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ON DYNAMICS OF SUBSTRUCTURED SHELLS[1]

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The contribution outlines a general modelling method leading from the general equations of thin shells with locally periodic structure to the averaged equations with slowly varying coefficients. The proposed model describes the effect of periodicity cell size on the overall shell response and hence can be applied to the dispersive analysis in dynamic problems.

Key words: shells, microstructure, modelling, dynamics

1. Introduction

It is known that 2D-theories of shells, derived from three dimensional problems of solid mechanics, are valid if a shell is sufficiently thin. This requirement makes it possible to formulate 2D-constitutive shell and plate equations as physically reasonable approximations of three dimensional equations, obtained by means of the well known Kirchhoff hypothesis or using the asymptotic approach to derive 2D-theories of plates. Hence, the undeformed shell mean thickness h plays an important role of a certain small length parameter (here and in the sequel the plate will be treated as a special case of the shell); on the basis of this parameter estimation of orders of various terms in shell equations can be performed as shown by Koiter, Simmonds (1973) and Pietraszkiewicz (1979).

By a substructured shell we shall mean a thin shell endowed with a material inhomogeneity and/or a variable thickness having what will be called a locally periodic structure in directions tangent to the undeformed shell midsurface $\mathcal{M}$. It means that every medium-size piece of the shell constituting a

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shallow shell, with a sufficient accuracy, can be described as having a periodic structure related to the Cartesian coordinates on a certain plane tangent to $\mathcal{M}$. This situation takes place for shells made of composite materials or reinforced by systems of fibres or stiffeners. Hence, to every point $\boldsymbol{x} \in \mathcal{M}$ we assign the plane $\mathcal{T}_{\boldsymbol{x}}$ tangent to $\mathcal{M}$ at $\boldsymbol{x}$ and periods $l_\alpha(\boldsymbol{x})$, $\alpha = 1,2$ in direction of unit vectors $\boldsymbol{e}_\alpha(\boldsymbol{x})$ on $\mathcal{T}_{\boldsymbol{x}}$. Moreover, we assume that $\boldsymbol{d}_\alpha(\boldsymbol{x}) \equiv l_\alpha(\boldsymbol{x})\boldsymbol{e}_\alpha(\boldsymbol{x})$, $\boldsymbol{x} \in \mathcal{M}$, are slowly varying sufficiently regular functions. Define by l the mean value of functions $l_1(\cdot)$, $l_2(\cdot)$. This value will be treated as a substructure length parameter, provided that it is sufficiently small compared to the minimum shell midsurface curvature radius as well as to the characteristic wavelength of midsurface deformation patterns.

Thus, we jump to a conclusion that in the description of thin substructured shells two small parameters are present: mean shell thickness h and substructure length parameter l. The cases $l \ll h$ and $l \approx h$ require a special treatment since then the Kirchhoff hypothesis cannot be applied to the derivation of 2D-theory. On the other hand the case $l \gg h$ can be investigated within the framework of the known 2D-theory which will be referred to as the mezostructured shell 2D-theory. Notice, that the situation discussed above is similar to that investigated for plates by Kohn and Vogelius (1984). In the sequel we shall confine ourselves to mezostructured shells, where l will be referred to as a mezostructure length parameter. 2D-equations of mezostructured shells have a form similar to that of homogeneous shells but involving rapidly varying and often discontinuous functional coefficients. That is why the aforementioned equations cannot be directly applied to the numerical analysis of many special problems.

