioned above. The probabilities according to Zadeh of these events are a measure of the structural reliability which takes into account both types of uncertainty: randomness and imprecision.

The above probabilities are determined using the FORM. However, in this case the Hasofer-Lind index β is a fuzzy set of $\xi \in R$, for example, of the type shown in Fig. 3.

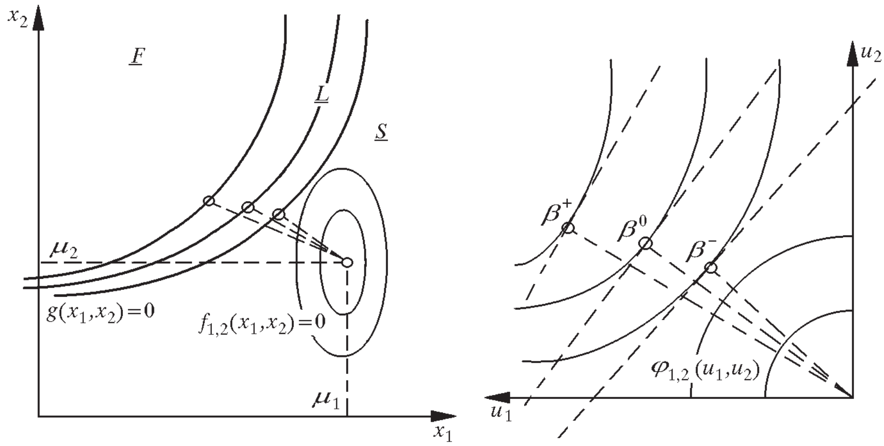

Fig. 2. Fuzzy areas of the undesirable states $\underline{F}$, limit states $\underline{L}$ and desirable states $\underline{S}$ in the system of basic random variables X_1, X_2 and the system of standardised random variables U_1, U_2

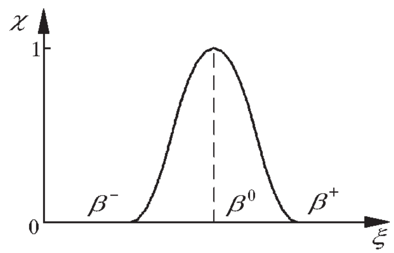

Fig. 3. Fuzzy Hasofer-Lind's index of the type $\underline{\beta} = \{\text{about } \beta^0 \text{ in } < \beta^-, \beta^+ >\}$

2. Formulation of the problem

Let us apply this concept to assessing the state of structural reliability. The strict criterion defined by the limit state function

$$g(\boldsymbol{q}(\boldsymbol{X}), \boldsymbol{X}) = \lambda_k(\boldsymbol{q}(\boldsymbol{X}), \boldsymbol{X}) - \lambda(\boldsymbol{X}) \tag{2.1}$$

where

λ – load multiplier of a structural system under one-parameter load $\boldsymbol{Q} = \lambda \boldsymbol{Q}^0$, with $\boldsymbol{Q}^0$ being the comparative load

λ_k – critical load multiplier corresponding to the global loss of the system stability
$\boldsymbol{X}$ – vector of basic random variables
$\boldsymbol{q}(\boldsymbol{X})$ – vector of displacements,

divides structural states into entirely desirable ones – before the load parameter λ reaches the critical value λ_k – and into entirely undesirable ones – after this value has been exceeded. However, as the nonlinear analysis shows, the weakening of a construction as the load grows is not an abrupt but continuous process. Thus, in our reliability analysis, we will use a fuzzy criterion of stability, taking into account the fuzziness of the areas, into which the sample space is divided by the limit state function. This criterion would describe a gradual – but not abrupt – increase in the risk of structural instability. This criterion would be based on a parameter which describes the state of a structure and varies with load growth.

