VIBRATIONS AND STABILITY OF A NON-CONSERVATIVELY COMPRESSED PRISMATIC COLUMN UNDER NONLINEAR CREEP CONDITIONS

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The aim of this paper is to investigate the influence of the physical nonlinearity of the material under creep conditions on vibrations and stability of a non-conservatively compressed column. Especially, behaviour of the characteristic curves is examined for a column compressed by a tangential force. The additional effects such as: compressibility of the column axis, rotational inertia and external damping are taken into account.

Key words: column, nonlinear creep, vibrations, stability

1. Introduction

There exists comprehensive literature devoted to the stability of linearly elastic structural systems, mainly columns, loaded by non-conservative forces (Bolotin, 1961; Gajewski and Życzkowski, 1988). Considerably fewer papers take into account the rheological properties of material. In particular, some of them consider a very interesting destabilization phenomenon due to the internal damping of material. The early papers dealing with the above problem are; e.g., Zoriy and Leonov (1961), Herrmann and Jong (1965), Dżygadło and Solarz (1970), Gajewski and Życzkowski (1972), Gajewski (1972). The internal damping of material has been characterized by a linear rheological model of visco-elastic material of the Kelvin-Voigt type.

The influence of internal damping of material on optimum shapes of structural elements was taken into account by several authors; e.g., Plaut (1972, 1975), Claudon (1975, 1978), Claudon and Sunakawa (1981), Langthjem (1993). Material was also described by means of the Voigt-Kelvin linear model. Many other problems of analysis and synthesis of columns compressed
by follower forces in view of their stability have been discussed by Bogacz and Janiszewski (1987).

Stability of columns compressed by a non-conservative concentrated force under nonlinear creep conditions was investigated by Kowalski (1975) in his Ph.D. Thesis (under supervision of M.Życzkowski). Some elements of the work have been presented by Życzkowski and Kowalski (1984). The authors took into account the Norton and Zhukov-Rabotnov-Churikov (1953) nonlinear models of material. The dependence of the critical force causing loss of static and kinetic stability on the so-called tangency coefficient has been considered.

Some problems of the optimal design of conservative systems with respect to nonlinear creep stability were investigated:
– for bars by Życzkowski and Wojdanowska (1972), Błachut and Życzkowski (1984), Wróblewski (1989)
– for trusses by Wojdanowska and Życzkowski (1972), Wojdanowska (1974)
– for arches by Wróblewski and Życzkowski (1989)

A comprehensive review of papers (187 references) devoted to the optimal structural design under creep conditions was presented by Życzkowski (1996). Recently, an attempt at optimization of a column compressed by a non-conservative force in nonlinear creep conditions was presented by Gajewski (1997).

The principal aim of the present work is to examine the influence of nonlinear creep of material on vibrations and stability of column compressed by the tangential force. Especially, behaviour of the characteristic curves will be considered. We will also take into account the additional effects; namely, compressibility of the axis, rotational inertia and external damping.

2. Nonlinear creep law

Following the state equation hypothesis of Davenport (1938) it is assumed that the stress and strain components of the basic uniaxial stress state are interrelated by the following creep law, accounting for the strain hardening

\[ \Phi(\sigma, p, \dot{p}) = 0 \]  

(2.1)

where

\( p \) – creep strain, \( p = \varepsilon - \sigma / E \)
\( \varepsilon \) – full strain
\( \sigma \) - stress
\( E \) - elastic modulus
\( \Phi \) - given material function

and the bar over a symbol denotes dimensional quantities. Among various creep stability theories the Rabotnov-Shesterevov (1957) theory of perfect, straight bars seems to be most suitable here. Generally, it can be assumed that during vibrations of a system (or as a result of buckling), the stress and strain components in a basic state are subject to small variations and the creep law (2.1) can be linearized with respect to them. The behaviour of the variations determines the stability of the basic state (precritical) at a critical time \( t^* \). In the basic state the relation between stress \( \bar{\sigma} \) and strain \( \bar{\varepsilon} \) is determined by Eq (2.1), while the tangent creep modulus \( \bar{E}_{tc} \) should be evaluated on the basis of Rabotnov-Shesterevov theory from the following equation

\[
\left. \frac{\partial \Phi}{\partial \bar{\sigma}} \right|_0 \delta \bar{\varepsilon} + \left. \frac{\partial \Phi}{\partial \bar{\sigma}} \right|_0 \delta \bar{\sigma} + \left. \frac{\partial \Phi}{\partial \bar{\sigma}} \right|_0 \delta \bar{\varepsilon} = 0
\]  

(2.2)