The aim of this contribution is to propose a certain general approach leading from 2D-equations of mezostructured shells to the averaged 2D-equations with slowly varying functional coefficients. Moreover, the obtained equations have to describe the effect of mezostructure length parameter on the overall dynamic shell behaviour (a length-scale effect). This fact is important in dynamic problems where we have to calculate higher vibration frequencies and analyze the dispersive properties of the system. For this reason the asymptotic homogenization methods, neglecting the length-scale effect, will be not used in this paper as a tool of modelling. Notice, that the asymptotic approach to averaging of 2D-shell equations, based on that used by Kohn and Vogelius (1984), has been proposed by Lewiński and Telega (1988) (cf also Lutoborski, 1985). Unfortunately, the results have been obtained under rather artificial assumption that the shell stiffnesses are periodic with respect to a certain curvilinear coordinate system on $\mathcal{M}$; the above assumption eliminates even such fundamental composite shell problems as those related to shells reinforced by a system of fibres or stiffeners with a constant cross section. This drawback is implied by a very restrictive definition of the concept of locally periodic structure used by Kohn and Vogelius (1984) and in related papers. That is why in this contribution a more general notion of a locally periodic function and hence a more general description of a local periodic shell structure will be taken as a basis of analysis.

2. 2D-equations of mezostructured shells

For the sake of simplicity we shall investigate the simplified linear Kirchhoff-Love theory of thin elastic shells in which the terms depending on the second metric tensor of $\mathcal{M}$ are neglected in the formulae for curvature changes (cf Koiter, 1960). The governing 2D-equations of this theory can be written down in the matrix form

$$L\boldsymbol{s} - \mu\ddot{\boldsymbol{u}} + \boldsymbol{f} = \boldsymbol{0} \qquad\qquad \boldsymbol{s} = \mathsf{C} : E\boldsymbol{u} \tag{2.1}$$

where $\boldsymbol{u} = [\boldsymbol{v}, w]^\top$ are the midsurface shell displacements (in tangent and normal directions, respectively), $\boldsymbol{s} = [\boldsymbol{n}, \boldsymbol{m}]^\top$ are the stress resultants and stress couples, respectively, $\boldsymbol{f}$ stands for the external force components, $\mathsf{C} = \mathrm{diag}\{\mathsf{D}, \mathsf{B}\}$, where D, B are the membrane and bending stiffness tensors, respectively, μ is the shell mass density per midsurface unit area and L, E are differential operators

$$L = \left\{ \begin{array}{cc} \nabla\cdot & 0 \\ \boldsymbol{b}: & (\nabla\otimes\nabla): \end{array} \right\} \qquad\qquad E = \left\{ \begin{array}{cc} \nabla\otimes & -\boldsymbol{b} \\ 0 & -(\nabla\otimes\nabla) \end{array} \right\} \tag{2.2}$$

with ∇ as the covariant derivative on the midsurface $\mathcal{M}$ and $\boldsymbol{b}$ as the midsurface second metric tensor. For mezostructured composite shells $\mu(\cdot)$ and $\mathsf{C}(\cdot)$ are rapidly varying discontinuous functions; that is why Eqs (2.1) cannot be directly applied to the numerical analysis of special problems. To overcome this difficulty we shall derive from Eqs (2.1) a certain averaged form of the mezostructured shell theory. This could be done by the asymptotic approach using the concept of G-convergence, cf Jikov et al. (1994). However, using this approach we obtain the resulting averaged 2D-equations in the form independent of the mezostructure length parameter l. Hence, the problem arises how to derive from Eqs (2.1) the 2D-equations with slowly varying coefficients depending on the mezostructure length parameter. In order to solve this problem we shall apply the modelling procedure based on the concept of internal variables proposed by Woźniak (1997).

