JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
1, 36, 1998

# WEIGHT FUNCTIONS OF $K_1$ AND $K_2$ FOR A SINGLE RADIAL CRACK EMANATING FROM A CIRCULAR HOLE

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Weight functions of the stress intensity factors $K_1$ and $K_2$ for a single radial crack emanating from a circular hole are derived using the boundary element method (BEM) together with the Bueckner type singular complex stress function $Z(z)$ applied in the vicinity of the crack tip. Both weight functions, corresponding to the loading modes $I$ and $II$ respectively, are valid for any ratio of the crack length $a$ to hole radius $R$. They are represented in the unified form convenient for creating the weight function database. The accuracy of present solutions has appeared to be much better than 99% verified by comparing the values of $K_1$ and $K_2$ obtained by means of the unitary weight function method to the BEM results and to some particular $K_1$ and $K_2$ solutions found in the literature.

*Key words:* crack, stress intensity factor, weight function, circular hole

## 1. Introduction

Stress intensity factor $K$ is one of the most important parameters of fracture mechanics, widely used in engineering for estimating conditions of brittle fracture, fatigue life and corrosion cracking of a structural element subjected to static or variable loading. Since the value of $K$ depends on many factors such as: shape of the body, crack geometry and location, loading and displacement conditions, temperature field etc., there exists a great number of particular $K$ solutions and many methods of obtaining $K$ values which can be found in the literature e.g. Murakami (1987), Sih (1973a,b), Tada et al. (1973).

The weight function method, suggested by Bueckner (1970), (1973) and Rice (1972) is the most versatile one, due to the possibility of putting together different linear elastic stress fields resulting from external loads, temperature fields, residual stresses, etc., as far as the weight function, proper for the desired geometrical form, is known. Since the resultant stress distribution along the potential crack path of the uncracked body is known, the related stress intensity factors $K_1$, $K_2$ and $K_3$ are determined by a simple integration

$$K_j = \int_0^a \sigma_{1j}(x) m^{(j)}(x,a)\, dx \tag{1.1}$$

where $a$ is the crack length, $m^{(j)}(x,a)$ represents known weight functions adequate for the cracked body and $\sigma_{1j}(x)$ stand for the components of stress tensor released on the crack faces in the directions corresponding to three loading modes $j = 1,2,3$.

Some improvements of the weight function method were presented by Molski (1994a) offering a unified parametric description of any weight function with subsequent applications to the determination of stress intensity factors, valid for any stress distributions released along the crack surfaces. Some examples of such solutions can be found e.g. in Molski (1994a,b), (1996), (1997) for various cracked elements.

In the present study a plane elastic problem of a single radial crack emanating from a circular hole and loaded in opening and sliding modes, as shown in Fig.1, has been analysed.

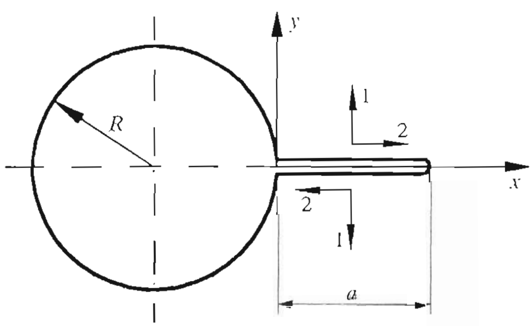


Fig. 1. Single radial crack emanating from a circular hole and subjected to a multiaxial in-plane loading on the surfaces

The normal $\sigma_{11}(x,0)$ and shear $\sigma_{12}(x,0)$ stress components that appear along the potential crack path of uncracked body, are released on the crack faces and form a multiaxial stress state, described at the vicinity of the crack tip by two independent stress intensity factors $K_1$ and $K_2$, corresponding to the modes $I$ and $II$, respectively. Thus, the aim of the present work is to determine the weight function corresponding to each mode.