The fuzzy criterion of a snap-through stability loss could be based, for example, on the so-called scalar rigidity parameter C_S, which is used to control the incremental-iterative process when the equilibrium path is determined in a non-linear analysis of bar structures (Bergan and Søreide, 1973). It is defined by the so-called scalar measure of construction rigidity

$$S^{(i)} = \frac{1}{\Delta\overline{W}^{(i)}} \tag{2.2}$$

where

$$\Delta\overline{W}^{(i)} = \Delta\overline{\boldsymbol{q}}^{(i)\top}\Delta\overline{\boldsymbol{Q}}^{(i)} \tag{2.3}$$

is the so-called normalised incremental work of the normalised vector of the load increments $\Delta\overline{\boldsymbol{Q}}^{(i)}$ on a normalised vector of the displacement increments $\Delta\overline{\boldsymbol{q}}^{(i)}$, defined as follows

$$\Delta\overline{\boldsymbol{q}}^{(i)} = \frac{\Delta\boldsymbol{q}^{(i)}}{\|\Delta\boldsymbol{Q}^{(i)}\|} \qquad\qquad \Delta\overline{\boldsymbol{Q}}^{(i)} = \frac{\Delta\boldsymbol{Q}^{(i)}}{\|\Delta\boldsymbol{Q}^{(i)}\|} \tag{2.4}$$

where

$\Delta\boldsymbol{Q}^{(i)}$ – vector of one-parameter load increment in the ith step of the increment $\Delta\boldsymbol{Q}^{(i)} = \Delta\lambda^{(i)}\boldsymbol{Q}^0$
$\|\Delta\boldsymbol{Q}^{(i)}\|$ – norm of the load increment vector.

At high construction rigidity, at a given value of the load increment $\Delta\boldsymbol{Q}^{(i)}$, there is a relatively small displacement increment $\Delta\boldsymbol{q}^{(i)}$, and thus it gives a relatively small value of the normalised incremental work $\Delta\overline{W}^{(i)}$ and a relatively high value of the parameter $S^{(i)}$. The ratio of this parameter value in ith step of the increment to its initial value $S^{(0)}$ is a measure of how the structural rigidity decreases with an increase in the load parameter λ

$$C_S = \frac{S^{(i)}}{S^{(0)}} = \frac{\Delta\lambda^{(i)}}{\Delta\lambda^{(0)}} \frac{\Delta \boldsymbol{q}^{(0)\top} \boldsymbol{Q}^0}{\Delta \boldsymbol{q}^{(i)\top} \boldsymbol{Q}^0} = \Big(\frac{\Delta\lambda^{(i)}}{\Delta\lambda^{(0)}}\Big)^2 \frac{\Delta \boldsymbol{q}^{(0)\top} \mathbf{K}^{(0)} \Delta \boldsymbol{q}^{(0)}}{\Delta \boldsymbol{q}^{(i)\top} \mathbf{K}^{(i)} \Delta \boldsymbol{q}^{(i)}} \tag{2.5}$$

For the first iteration, the parameter C_S assumes the value 1 and decreases as the construction weakness until it reaches the value 0 in the limit point when the tangent rigidity matrix is singular, and then it grows as the construction strengthens. On stable sectors of the stability track it is positive, on the unstable ones – negative. This interrelationship is shown in Fig. 4.

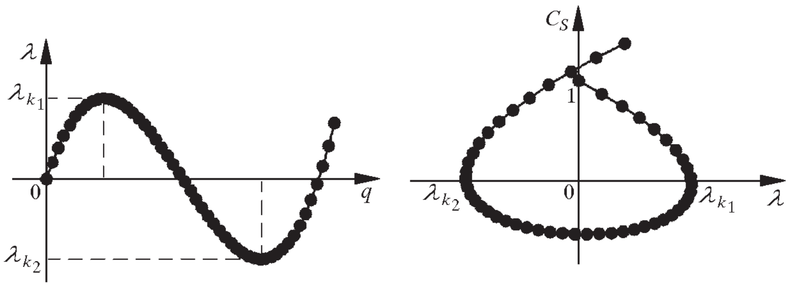

Fig. 4. Equilibrium path and scalar rigidity parameter of the Mises truss

3. Numerical examples

Let us assume then that the limit state function is as follows

$$g(\boldsymbol{q}(\boldsymbol{X}), \boldsymbol{X}) = \xi - \frac{\lambda(\boldsymbol{X})}{\lambda_k(\boldsymbol{q}(\boldsymbol{X}), \boldsymbol{X})} \qquad \xi \in \underline{\xi} \qquad 0 \leqslant \xi \leqslant 1 \tag{3.1}$$