3. Basic concepts

Let the shell midsurface be smooth and to every $\boldsymbol{x} \in \mathcal{M}$ be uniquely assigned a vector basis $\boldsymbol{d}_\alpha(\boldsymbol{x})$, $\alpha = 1,2$, on the plane $\mathcal{T}_{\boldsymbol{x}}$ tangent to $\mathcal{M}$ at $\boldsymbol{x}$. The vector fields $\boldsymbol{d}_\alpha(\cdot)$ are assumed to be smooth and slowly varying; the exact explanation of the latter concept will be given below. Moreover, the values of $|\boldsymbol{d}_\alpha(\cdot)|$ have to be sufficiently small compared with the minimum characteristic length dimension of a midsurface $\mathcal{M}$ and sufficiently large compared with the maximum thickness of a shell. Define $\Delta(\boldsymbol{x}) = \{\boldsymbol{z} \in \mathcal{T}_{\boldsymbol{x}} : \ \boldsymbol{z} = \boldsymbol{x} + \eta^\alpha \boldsymbol{d}_\alpha(\boldsymbol{x}), \ \ \eta^\alpha \in (-0.5, 0.5)\}$. By $\mathcal{M}^0$ we denote a set of points on $\mathcal{M}$ such that for every $\boldsymbol{x} \in \mathcal{M}^0$ the plane element $\Delta(\boldsymbol{x})$ is the orthogonal projection on $\mathcal{T}_{\boldsymbol{x}}$ of some piece $\mathcal{M}_{\boldsymbol{x}}$ of $\mathcal{M}$. Obviously, every $\mathcal{M}_{\boldsymbol{x}}$, $\boldsymbol{x} \in \mathcal{M}^0$, is a midsurface of a certain shallow shell element and $\mathcal{M}^0$ is a part of midsurface $\mathcal{M}$ which does not comprise a certain near-boundary layer.

Let $\varphi(\cdot)$ be an arbitrary integrable function defined (almost everywhere) on $\mathcal{M}$. In every $\overline{\mathcal{M}_{\boldsymbol{x}}}$ this function will be treated as a function $\varphi(\boldsymbol{z})$ of a point $\boldsymbol{z} \in \overline{\Delta}(\boldsymbol{x})$; this situation is typical for parametrization of shallow shells. We assume that in a vicinity of every $\overline{\mathcal{M}_{\boldsymbol{x}}}$, $\boldsymbol{x} \in \mathcal{M}^0$, the material and inertial properties of a shell can be described with a sufficient accuracy by $\Delta(\boldsymbol{x})$-periodic functions and hence defined on $\mathcal{T}_{\boldsymbol{x}}$. The *averaged value* of φ over $\Delta(\boldsymbol{x})$ will be denoted by

$$\langle \varphi \rangle(\boldsymbol{x}) = \frac{1}{|\Delta(\boldsymbol{x})|} \int\limits_{\Delta(x)} \varphi(\boldsymbol{z})\, dz \qquad \boldsymbol{x} \in \mathcal{M}^0 \tag{3.1}$$

where $|\Delta(\boldsymbol{x})| = \mathrm{mes}\Delta(\boldsymbol{x})$. Hence the averaging operator $\langle \cdot \rangle$ is uniquely determined by the mapping $\Delta : \ \mathcal{M}^0 \ni \boldsymbol{x} \to \Delta(\boldsymbol{x})$. In the sequel all functions under consideration are assumed to be defined (almost everywhere) on $\mathcal{M}$ (they can also depend on the time coordinate t) and to satisfy the required regularity conditions. Moreover, an arbitrary function $\varphi(\cdot)\Big|\overline{\mathcal{M}_{\boldsymbol{x}}}$, i.e. a function φ with the domain restricted to $\overline{\mathcal{M}_{\boldsymbol{x}}}$, $\boldsymbol{x} \in \mathcal{M}^0$, will be treated as a function of $\boldsymbol{z} \in \overline{\Delta}(\boldsymbol{x})$. A function $\langle \varphi \rangle(\cdot)$ will be also denoted by $\langle \varphi \rangle$.

The leading concept of the modelling approach is that of the *tolerance space*, proposed by Zeeman (1962), as a pair $(\mathcal{S}, \approx)$, where $\mathcal{S}$ is a certain nonempty set and $\approx$ is a tolerance relation defined on $\mathcal{S} \times \mathcal{S}$. It has to be remembered that in the general case this relation is not transitive. Roughly speaking, the tolerance relation will be treated as a certain indiscernibility relation between elements of $\mathcal{S}$; $s_1 \approx s_2$ means that s_1 can be approximated with a sufficient accuracy by s_2 and *vice versa*. In the sequel the symbol $\approx$ will stand for a certain tolerance relation describing the accuracy of performed calculations or required measurements but it has to be remembered that in different formulas the symbol $\approx$ is related to different tolerances.