## 2. Determination of weight functions

Particular values of weight functions, for various $a/R$ ratios and for the loading modes $I$ and $II$, have been obtained numerically by means of the BEM technique (cf Portela and Aliabadi, 1993). The complex stress function $Z_j(z)$, given by Eq (2.1) and satisfying Bueckner's (1970), (1973) loading conditions, has been applied at a small distance from the crack tip. Thus stress and displacement fields can be derived from

$$Z_j(z) = \frac{B_j}{\sqrt{z^3}} \tag{2.1}$$

where $z = re^{i\varphi}$ is a complex number in the polar coordinate system $(r,\varphi)$ located at the crack tip. Bueckner's parameters $B_j = P_j\sqrt{c}/\pi$ correspond to a pair of self-equilibrated opposite forces $P_j$ applied to the crack surfaces at a small distance $c$ from the crack tip.

The numerical approach is similar to that described by Molski (1996), (1997). However, it should be noted that we look for the crack face relative displacements, as a result of known Bueckner's stress or displacement fields applied in the crack tip, Eq (2.1). The boundary element method, due to the accuracy and simplicity of data gathering, is an excellent numerical tool for this purpose.

Numerical calculations of weight functions for each mode and various ratios of the crack length $a$ to the hole radius $R$, have been carried out using the BEM software program Cracker (cf Portela and Aliabadi, 1993). The crack face relative displacements – opening $v_1^*(r,\pi)$ for the mode $I$ and sliding $u_2^*(r,\pi)$ for the mode $II$, obtained using the BEM, have been interpolated along the whole crack length and interpreted as the displacement weight functions. Since the weight functions in this case do not depend neither on the plane state of the body nor the elastic material constants $E$ and $\nu$, their values have been conveniently chosen as $E = 1$ and $\nu = 0$ to simplify the output data analysis.

## 3. Correction and unitary weight functions – numerical results

The next step consists in transforming numerically obtained crack face displacements $v_1^*(r,\pi)$ and $u_2^*(r,\pi)$ into the correction $F_j(s)$ and unitary $w^{(j)}(x/a,s)$ weight functions, following the procedure described by Molski

(1994a). Two different correction functions: $F_1(s)$ and $F_2(s)$, shown by the solid and dashed lines in Fig.2, have been obtained by numerical integration of the previously found and normalized displacement functions $v_1^*(r,\pi)$ and $u_2^*(r,\pi)$. They depend only on the parameter $s$, where $s = a/(a+R)$, and indicate the influence of the uniform, self-equilibrated normal $\sigma_{11}$ and shear $\sigma_{12}$ stresses applied or released directly on the crack surfaces, on $K_1$ and $K_2$ respectively.

It is worthwhile to note that particular values of the correction functions $F_j(s)$, i.e. for the uniform stress released on the crack faces, may be also calculated directly, making use of the principle of superposition. The problem cosists in generating the uniform stress fields – the equi-biaxial tension for the mode $I$ and the uniform shear for the mode $II$ – in the whole body without cracks. The first may be done by applying a constant load normal to all free, both internal and external, surfaces of the uncracked body. The second one is more laborious due to the fact that two properly chosen perpendicular loading components have to be applied to all free surfaces to generate the uniform shear, in direction of the crack plane. In both cases if a crack is present, uniform stresses are released along the crack sides and unknown correction functions $F_j$ are determined directly from the calculated stress intensity factors $K_1$ and $K_2$. It is obvious, that in this way we obtain another proof of the accuracy of the correction functions calculated in two different ways. In the present work both $F_j$ solutions for each mode – those obtained from integrating of the crack face relative displacements and those calculated directly – were almost the same with the maximal difference of about 0.2 percent.