where

$$\chi_{\underline{\xi}}\Big(\frac{\lambda}{\lambda_k}\Big) = C_S\Big(\frac{\lambda}{\lambda_k}\Big) \tag{3.2}$$

It divides the axis λ/λ_k into the following areas (Fig. 5):

— fuzzy area of the desirable states $\underline{S}$

$$\chi_{\underline{S}}\Big(\frac{\lambda}{\lambda_k}\Big) = \begin{cases} C_S\Big(\dfrac{\lambda}{\lambda_k}\Big) & \text{for} \quad \lambda < \lambda_k \\ 0 & \text{for} \quad \lambda \geqslant \lambda_k \end{cases} \tag{3.3}$$

— fuzzy area of the limit states $\underline{L}$

$$\chi_{\underline{L}}\Big(\frac{\lambda}{\lambda_k}\Big) = \begin{cases} 1 - C_S\Big(\dfrac{\lambda}{\lambda_k}\Big) & \text{for} \quad \lambda < \lambda_k \\ 0 & \text{for} \quad \lambda \geqslant \lambda_k \end{cases} \tag{3.4}$$

— crisp area of the undesirable states F

$$c_F\Big(\frac{\lambda}{\lambda_k}\Big) = \begin{cases} 0 & \text{for} \quad \lambda < \lambda_k \\ 1 & \text{for} \quad \lambda \geqslant \lambda_k \end{cases} \tag{3.5}$$

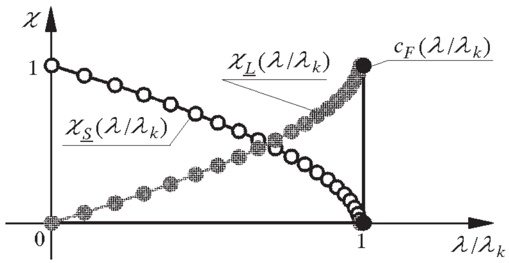

Fig. 5. Areas: desirable $\underline{S}$, limit $\underline{L}$ and undesirable F in the case of a snap-through type of loss of stability

As can be seen in Fig. 5, the area of desirable states does not appear in the degree equal 1. What is more, it will never appear whatever parameter – starting from 0 and varying with the load growth – will be assumed as the base of the fuzzy criterion of the limit state.

The following probabilities could serve as a measure of the reliability, taking into account both uncertainty types, i.e. the random uncertainty of $\boldsymbol{X}$ variables and the uncertainty of the accepted stability criterion:

- the probability P_F of a crisp event when the random load parameter $\lambda(\boldsymbol{X})$ hits the crisp area of the undesirable states F

$$P_F = P(\boldsymbol{X} \in F) = \int\limits_{-\infty}^{+\infty} \dots \int\limits_{-\infty}^{+\infty} c_F(\boldsymbol{x}) f_{1\dots n}(\boldsymbol{x})\, d\boldsymbol{x} \tag{3.6}$$

- the probability according to Zadeh $P_{\underline{-S}}$ of a fuzzy event when the random load parameter $\lambda(\boldsymbol{X})$ hits the fuzzy area $\underline{-S}$ – the complementary to the fuzzy area of the desirable states $\underline{S}$

$$P_{\underline{-S}} = P(\boldsymbol{X} \in \underline{-S}) = \int\limits_{-\infty}^{+\infty} \dots \int\limits_{-\infty}^{+\infty} \chi_{\underline{-S}}(\boldsymbol{x}) f_{1\dots n}(\boldsymbol{x})\, d\boldsymbol{x} \tag{3.7}$$

where

$$\chi_{\underline{-S}}(\boldsymbol{x}) = 1 - \chi_{\underline{S}}(\boldsymbol{x})$$

These probabilities will be the upper and lower probability estimates of the structural failure P_f

$$P_F \leqslant P_f \leqslant P_{\underline{-S}} \tag{3.8}$$

Unfortunately, as it will turn out further on, in the case of the so-defined limit state criterion, the structural reliability described by the fuzzy Hasofer-Lind index fails.