A differentiable function $F(\cdot)$ will be called *slowly varying*, $F \in SV(\Delta)$, if for every integrable function $\varphi(\cdot)$ satisfies conditions of the form

$$\langle \varphi F \rangle(\boldsymbol{x}) \approx \langle \varphi \rangle(\boldsymbol{x}) F(\boldsymbol{x}) \qquad \boldsymbol{x} \in \mathcal{M}^0 \tag{3.2}$$

and the similar conditions are also fulfilled by all derivatives of $F(\cdot)$.

An integrable function $f(\cdot)$ will be termed *locally periodic*, $f \in LP(\Delta)$, if $\langle f \rangle(\cdot) \in SV(\Delta)$ and if for every $\boldsymbol{x} \in \mathcal{M}^0$ there exist $\Delta(\boldsymbol{x})$-periodic function $f_{\boldsymbol{x}}(\cdot)$ (defined on $\mathcal{T}_{\boldsymbol{x}}$) which approximates $f(\cdot)\big|\overline{\mathcal{M}_{\boldsymbol{x}}}$. The function $f_{\boldsymbol{x}}(\cdot)$ will be called a *local periodic approximation* of $f(\cdot)$. It has to be emphasized that the concept of locally periodic function introduced here does not coincide with the definition of locally periodic structure used by Kohn and Vogelius (1984) and related papers on homogenization.

At last, by the *highly oscillating* function, $h \in HO(\Delta)$, we shall mean a differentiable function $h(\cdot)$ such that $h \in LP(\Delta)$ and for every $F \in SV(\Delta)$ the conditions

$$\langle \nabla(Fh) \rangle(\boldsymbol{x}) \approx \langle F \nabla h \rangle(\boldsymbol{x}) \qquad \boldsymbol{x} \in \mathcal{M}^0 \tag{3.3}$$

are assumed to hold. The detailed discussion of the above relation can be found in Woźniak (1997), where an alternative approach to Eq (3.3) was applied.

4. Internal variable model

We begin with Eqs (2.1) where the conditions $\mu \in LP(\Delta)$ and $\mathbf{C} \in LP(\Delta)$ for some $\Delta : \mathcal{M}^0 \ni \boldsymbol{x} \to \Delta(\boldsymbol{x})$ are assumed to hold. The idea of the internal variables is based on assumptions which restrict the class of displacement fields $\boldsymbol{u}(\cdot)$ and external loadings $\boldsymbol{f}(\cdot)$ in (2.1) to certain subclasses. The subclass of displacement fields comprise motions with wavelengths of an order much larger then l on which there are superimposed disturbances of displacements caused by the highly oscillating character of the shell mezostructure.

The *first modelling assumption* states that every averaged displacement field $\boldsymbol{U}(\cdot) = \langle\mu\rangle^{-1}\langle\mu\boldsymbol{u}\rangle(\cdot)$ under consideration is a slowly varying function, $\boldsymbol{U} \in SV(\Delta)$, and every displacement disturbance field defined by $\boldsymbol{d}(\cdot) = \boldsymbol{u}(\cdot) - \langle\boldsymbol{U}\rangle(\cdot)$ is assumed to be a highly oscillating function, $\boldsymbol{d} \in HO(\Delta)$. Notice that $\langle\mu\boldsymbol{d}\rangle = \boldsymbol{0}$.

