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THE EFFECTS OF CONSTRAINED CROSS-SECTIONAL WARPING ON THE BENDING OF BEAMS

SŁAWOMIR JANECKI

Institute of Fluid Flow Machinery, Polish Academy of Sciences, Gdańsk

e-mail: tja@imppan.imp.gd.pg.gda.pl

The equations of motion for a bent beam of compact cross-section, are presented in this work. They are derived by using of the general method, applicable to one-dimensional models of continuum. The warping constraint of cross-sections, caused by shearing, has been taken into account. An additional parameter characterizing the form of a cross-section warping function is introduced. Two dimensionless shearing coefficients appear in the given equations. One of them characterizes the constrained shearing, and the other one the free shearing. It has been shown that these coefficients do not depend upon the form of the warping function. In a particular case, if the constraint cross-sectional warping does not appear, then the equation of motion and the constitutive relations are the same as in Timoshenko's theory.
A series of important examples illustrating the bending theory of beams is presented in the paper. The form of the warping function and the warping constraint of an arbitrary, compact cross-section has been taken into account in these examples. In a particular case, an equation defining the parameter magnitudes of the thickness shear mode, for a simply supported beam of an arbitrary cross-section, is given.
A critical analysis the works of Bickford (1982), Ewing (1990), Leung (1990) and Levinson (1981) has been made.

Key words: beams, shear effect, vibration

1. Introduction

In the assessment of dynamic behaviour of many structural elements, they are substituted by mathematical, one-dimensional models of a continuous medium. The classical theory of these models is based upon the hypotheses, crucial aspects of which are the internal kinematic or kinetic constraints superimposed onto the motion and the state of stress of the material body. Depending on the assumed hypotheses, one obtains different models and equations describing the motion of the body. Some of them lead to equations which are not useful when describing the wave phenomena; the others, hovever, allow for the description of these phenomena. The Euler-Bernoulli equations when applied to beam theory, represent an example of the first of the above mentioned equations, whilst the Timoshenko equations are an example of the second type. The Euler-Bernoulli model of the beam does not include the effects caused by the cross-sections rotation and the effects due to the shear stresses. Rayleigh (1945) introduced a correction, considering the rotary inertia of the beam cross-sections, whilst Timoshenko (1921) made some corrections accounting for the influence of the shearing forces on the bending of the beam.

Many investigators have dealt with problems connected with the model proposed by Timoshenko. The wave character of the motion equations were investigated by Flugge (194), Kruszewski (1949), Prescott (1942), the natural vibration of the beam by Anderson (1953), Boley and Chao (1955), Huang (1961), Hurty and Rubinstein (1964), Kruszewski (1949), Wang (1970), and the forced vibration by Hermann (1955), Volterra and Zachmanoglou (1957). A series of works Barr (1959), Cowper (1966), Hutchinson (1981), Hutchinson and Zillmer (1986), Mindlin and Deresiewicz (1954), Spence and Seldin (1970). Stephen (1978), (1980), Stephen and Levinson (1979), is devoted to determination of the shear coefficients of different cross-sections of the beam, appearing in the Timoshenko equations. Barr (1959), admitting the distortion of cross-sections of the beam, caused by the influence of shear, introduced the "distortion coefficient", being an equivalent to the coefficient of shear which appears in the Timoshenko equations. There are many other works dealing with an improvement of the beam model. Using the three-dimensional linearized theory of elasticity, as a starting point new formulations have been obtained by Aalami and Atzori (1974), Cowper (1966), Gross (1969), Janecki (1977), Levinson (1981), Krishna Murty (1970), Renton (1991), Stephen and Levinson (1979). Volterra and Zachmanoglou (1957). In his pioneering work Cowper (1966) derived equations of the beam vibration. Assuming that the cross-sectional warping is the same as in the cantilever beam under a single transverse load at the tip. or a uniformly loaded beam, Cowper defined the relationships between the bending moment and the mean rotation of the cross-section, as well as the relationship between the shearing force and the mean angle of shear. Moreover, he established the formula determining the coefficient of shear that appears in the equations of the Timoshenko's theory. Aalami and Atzori (1974) expressed the opinion, without any supporting arguments, that the frequencies of the Timoshenko beam bending vibration cannot be determined accurately enough with only one coefficient correcting the nonuniform distribution of stresses in the cross-section of the beam. Intuitively, they derived two correcting coefficients in the physical relationships for the sectional internal forces, one of them for the shear force and the second one for the bending moment. The values of these coefficients for different shapes of cross-sections were determined, on the basis of the ad hoc assumed additional hypotheses which are not included in the presented theory. In a similar way, Stephen and Levinson (1979), proceeding analogicaly to Cowper (1966), introduced two coefficients into the equation of the beam. One of them is connected with the longitudinal warping of beam cross-sections and appears in the constitutive relationship for the transverse force and the angle of shear. The second one depends upon the distribution of normal stresses in the beam cross-section and is found in the relationship between the bending moment and the curvature of the beam axis. The values of the derived coefficients result directly from the theory; it is then internally coherent. Levinson (1981) intuitively assumed the kinematic hypothesis determining the warping of cross-sections. Subsequently, he derived the equation of motion with two independent variables for a beam of narrow, rectangular cross-section.