Numerical values of the correction functions $F_1(s)$ and $F_2(s)$, interpolated by polynomials, are given below for the modes $I$ and $II$ and are also shown in Fig.2

$$F_1(s) = 1.1215 - 0.882s + 1.325s^2 - 1.393s^3 + 0.365s^4 + 0.737s^5 - 0.5664s^6 \tag{3.1}$$
$$F_2(s) = 1.1215 - 0.202s - 0.3074s^2 + 2.116s^3 - 5.522s^4 + 5.747s^5 - 2.246s^6$$

The stress intensity factors $K_1$ and $K_2$ are expressed now by Eqs (3.2), true for the uniform stresses $\sigma_{11} = \text{const}$ and $\sigma_{12} = \text{const}$ released directly on the crack surfaces

$$K_1 = \sqrt{\pi a}\,\sigma_{11} F_1(s) \tag{3.2}$$
$$K_2 = \sqrt{\pi a}\,\sigma_{12} F_2(s)$$

As shown by Molski (1994a,b), any form of the conventional weight function can be normalized and transformed into $F_j(s)$ and $w^{(j)}(x/a,s)$, which

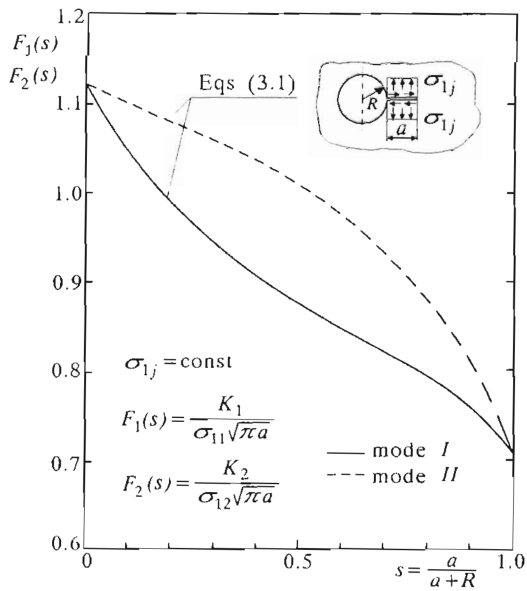


Fig. 2. Correction functions $F_j(s)$ for modes $I$ and $II$

has the following special feature

$$\int_0^1 w^{(j)}\left(\frac{x}{a}, s\right) d\frac{x}{a} = 1 \tag{3.3}$$

Since the correction functions $F_j(s)$ express the shape effect of the body on the stress intensity factors $K_j$ for the uniform stress applied or released directly on the crack surface, the unitary weight function, having the features of dimentionless weight density function, qualifies the load distributed along the whole crack length and estimates its contribution to the stress intensity factor value.

Thus, in the case of non-uniform stresses $\sigma_{1j}(x)$ distributed along the crack sides, the stress intensity factors $K_j$ are

$$K_j = \sqrt{\pi a}\; F_j(s) \int_0^1 \sigma_{1j}\left(\frac{x}{a}\right) w^{(j)}\left(\frac{x}{a}, s\right) d\frac{x}{a} \tag{3.4}$$

where the intergrand may be interpreted as an equivalent constant stress $(\sigma_{eq})_j$ that would give the same $K_j$ values as a true non-uniform one.

For numerical purposes, in the case of non-uniform stress along the crack faces, it is more convenient to use the fractional values of the unitary weight function integrals $\Omega_i(s)$ (cf Molski, 1994a). Theses values are obtained by dividing the whole crack length $a$ into ten equal segments $i$ and integrating the unitary weight function numerically for each segment, starting from the crack end opposite to the considered crack tip. The courses of fractional integral values $\Omega_{1i}(s)$ and $\Omega_{2i}(s)$ of the unitary weight functions versus shape parameter $s$ are shown in Fig.3 and Fig.4. These values are interpreted as weight coefficients, valuating the stresses released along each one tenth of the crack segment.