The next step in modelling is related to the specification of the proper class of displacement disturbances. To this end for every $\boldsymbol{x} \in \mathcal{M}^0$ we shall introduce the local periodic approximations $\mathsf{C}_{\boldsymbol{x}}(\cdot)$ and $\mu_{\boldsymbol{x}}(\cdot)$ of locally periodic functions $\mathsf{C}(\cdot)$ and $\mu(\cdot)$, respectively. Denoting $\overline{\boldsymbol{f}} = \boldsymbol{f} + L(\mathsf{C} : E\boldsymbol{U}) - \mu\ddot{\boldsymbol{U}}$ and using Eqs (2.1) we formulate in $\overline{\Delta}(\boldsymbol{x})$ the *local problem*

$$L(\mathsf{C}_{\boldsymbol{x}} : E\boldsymbol{d}_{\boldsymbol{x}}) - \mu_{\boldsymbol{x}}\ddot{\boldsymbol{d}}_{\boldsymbol{x}} + \overline{\boldsymbol{f}} = \boldsymbol{0} \tag{4.1}$$

for a $\Delta(\boldsymbol{x})$-periodic function $\boldsymbol{d}_{\boldsymbol{x}}(\cdot,t)$ such that $\langle\mu_{\boldsymbol{x}}\boldsymbol{d}_{\boldsymbol{x}}\rangle = \boldsymbol{0}$. The above equation describes vibrations $\boldsymbol{d}_{\boldsymbol{x}}(\boldsymbol{z},t)$, $\boldsymbol{z} \in \overline{\Delta}(\boldsymbol{x})$ of the shell element with midsurface $\overline{\mathcal{M}_{\boldsymbol{x}}}$ under forces $\overline{\boldsymbol{f}}$ and periodic boundary conditions. To solve this problem we shall use the orthogonalization method known in structural dynamics. Using this method we have to formulate in $\overline{\Delta}(\boldsymbol{x})$ the eigenvalue problem

$$L(\mathsf{C}_{\boldsymbol{x}} : E\boldsymbol{h}_{\boldsymbol{x}}) + \omega^2\mu_{\boldsymbol{x}}\boldsymbol{h}_{\boldsymbol{x}} = \boldsymbol{0} \tag{4.2}$$

for a $\Delta(\boldsymbol{x})$-periodic function $\boldsymbol{h}_{\boldsymbol{x}}(\cdot)$ such that $\langle\mu_{\boldsymbol{x}}\boldsymbol{h}_{\boldsymbol{x}}\rangle(\boldsymbol{x}) = \boldsymbol{0}$; here ω is an eigenvalue related to this functions. The expected principal modes $\boldsymbol{h}_{\boldsymbol{x}}(\cdot)$ of natural vibrations of the system described by Eqs (4.1) can be obtained, in most cases only in the approximate form represented by certain $\Delta(\boldsymbol{x})$-periodic functions $\boldsymbol{h}_{\boldsymbol{x}}{}^A(\cdot)$, $A = 1,2,\dots$. Hence an approximate solution to Eqs (4.1) can be expected in the form of a finite sum $\boldsymbol{d}_{\boldsymbol{x}} = \boldsymbol{h}_{\boldsymbol{x}}{}^A(\boldsymbol{z})Q_A(\boldsymbol{x},t)$, $\boldsymbol{z} \in \overline{\Delta}(\boldsymbol{x})$ (here and in the sequel A runs over $1,\dots,N$, summation convention over A holds) where $Q_A(\boldsymbol{x},\cdot)$ are unknown functions of time and $\langle\mu_{\boldsymbol{x}}\boldsymbol{h}_{\boldsymbol{x}}{}^A\rangle = \boldsymbol{0}$. These functions are governed by the orthogonality conditions

$$\langle[L(\mathsf{C}_{\boldsymbol{x}} : E\boldsymbol{d}_{\boldsymbol{x}}) - \mu_{\boldsymbol{x}}\ddot{\boldsymbol{d}}_{\boldsymbol{x}} + \overline{\boldsymbol{f}}] \cdot \boldsymbol{h}_{\boldsymbol{x}}{}^A\rangle(\boldsymbol{x}) = 0 \qquad \boldsymbol{x} \in \mathcal{M}^0 \tag{4.3}$$

where $\boldsymbol{d}_{\boldsymbol{x}} = \boldsymbol{h}_{\boldsymbol{x}}{}^A(\boldsymbol{z})Q_A(\boldsymbol{x},t)$, $\boldsymbol{z} \in \overline{\Delta}(\boldsymbol{x})$. The above conditions lead to a system of N ordinary differential equations for $Q_A(\boldsymbol{x},\cdot)$, which can be formulated independently for every $\boldsymbol{x} \in \mathcal{M}^0$.