Because the mean value of the load parameter λ meets the condition

$$0 < \mu_\lambda < \mu_{\lambda_k} \tag{3.9}$$

the fuzzy index $\underline{\beta}$ will be a fuzzy set of the type shown in Fig. 6

$$\underline{\beta} = \{\text{a little bit less than } \beta^0 \text{ in } <\beta^-, \beta^0>\}$$

where $\beta^- < 0$, $\beta^0 > 0$.

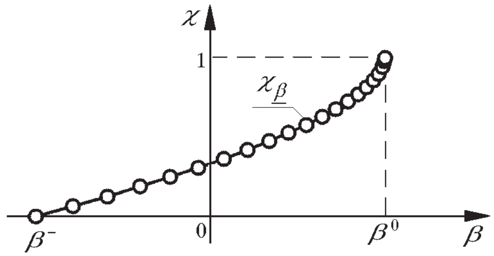

Fig. 6. Fuzzy reliability index $\underline{\beta}$ in the case of a snap-through problem

Fig. 7 show the crisp area of the undesirable states F and the fuzzy area of the limit states $\underline{L}$ in the space of two random normal and uncorrelated variables: the load parameter λ and rigidity parameter EA in the case of the Mises truss. The mean values of random variables and, as a result, the origin of the system of random standardised variables will be inside the fuzzy limit area $\underline{L}$. Thus, the approximations on which the FORM method is based will make it impossible to obtain reliable results. However, the concept of the fuzzy stability criterion seems attractive enough as it enables modelling the continuous weakening of a construction in the loading process.

However, we can use another criterion of the stability. We can assume that the limit state of a structure appears with these value of the load parameter λ,

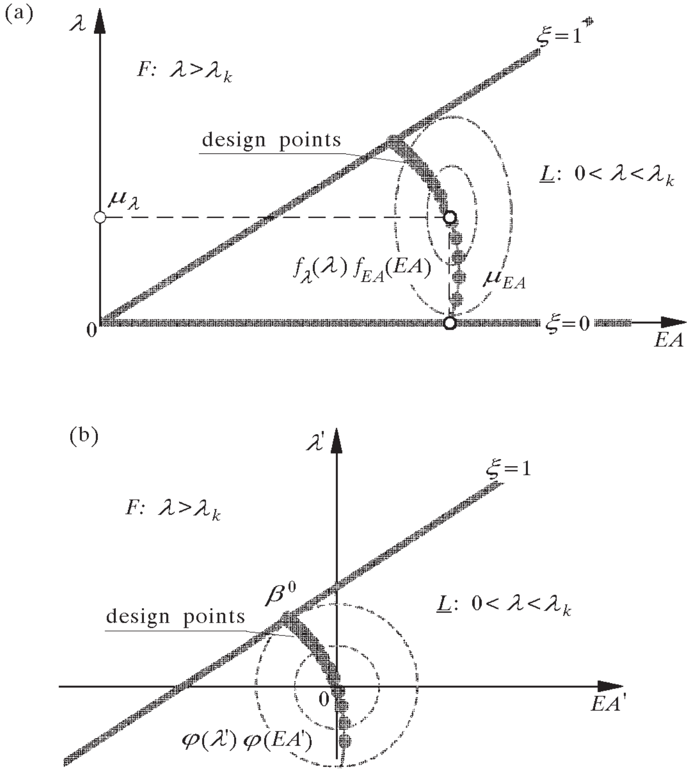

Fig. 7. Areas of the undesirable states F and limit states $\underline{L}$: (a) in the system of the two random variables λ and EA, (b) in the system of the random standardised variables λ' and EA'

when the scalar parameter of the stiffness is equal 0.5

$$\xi_k = \frac{\lambda}{\lambda_k}: \quad C_s(\xi_k) = \frac{1}{2}$$

when the structure becomes "more soft than stiff" because of large axial forces.