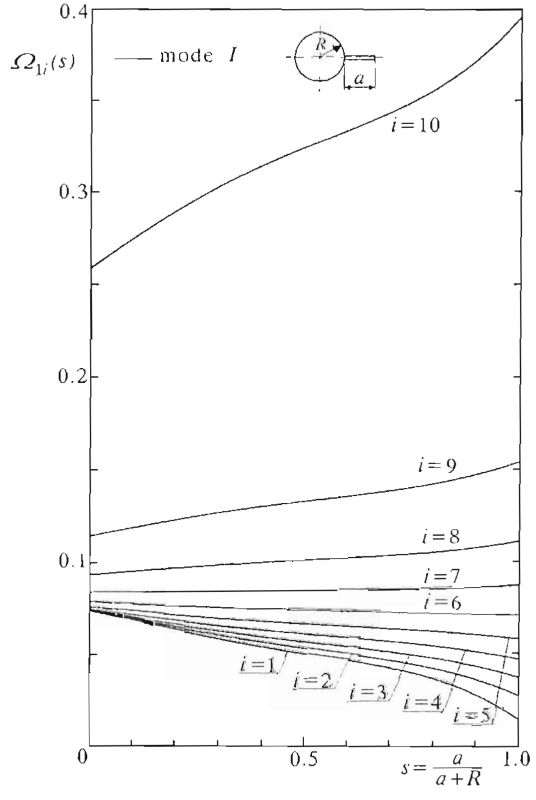


Fig. 3. Fractional values of the unitary weight function integrals (weight coefficients) for mode $I$: $\Omega_{1i}(s)$ versus $s$

Once obtained fractional integral values $\Omega_{1i}(s)$ and $\Omega_{2i}(s)$ have been interpolated by polynomials and incorporated, together with the correction

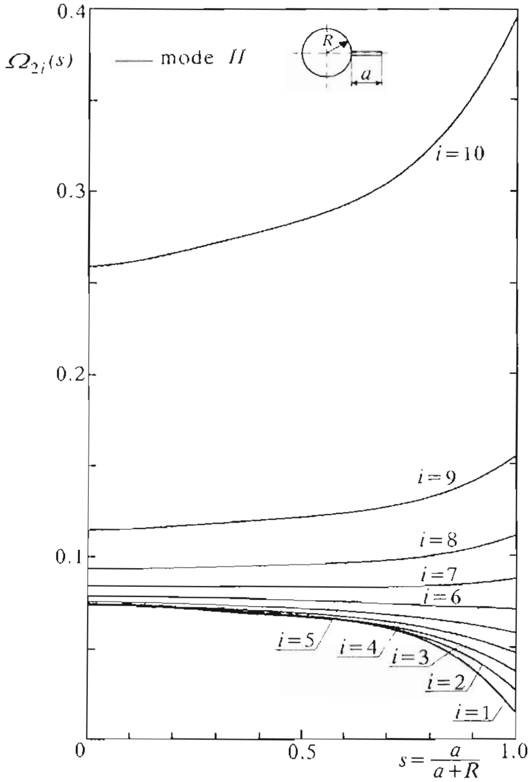


Fig. 4. Fractional values of the unitary weight function integrals (weight coefficients) for mode $II$. $\Omega_{2i}(s)$ versus $s$

functions $F_j(s)$, into the main program (Molski and Truszkowski, 1995) able to calculate stress intensity factors $K_j$ in accordance with Eq (3.4), for any load related to the crack faces. All normalized $K_j$ values, obtained in this way and used below for comparative studies, are indicated as Unitary Weight Function Method (UWFM) results.

## 4. Accuracy verification

To estimate the accuracy of the present UWFM approach, the calculated stress intensity factors $K_1$ and $K_2$ are compared to particular reference solutions from the literature (cf Isida et al., 1985; Murakami, 1987; Tweed and Rooke, 1973) and also to the BEM results obtained by the author for the same problems by means of a direct modelling. Uniform loads: $\sigma_x$ and $\sigma_y$ for the mode $I$ and $\tau$ for the mode $II$ have been applied to the wide plate, containing a circular hole with a single radial crack, sufficiently far from the cracked area. In the case of mode $I$ – two different loading conditions have been analysed:

1. Equi-biaxial tension: $\sigma_x = \sigma_y = \sigma$, $(\lambda = 1)$, where $\lambda = \sigma_x/\sigma_y$
2. Simple tension: $\sigma_x = 0$, $\sigma_y = \sigma$, $(\lambda = 0)$.