Eqs (4.3) involve an unknown averaged displacement field $\boldsymbol{U}(\cdot)$. Because $\boldsymbol{U} \in SV(\Delta)$ we also introduce the following solvability conditions implied by Eqs (4.1)

$$\langle L(\mathsf{C}_{\boldsymbol{x}} : E\boldsymbol{d}_{\boldsymbol{x}}) - \mu_{\boldsymbol{x}}\ddot{\boldsymbol{d}}_{\boldsymbol{x}} + \overline{\boldsymbol{f}}\rangle(\boldsymbol{x}) = 0 \qquad \boldsymbol{x} \in \mathcal{M}^0 \tag{4.4}$$

which have to be considered together with Eqs (4.4).

The *second modelling assumption* states that the finite sums $\boldsymbol{d_x}(\boldsymbol{z},t) = \boldsymbol{h_x}^A(\boldsymbol{z})Q_A(\boldsymbol{x},t)$, $\boldsymbol{z} \in \overline{\Delta}(\boldsymbol{x})$, in Eqs (4.3), (4.4), describe, with a sufficient accuracy, the local periodic approximations of the displacement disturbances $\boldsymbol{d}(\cdot,t)$ defined on $\mathcal{M}$.

This assumption restricts both the class of external loadings $\boldsymbol{f}(\cdot)$ in Eqs (2.1) and the class of displacement disturbances. Since $\boldsymbol{d} \in HO(\Delta)$ it follows that $Q_A(\cdot,t) \in SV(\Delta)$ and there exist functions $\boldsymbol{h}^A \in HO(\Delta)$ such that $\boldsymbol{h_x}^A$ are periodic local approximations of $\boldsymbol{h}^A$ for every $\boldsymbol{x} \in \mathcal{M}^0$. The functions $\boldsymbol{h}^A(\cdot)$, $A = 1,...,N$, will be referred to as the shape functions.

Taking into account the first and second modelling assumptions, from Eqs (4.3), (4.4) after rather lengthy calculations we arrive at the following averaged form of equations of motion

$$\boldsymbol{L}\boldsymbol{S} - \langle\mu\rangle\ddot{\boldsymbol{U}} + \langle\boldsymbol{f}\rangle = \boldsymbol{0} \tag{4.5}$$

with constitutive equations of the form

$$\boldsymbol{S} = \langle\mathsf{C}\rangle : \boldsymbol{E}\boldsymbol{U} + \langle\mathsf{C} : \boldsymbol{E}\boldsymbol{h}^A\rangle Q_A \tag{4.6}$$

and what are called the dynamic evolution equations

$$\langle\mu\boldsymbol{h}^A \cdot \boldsymbol{h}^B\rangle\ddot{Q}_B + \langle\boldsymbol{E}\boldsymbol{h}^A : \mathsf{C} : \boldsymbol{E}\boldsymbol{h}^B\rangle Q_B + \langle\boldsymbol{E}\boldsymbol{h}^A : \mathsf{C}\rangle : \boldsymbol{E}\boldsymbol{U} + \langle\boldsymbol{f} \cdot \boldsymbol{h}^A\rangle = 0 \tag{4.7}$$

where $A = 1,...,N$. The above formulae represent the system of $3 + N$ equations for averaged displacements (three components of $\boldsymbol{U}(\cdot)$) and N extra unknowns $Q_A(\cdot)$, $A = 1,...,N$. It can be easily seen that for $Q_A(\cdot)$ we have derived the system of ordinary differential equations (4.7) involving only time derivatives of $Q_A(\cdot)$; that is why $Q_A(\cdot)$ have been called *internal variables* (or macro-internal variables, cf Woźniak (1997)). Thus, we conclude that internal variables do not enter the displacement boundary conditions. Hence, the number and form of these conditions is similar to those of the well known simplified 2D-theory of thin linear-elastic shells which is governed by Eqs (2.1).