Assuming that the variations of stress and strain components are subject to small linear vibrations of complex frequency \( \bar{\Omega} \)

\[
\delta \bar{\varepsilon} = \delta \bar{\varepsilon}^a e^{\bar{\Omega} t} \quad \delta \bar{\sigma} = \delta \bar{\sigma}^a e^{\bar{\Omega} t}
\]  

(2.3)

and substituting them into Eq (2.2) we obtain the tangent creep modulus

\[
\bar{E}_{tc} = \frac{\delta \bar{\sigma}^a}{\delta \bar{\varepsilon}^a} = \frac{\bar{\Omega} \left. \frac{\partial \Phi}{\partial \bar{\sigma}} \right|_0 + \left. \frac{\partial \Phi}{\partial \bar{\varepsilon}} \right|_0}{\bar{\Omega} \left. \frac{\partial \Phi}{\partial \bar{\varepsilon}} \right|_0 + \left. \frac{\partial \Phi}{\partial \bar{\sigma}} \right|_0 - \left. \frac{\partial \Phi}{\partial \bar{\sigma}} \right|_0}
\]  

(2.4)

In this paper, the commonly used strain hardening creep law suggested by Rabotnov (1966) has been adopted

\[
\Phi = \dot{\bar{\sigma}} - \Gamma \bar{\sigma}^n = \dot{\bar{\sigma}} - \Gamma \bar{\sigma}^n p^{-\mu}
\]  

(2.5)

where \( \mu, n, \Gamma \) denote the material constants (temperature dependent). Moreover, all the results are obtained using material constants for copper at a temperature 200°C, \( n = 32.8, \mu = 9.52, \bar{E} = 1.22 \cdot 10^5 \text{ MPa}, \Gamma = 2.18 \cdot 10^{-113} \text{ (MPa)}^{-n} \text{h}^{-1} \) (see: Zhukov et al., 1953).

In the basic precritical state under the assumptions of constant stress \( \bar{\sigma}(t) = \text{const} \) and with the initial condition \( p(0) = 0 \), we obtain from Eqs (2.1) and (2.5)

\[
\varepsilon_0 = \frac{\bar{\sigma}_0}{\bar{E}} \left\{ 1 + \bar{E}[(1 + \mu) \Gamma t^*]^{1/1+\mu} \bar{\sigma}_0^{\frac{n-1-\mu}{1+\mu}} \right\}
\]  

(2.6)
and the "secant modulus"

\[
E_{sc} = \frac{E_{sc}}{E_0} = \frac{\bar{E}}{E_0 \left\{ 1 + \bar{E}[(1 + \mu)\Gamma t_\ast]^\frac{1}{1+\mu} |\bar{\sigma}|^\frac{n-1-\mu}{1+\mu} \right\}}
\]  

(2.7)

According to the Rabotnov-Shesterikov (1957) theory, the "tangent modulus" for the nonlinear creep law (2.5) can be written in the form

\[
E_{tc} = \frac{\bar{E} \left(1 + \frac{1+\mu}{\mu} \tilde{t}_\ast \bar{\Omega} \right)}{E_0 \left\{ 1 + \frac{1+\mu}{\mu} \tilde{t}_\ast \bar{\Omega} + \frac{n(1+\mu)}{\mu} \Gamma t_\ast \bar{E}[(1 + \mu)\Gamma t_\ast]^{-\frac{1}{1+\mu}} |\bar{\sigma}|^\frac{n-1-\mu}{1+\mu} \right\}}
\]

(2.8)

It is a function of the critical time \( \tilde{t}_\ast \) and complex frequency of vibration \( \bar{\Omega} = \tilde{\delta} + i\tilde{\omega} \). \( E_0 \) is a certain constant of the stress dimension.

3. State equation

The vibrations and stability analysis of a non-prismatic column compressed by a non-conservative force requires first the calculation of displacement under uniaxial compression (precritical state) and then the application of the kinetic stability criterion, making use of small superposed vibrations. The general equations of precritical and vibration states were derived and presented in the monograph by Gajewski and Życzkowski (1988), where the effects of: extensibility of the axis, shear deformations, rotational inertia and nonlinear properties of the material were introduced. The equations of small vibrations of complex frequency \( \bar{\Omega} \) superposed on the momentless precritical state can be transformed to four complex or to eight real linear ordinary differential equations with the appropriate boundary conditions. In the case of cantilever compressed by a follower force \( P \) it is a non-self-adjoint boundary value problem. For a prismatic column the equations of vibration state can be reduced one fourth order differential equation of constant complex coefficients

\[
v^{IV} + f v'' + g v = 0
\]

\[
f = \tilde{E}_{tc}[\varepsilon_{01} P(1 + hP) - r\alpha\bar{\rho}\Omega^2] - \varepsilon_{01} h\bar{\rho}\Omega^2
\]

\[
\tilde{E}_{tc} = E_{tc}^{-1}
\]