For all three cases being considered here, the normal and shear stresses distributed along the potential crack path of uncracked body are well known from the theory of elasticity (cf Timoshenko and Goodier, 1951) and are shown in Fig.5.

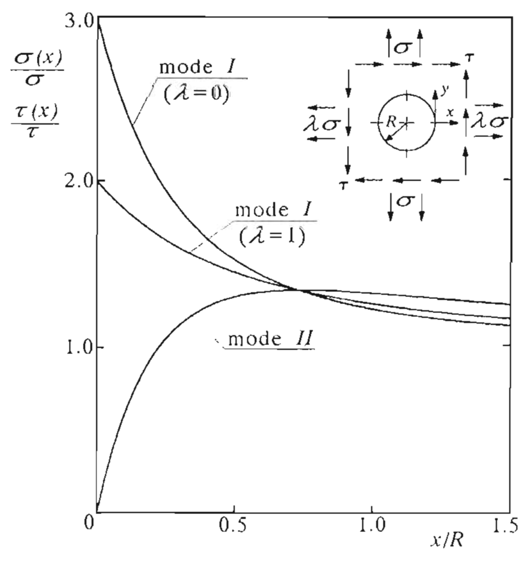


Fig. 5. Elastic stress distributions along the potential crack path resulting from various loading conditions of the uncracked body

These stresses, together with the weight coefficients $\left(\Omega_i(s)\right)_j$ and correction functions $F_j(s)$, enter the integration (3.4) to calculate the stress intensity factors $K_j$ for various $a/R$ ratios. The numerical results obtained in the present UWFM approach, as well as the reference values mentioned above, are shown in Table 1 and Table 2 for two loading modes, respectively. New dimentionless correction functions $Y_1(s)$ and $Y_2(s)$ are defined by Eq (4.1).

Their values depend on the loading conditions imposed on the body

$$Y_1 = \frac{K_1}{\sigma\sqrt{\pi a}} \qquad Y_2 = \frac{K_2}{\tau\sqrt{\pi a}} \tag{4.1}$$

**Table 1.** Mode $I$ correction functions $Y_1$ obtained using various methods

| $a/R$ | $Y_1$ ($\lambda = 0$) | | | | $Y_1$ ($\lambda = 1$) | | | |
|---|---|---|---|---|---|---|---|---|
| | [4,11] | [20] | BEM | UWFM | [4,11] | [20] | BEM | UWFM |
| 0.01 | 3.293 | 3.291 | – | 3.292 | 2.213 | 2.212 | – | 2.212 |
| 0.10 | 2.772 | 2.771 | 2.781 | 2.767 | 1.989 | 1.988 | 1.992 | 1.985 |
| 0.20 | 2.374 | 2.373 | 2.379 | 2.370 | 1.807 | 1.807 | 1.810 | 1.804 |
| 0.30 | – | 2.092 | 2.095 | 2.090 | – | 1.671 | 1.674 | 1.670 |
| 0.50 | 1.728 | 1.727 | 1.730 | 1.727 | 1.480 | 1.480 | 1.481 | 1.481 |
| 1.00 | 1.306 | 1.306 | 1.308 | 1.307 | 1.226 | 1.226 | 1.227 | 1.227 |
| 1.50 | – | 1.127 | 1.130 | 1.128 | – | 1.097 | 1.098 | 1.098 |
| 2.00 | – | 1.030 | 1.034 | 1.032 | – | 1.020 | 1.021 | 1.021 |
| 5.00 | – | 0.845 | 0.847 | 0.847 | – | 0.850 | 0.851 | 0.852 |