In order to obtain the explicit form of Eqs (4.5) $\div$ (4.7) denote $\boldsymbol{U} = [\boldsymbol{V},\boldsymbol{W}]^\top$, $\boldsymbol{f} \equiv [f_1, f_2]^\top$, $f \equiv f_3$, where $\boldsymbol{f}$ and f are the external loadings (per midsurface unit area), tangent and normal to $\mathcal{M}$, respectively. Let us assume the finite sums $\boldsymbol{h}^A Q_A$ in the simplest form $[\boldsymbol{h}Q, gP]^\top$, where Q, P are internal variables and $\boldsymbol{h} = [h_1,h_2]^\top$, g are locally periodic functions describing the expected shape of displacement disturbances. Under the extra denotation $\mathsf{E} \equiv \nabla \otimes \boldsymbol{V} - \boldsymbol{b}\boldsymbol{W}$ from Eqs (4.5) ÷ (4.7) we obtain the following equations of motion

$$
\begin{aligned}
&\nabla \cdot \boldsymbol{N} - \langle \mu \rangle \ddot{\boldsymbol{V}} + \langle \boldsymbol{f} \rangle = \boldsymbol{0} \\
&(\nabla \otimes \nabla) : \boldsymbol{M} + \boldsymbol{b} : \boldsymbol{N} - \langle \mu \rangle \ddot{\boldsymbol{W}} + \langle f \rangle = 0
\end{aligned}
\tag{4.8}
$$

and the dynamic evolution equations

$$
\begin{aligned}
&\langle \mu \boldsymbol{h} \cdot \boldsymbol{h} \rangle \ddot{Q} + H + \langle \boldsymbol{f} \cdot \boldsymbol{h} \rangle = 0 \\
&\langle \mu g^2 \rangle \ddot{P} + G + \langle f g \rangle = 0
\end{aligned}
\tag{4.9}
$$

where the averaged stress resultants $\boldsymbol{N}$ and averaged stress couples $\boldsymbol{M}$ as well as H, G are given by the constitutive equations

$$
\begin{aligned}
\boldsymbol{N} &= \langle \mathsf{D} \rangle : \mathsf{E} + \langle \mathsf{D} : (\nabla \otimes \boldsymbol{h}) \rangle Q - \boldsymbol{b} : \langle \mathsf{D} g \rangle P \\
\boldsymbol{M} &= -\langle \mathsf{B} \rangle : (\nabla \otimes \nabla) \boldsymbol{W} - \langle \mathsf{B} : (\nabla \otimes \nabla) g \rangle P \\
H &= \langle (\nabla \otimes \boldsymbol{h}) : \mathsf{D} \rangle : \mathsf{E} + \langle (\nabla \otimes \boldsymbol{h}) : \mathsf{D} : (\nabla \otimes \boldsymbol{h}) \rangle Q - \boldsymbol{b} : \langle g \mathsf{D} : (\nabla \otimes \boldsymbol{h}) \rangle P \\
G &= -\boldsymbol{b} : \langle \mathsf{D} g \rangle : \mathsf{E} + \langle (\nabla \otimes \nabla) g : \mathsf{B} \rangle : (\nabla \otimes \nabla) \boldsymbol{W} - \boldsymbol{b} : \langle g \mathsf{D} : (\nabla \otimes \boldsymbol{h}) \rangle Q + \\
&+ \Big(\langle (\nabla \otimes \nabla) g : \mathsf{B} : (\nabla \otimes \nabla) g \rangle + \boldsymbol{b} : \langle g^2 \mathsf{D} \rangle : \boldsymbol{b} \Big) P
\end{aligned}
\tag{4.10}
$$