The membership function of the fuzzy safe area $\chi_{\underline{S}}$ can be obtained by "cutting" its original form at ξ_k value (the so called critical point of a fuzzy number) and then normalising. The membership function of the fuzzy limit area $\chi_{\underline{L}}$ can be found as the complement of $\chi_{\underline{S}}$ (Fig. 8)

$$\chi_{\underline{S}}\Big(\frac{\lambda}{\lambda_k}\Big) = \begin{cases} 1 & \text{for} \quad \dfrac{\lambda}{\lambda_k} \leqslant \xi_k \\ C_S\Big(\dfrac{\lambda}{\lambda_k}\Big)\dfrac{1}{C_S(\xi_k)} & \text{for} \quad \xi_k < \dfrac{\lambda}{\lambda_k} < 1 \\ 0 & \text{for} \quad \dfrac{\lambda}{\lambda_k} \geqslant 1 \end{cases}$$
$$\chi_{\underline{L}}\Big(\frac{\lambda}{\lambda_k}\Big) = \begin{cases} 1 - \chi_{\underline{S}}\Big(\dfrac{\lambda}{\lambda_k}\Big) & \text{for} \quad \dfrac{\lambda}{\lambda_k} < 1 \\ 0 & \text{for} \quad \dfrac{\lambda}{\lambda_k} \geqslant 1 \end{cases} \tag{3.10}$$

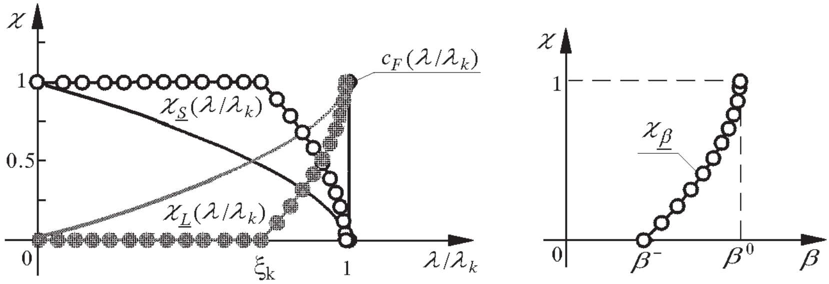

Fig. 8. Areas: safe $\underline{S}$, limit $\underline{L}$ and failure F in the snap-through problem; fuzzy Hasofer-Lind's reliability index $\underline{\beta}$ = {a little bit less than β^0 in $<\beta^-, \beta^0>$}

Now, the fuzzy Hasofer-Lind reliability index is a positive fuzzy number (Fig. 8).

The mean values of the random variables and also the origin of the standard random variables coordinates belong to the fuzzy safe area $\underline{S}$ in the degree equal 1 (Fig. 9).

Probabilities according to Zadeh (3.6) and (3.7) are the estimation of failure hazard (3.8). For example, in the case of $\mu_{EA} = 1000\,\mathrm{kN}$, $V_{EA} = 0.1$ and $\mu_\lambda = 3.5\,\mathrm{kN}$, $V_\lambda = 0.3$ and in the case of the slope of the truss $\gamma = 15°$, this estimation is as follows