(3.1)

\[
g = \varepsilon_{01} \tilde{E}_{tc} \left\{ \gamma \varepsilon_{01} P(1 + hP)^2\Omega + [\varepsilon_{01}(1 + hP) + r\alpha h\bar{\rho}\Omega^2]\bar{\rho}\Omega^2 \right\}
\]
where the following dimensionless quantities have been introduced:

— independent variables

\[ x = \frac{s}{l} \quad t = \frac{\bar{t}}{t_0} \]

— external force

\[ P = \frac{\overline{P}l^2}{E_0 I_0} \]

— constant parameters

\[ \alpha = \frac{I_0}{A_0 \ell^2} \quad t_0 = \sqrt{\frac{\rho_0 A_0 \ell^4}{E_0 I_0}} \quad t_* = \frac{\bar{t}_*}{t_0} \quad \bar{\rho} = \frac{\rho}{\rho_0} \]

— elongation of the axis

\[ \varepsilon_{00} = -\frac{\alpha P}{E_{sc}} \quad \varepsilon_{00} = -\alpha P \left( 1 + TP \frac{n-1-\mu}{1+\mu} \right) \quad \varepsilon_{01} = 1 + \varepsilon_{00} \]

— quantities connected with the physical law

\[ \begin{align*}
T &= T_0 \left( \frac{T}{\tau_0} \right)^{\frac{1}{1+\mu}} \left( \frac{\alpha}{\alpha_0} \right)^{\frac{n-1-\mu}{1+\mu}} \quad T_0 = e \left[ (1 + \mu) \Gamma E_0^n \tau_0 \right] \quad T_0 = e \left[ \frac{E}{E_0} \right]^{\frac{1}{1+\mu}} \frac{n-1-\mu}{1+\mu} \\
E_{tc} &= e \frac{1 + \frac{1+\mu}{\mu} t_* \Omega}{1 + \frac{1+\mu}{\mu} t_* \Omega + \frac{n}{\mu} TP \frac{n-1-\mu}{1+\mu}} \quad e = \frac{E}{E_0} \\
\tau &= t_0 t_* = \bar{t}_* \quad \tau_0 = 3600 \text{ s} \quad \alpha_0 = 10^{-4}
\end{align*} \]

\( A_0 \) denotes the cross-sectional area and \( I_0 \) denotes the moment of inertia of the prismatic column cross-sections, \( \rho_0 \) is a constant of density dimension, \( t_0 \) denotes a certain constant which may be treated as a unit of time and \( \eta, \gamma \) denote the tangency coefficient and external viscous damping coefficient, respectively. The parameter \( \tau \) characterises the magnitude of the cross-sectional rotational inertia while the function \( h(\Omega) \) characterizing the shear effects is defined by the equation

\[ h(\Omega) = \frac{\alpha \varepsilon_{01}}{k_1 E_{tc} - \alpha \varepsilon_{01} P} \quad (3.2) \]

where \( k_1 \) denotes the shear coefficient. Taking into account the physical constants for copper, given above, and \( e \equiv 1 \) we have \( T_0 = 0.781408 \).
In further parts of this work we neglect the shear effect \((h = 0)\), which considerably simplifies numerical calculations. In such a case the boundary conditions for the cantilever column can be written in the form

\[
\begin{align*}
v(0) &= 0 \\
v'(0) &= 0 \\
v''(1) &= 0 \\
v'''(1) + c_3v'(1) &= 0
\end{align*}
\] (3.3)

where

\[
c_3 = \tilde{E}_{lc}[\varepsilon_0(1 - \eta)P - r\alpha\rho \Omega^2]
\] (3.4)

The solution of Eq (3.1) can be looked for in the following form

\[
v(x) = A \sin s_1 x + B \cos s_1 x + C \sin s_2 x + D \cos s_2 x
\] (3.5)

where \(s_1\) and \(s_2\) fulfil the algebraic complex characteristic equation

\[
s^4 + fs^2 + g = 0
\] (3.6)