Eqs (4.8) ÷ (4.10) represent the first approximation of general equations (4.5) ÷ (4.7). The basic unknowns are the averaged displacements $\boldsymbol{W}(\cdot)$, $\boldsymbol{V}(\cdot)$, respectively, normal and tangent to the midsurface and internal variables $Q(\cdot)$, $P(\cdot)$. The terms involving $\langle \mu \boldsymbol{h} \cdot \boldsymbol{h} \rangle$, $\langle \mu g^2 \rangle$ in Eqs (4.9) are of an order $\mathcal{O}(l^2)$, $\mathcal{O}(l^4)$, respectively, and describe the effect of mezostructure size on the dynamic shell behaviour, the terms with $\langle g \mathsf{D} \rangle$, $\langle g^2 \mathsf{D} \rangle$ in Eqs (4.10) are of an order $\mathcal{O}(l^2)$, $\mathcal{O}(l^4)$, respectively, and describe this effect on the shell response also in quasi-stationary problems. The detailed discussion of the above equations, representing a special case of what is called the *internal variable 2D-model* of thin mezostructured shells, will be given in a forthcoming paper, cf Tomczyk (1998).

It has to be emphasized that solutions $\boldsymbol{U}(\cdot)$, $Q_A(\cdot)$ to initial-boundary value problems for Eqs (4.5) ÷ (4.7) have a physical sense only if they are represented by sufficiently regular slowly varying functions; the pertinent conditions can be verified *a posteriori*. For Kirchhoff plates the above equations reduce to the form which was introduced by Jędrysiak and Woźniak (1995); they were investigated by Jędrysiak (1998a,b) and by Baron and Jędrysiak (1998). A similar approach was applied by Michalak and Woźniak (1996) and Michalak (1998) to the modelling and analysis of wavy-plates. The internal variable 2D-models for medium thickness plates were studied by Baron and Woźniak (1995) as well as Baron and Jędrysiak (1998) and for beams by Mazur-Śniady (1993).

5. Conclusions

We begin with the main advantages of the 2D-model outlined in Section 3. Because the averaged locally periodic functions are slowly varying we conclude that all coefficients in Eqs (4.5) $\div$ (4.10) are slowly varying functions and hence their values can be calculated only at some points of $\mathcal{M}$ and obtained using an extrapolation method in all other points. Solutions to the pertinent initial-boundary value problems can be obtained by means of the well known numerical procedures which is rather difficult if we deal with Eqs (2.1). Moreover, the obtained internal variable model is able to describe the effect of the mezostructure size (i.e. the size of cells $\Delta(\boldsymbol{x})$, $\boldsymbol{x} \in \mathcal{M}^0$) on the global dynamic shell behaviour. At last the values of functional but slowly varying coefficients in Eqs (4.5) $\div$ (4.10) can be derived by a simple calculation of averages at some points $\boldsymbol{x}$ belonging to $\mathcal{M}^0$ provided that the systems of functions $\boldsymbol{h_x}^A(\cdot)$ for these points were previously determined. The main drawback of the model lies in the specification of functions $\boldsymbol{h_x}^A(\cdot)$, which represent approximate solutions to the eigenvalue problems (4.2) with periodic boundary conditions on *the cells $\Delta(\boldsymbol{x})$. These functions describe free vibrations of shell elements rela-*ted to pertinent cells. It is known that the exact form of these vibrations can be found for rather simple cell structures. In many cases the approximation of expected natural modes of free vibrations has to be based on the experience of the researcher and knowledge of free vibrations in similar elements. However, some recently obtained benchmark solutions, see the references at the end of Section 4, show that the internal variable model can be successfully applied to the vibration analysis of mezostructured plates and shells.

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O dynamice powłok z wewnetrzną strukturą

Streszczenie

W pracy przedstawiono metodę prowadzącą od 2D-modelu cienkich powłok o lokalnie periodycznej strukturze do uśrednionych równań o wolno zmiennych współczynnikach. Otrzymane równania opisują wpływ wielkości mikrostruktury na reakcje powłoki i są dogodne do zastosowania w analizie zagadnień dynamicznych.

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