$$0.0004 \leqslant P_f \leqslant 0.0109$$

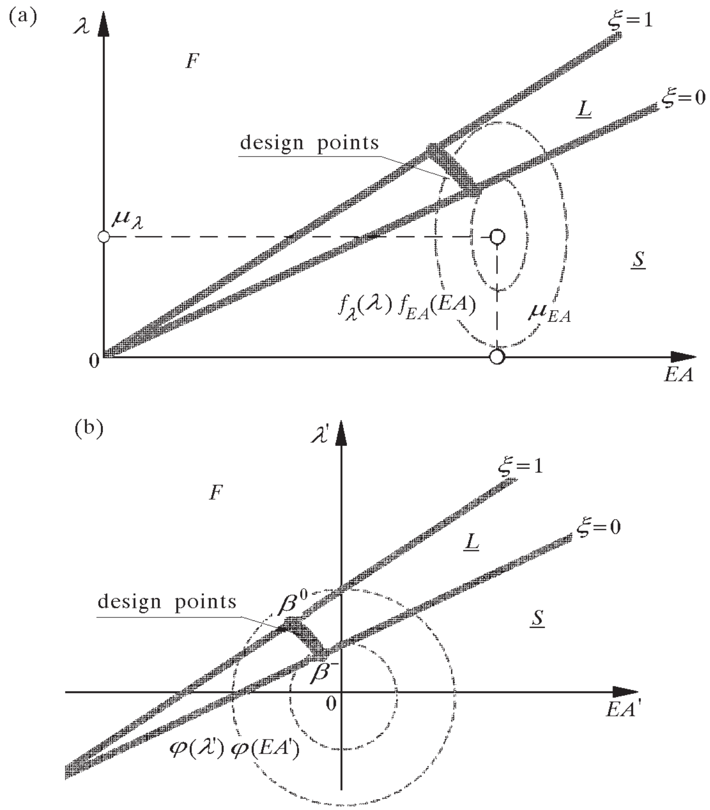

Fig. 9. Areas: safe $\underline{S}$, limit $\underline{L}$ and failure F: (a) in the random variables λ and EA coordinates, (b) in the standard random variables λ' and EA' coordinates

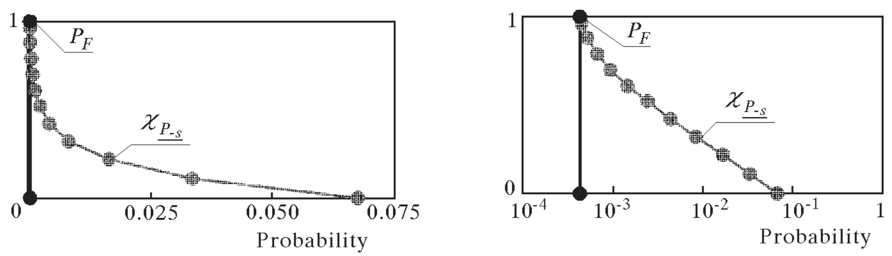

Fig. 10. Fuzzy probabilities of the failure according to Yager

The fuzzy probabilities according to Yager (Kacprzyk, 1986) of the fuzzy event $\underline{-S}$ estimate the failure hazard in Fig. 10.

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Ocena stateczności konstrukcji w ujęciu teorii zbiorów rozmytych

Streszczenie

Losowość i nieprecyzyjność są dwoma pojęciami o różnych, często mylonych znaczeniach. Losowość jest przedmiotem badań teorii prawdopodobieństwa, natomiast analizą nieprecyzyjności zajmuje się teoria zbiorów rozmytych. W pracy zastosowano teorię niezawodności przy użyciu wskaźnika Hasofera-Linda do oszacowania prawdopodobieństwa awarii konstrukcji narażonych na utratę statecznej postaci równowagi. W odróżnieniu od klasycznej analizy niezawodności funkcję stanu granicznego określono jednak przy zastosowaniu teorii zbiorów rozmytych, formułując trzy obszary rozmyte: stanów pożądanych, stanów granicznych i stanów niepożądanych. Do obliczenia odpowiednich prawdopodobieństw użyto aproksymacyjnej metody FORM. Takie postępowanie umożliwia uwzględnienie w analizie niezawodności osłabiania się konstrukcji jeszcze przed osiągnięciem punktu granicznego na ścieżce równowagi.

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