**Table 2.** Mode $II$ correction functions $Y_2$ obtained using various methods

| $a/R$ | $Y_2$ | | |
|---|---|---|---|
| | [4] | BEM | UWFM |
| 0.01 | 0.053 | – | 0.053 |
| 0.05 | 0.244 | 0.243 | 0.243 |
| 0.10 | 0.436 | 0.436 | 0.434 |
| 0.20 | 0.712 | 0.713 | 0.708 |
| 0.50 | 1.085 | 1.087 | 1.082 |
| 1.00 | 1.200 | 1.202 | 1.199 |
| 1.50 | – | 1.179 | 1.176 |
| 2.00 | – | 1.134 | 1.131 |
| 5.00 | – | 0.950 | 0.949 |

For all normalized $K_j$ values shown in Tables 1 and 2 the agreement is excellent. Since the reference $K$ solutions obtained by Isida at al. (1985) and by Tweed and Rooke (1973) are estimated to be in error of about $0.1\% \div 0.2\%$ and the accuracy of the BEM results is at least $99.5\%$ – it may be concluded that in the presented cases the error of the UWFM approach do not exceed one percent.

## 5. Conclusions

Boundary element method BEM together with the complex stress function $Z(z)$ describing a singularity of Bueckner's type at the crack tip, have appeared to be a very effective and accurate numerical tool for determining stress intensity factor weight functions of mode $I$ and mode $II$, for the problem of a single radial crack emanating from a circular hole. Both weight function solutions are quite different and valid in the whole range of $s$ parameter, i.e. for $0 \leq s \leq 1$, coinciding with the solution for a single edge crack in a half plane at the left extreme ($s = 0$) and with Griffith's crack at the right one ($s = 1$).

The uniform load correction functions $F_j$ and weight coefficients $\Omega_{1i}$ and $\Omega_{2i}$ depend only on the parameter $s$. which describes the relation between the crack length and the hole radius $R$.

The accuracy of stress intensity factors, for both loading modes being analysed here, has appeared to be very satisfactory with the maximal error much lower than one percent compared to some particular solutions known from the literature and to the boundary element results obtained by the author. Despite of the fact that the accuracy has been verified for well known normal and shear stresses released along the crack faces, similar errors of maximum 1% can be expected for any other loading functions, including residual and thermoelastic stress fields, as far as their distributions along the potential crack path can be properly determined.

*Acknowledgements*

The investigation presented in this paper is a part of grant No. 7 S101 040 04 sponsored by the State Committee for Scientific Research (KBN).

Numerical results have been obtained using the boundary element software program Cracker from Wessex Institute of Technology, UK

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### Funkcje wagowe współczynników $K_1$ i $K_2$ dla pojedynczej szczeliny wychodzącej z okrągłego otworu

Streszczenie

Analizowano płaskie zagadnienie pojedynczej szczeliny wychodzącej promieniowo z okrągłego otworu w materiale podlegającym prawu Hooke'a. Wyznaczono funkcje wagowe dla współczynników $K_1$ i $K_2$ stosując metodę elementu brzegowego MEB w połączeniu z zespoloną funkcją naprężeń $Z(z)$ zadaną w sąsiedztwie wierzchołka szczeliny. Otrzymane rozwiązania przedstawiono w postaci funkcji korekcyjnych i jednostkowych funkcji wagowych o zunifikowanym zapisie, pozwalającym zastosować je do wspomaganych komputerowo symulacji rozwoju pęknięć kruchych, zmęczeniowych i korozyjnych, przy dowolnym rozkładzie obciążenia uwalnianego na powierzchni szczeliny. Maksymalny błąd obliczonych tą metodą współczynników $K_1$ i $K_2$ nie przekraczał jednego procenta.

*Manuscript received February 19, 1997; accepted for print July 17, 1997*