Substituting Eq (3.5) into the boundary conditions (3.3) we obtain a system of four linear and homogeneous equations on constants \(A, B, C\) and \(D\). The principal determinant of these equations equalled to zero allows one to calculate the relation between complex frequencies of vibrations and compressive force \(P\) and other parameters of the problem (characteristic curves). It can be written in the form analogous to the equation presented by Gajewski (1972)

\[
F_1 - c_3F_2 = 0
\] (3.7)

and

\[
F_1 = (s_1^4 + s_2^4) - s_1s_2(s_1^2 + s_2^2) \sin s_1 \sin s_2 - 2s_1^2s_2^2 \cos s_1 \cos s_2
\]

\[
F_2 = (s_1^2 + s_2^2) - 2s_1s_2 \sin s_1 \sin s_2 - (s_1^2 + s_2^2) \cos s_1 \cos s_2
\]

After separating the real and imaginary parts in Eq (3.7) we obtain two real algebraic equations, which determine the real \(\delta\) and imaginary \(\omega\) parts of complex frequency \(\Omega\).

4. Numerical calculations and results

Using the Newton-Raphson method to solve Eq (3.7) numerical calculations have been made for various values of parameters characterising extensibility of axis, critical time, external damping and cross-sectional rotational
Fig. 1. Characteristic curves versus the critical time parameter $\tau$ for: $\eta = 1$, $\alpha = 10^{-5}$, $\gamma_0 = 0$, $\tau = 0$

Fig. 2. Characteristic curves versus the extensibility parameter $\alpha$ for: $\eta = 1$, $\tau = 1$ s, $\gamma_0 = 0$, $r = 0$

Fig. 3. Characteristic curves versus the rotational inertia parameter $r$ for: $\eta = 1$, $\alpha = 10^{-5}$, $\tau = 1$ s, $\gamma_0 = 0$
inertia. All the results presented in this paper are obtained for the tangential force, i.e., for $\eta = 1$ with the shear effects neglected. Fig.1 shows that the critical time parameter $\tau$ mostly influences the $(P - \omega)$ curves. In a minor degree it changes the critical magnitude of the force $P$ (for which $\delta = 0$). A similar situation occurs when the characteristic curves depend on the extensibility parameter $\alpha$. As it is seen in Fig.2 the critical value of the force $P$ is nearly independent of the parameter $\alpha \in (10^{-5}, 10^{-4})$.

The cross-sectional rotational inertia parameter $\tau$ exerts essential influence on the characteristic curves $(P - \omega)$ and $(P - \delta)$. A notable decrease in the critical force with increasing parameter $\tau$ may be observed in Fig.3. The results obtained here are similar to those presented by Kounadis and Katsikadelis (1976) for a linearly elastic column.

Fig. 4. Characteristic curves versus the external damping parameter $\gamma_0$ for: $\eta = 1$, $\alpha = 10^{-5}$, $\tau = 1 \text{s}$, $\delta = 0$

Fig. 5. Characteristic curves versus the external damping parameter $\gamma_0$ for: $\eta = 1$, $\alpha = 10^{-5}$, $\tau = 3600 \text{s}$, $\delta = 0$
The external damping parameter $\gamma_0$ has a very small effect on the $(P - \omega)$ curves, however it strongly modifies the $(P - \delta)$ curves. As it is seen in Fig.4 and Fig.5 the external damping eliminates the destabilisation phenomenon. Such an effect has been previously observed for linearly visco-elastic columns by Herrmann and Jong (1966), Gajewski (1972), Gajewski and Życzkowski (1972), Seyranian (1987) and others.

5. Conclusions

The vibrations and stability of a tangentially compressed column in nonlinear creep conditions have been comprehensively examined. It has been assumed that the mechanical properties of the material of the column are characterised by the Zhukov-Rabotnov-Churikov (1953) law. Although the nonlinear creep law considered in this paper is essentially different from the linear visco-elastic models of material, the results presented here are close in character to those obtained for the Kelvin-Voigt model. Especially, the destabilizing phenomenon is quite similar to that very well known from the literature.

References


Drgania i stateczność kolumny pryzmatycznej ściskanej siłą niekonserwatywną w warunkach nieliniowego pełzania

Streszczenie

W pracy zbadano wpływ nieliniowości fizycznej materiału na drgania i stateczność niekonserwatywnie ściskanych kolumn pryzmatycznych w warunkach nieliniowego pełzania. W szczególności przedstawiono kształty tak zwanych krzywych charakterystycznych (zależności części rzeczywistej i urojonej zespolonej częstości drgań od wartości siły ściskającej) dla kolumny ściskanej siłą śiedzącą. W pracy uwzględniono również dodatkowe czynniki, wpływające na kształty krzywych, a mianowicie: ściśliwość osi kolumny, bezwładność obrotów przekrojów oraz wiskotyczne tłumienie zewnętrzne drgań.

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