These equations are of a Timoshenko-type with the coefficient of shear equal to 5/6. A certain inconsistency appears in that theory, and the point of it is that the coefficient used in the physical relationship between the transverse force and the mean angle of shear, amounts to 2/3. The difficulty mentioned above does not appear in Janecki (1977). Janecki (1977), Volterra and Zachmanoglou (1957) considered the theory of the rectangular cross-section beam, comprising the complex warping of cross-sections during the bending and shearing of the beam. The equations derived there allow for the possibility of obtaining an arbitrary, finite number of branches of the natural vibration frequency. Huang (1961) introduced an additional coefficient that determined the shape of the cross-sectional warping and proved that the frequencies of natural vibration, of the two-point simply supported beam did not depend upon the value of this additional coefficient. The above mentioned derivations are qualitatively conformable to the conclusions resulting from the Pochhamer-Chree theory (cf Abramson et al., 1958). In the papers discussed above (except for Janecki, 1977) some cross-sectional internal forces are intuitively introduced based upon experience resulting from the theories of Euler-Bernoulli and Timoshenko. The bending moment is introduced as the resultant first moment of the normal stresses, appearing in the longitudinal fibres of the beam and the trasverse shearing force as the resultant of the tangential stresses appearing in the beam cross-section. It is assumed implicitly that the beam cross-sections warp freely while bending and shearing. Such a situation, however, seems untrue, particularly for short beams of variable cross-sections. Therefore, it is necessary to take into consideration and investigation, the effect of constrained warping of cross-sections in the theory of bent and sheared beams. A similar situation occurs when the beam is twisted. In the Saint-Venant theory (cf Love, 1944) the cross-sections warp freely, while in Vlasov's theory (cf Vlassov, 1964) of thin-walled bars, the constrained warping of cross-sections is taken into account. In the latter case, some additional internal self-balanced forces appear in the form of a bimoment. It should be expected then that in the case of the constrained warping of cross-sections, occuring in bending and shearing, there should also appear the internal self-balanced forces. In order to derive some additional forces, a rational method must be worked out for defining the resultant cross-sectional internal forces. The method will be independent of the assumed hypotheses.

Such a general method for one-dimensional models has been described by Janecki (1981). The need for such a general method is mentioned by Bickford (1982), taking advantage of the assumptions of Levinson (1981), derived in by variational method, the motion equations of the bent beam. To arrive at an agreement of the equations obtained by means of variations, with those obtained from equilibrium conditions, Bickford derived, in an artificial way, the bending moment of a higher order. Applying this additional force, he unexpectedly obtained unusual results (cf Levinson, 1985), in particular, for the lower branch of the relationship between the phase velocity and the wavelength. That result was due to an inconsistent procedure which will be explained.

2. The beam model

The beam, subjected to loadings, can simultaneously be bent, twisted and stretched. These deformations are responsible for the distributions of normal and tangential stresses that appear in the cross-sections of the beam. These distributions in turn, influence the magnitude of the displacements and the modes and frequencies of vibration. Taking these physical facts into account in mathematical models, depends to a great extent on the a'priori assumed hypotheses concerning the motion and stresses.

In order to investigate the influence of the warping function shape, as well as the effect of the constrained shearing on the transverse vibration, our attention will be concentrated on beams of bisymmetrical, uniform cross-sections. Then, there is no link between bending, torsion and stretching. For this reason, the equation of bending is decoupled as well.

Let us consider the bending of the beam in the plane $(0x_2x_3)$, Fig.1. Then

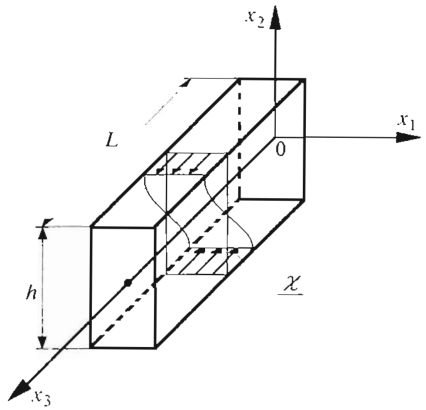

Fig. 1 The beam sketch, $0x_1x_2x_3$ – coordinate system

we have displacements

$$u_1 = 0 \qquad u_2 = u_2^0 \qquad u_3 = \omega_1 x_1 + \gamma_1 \chi_2 \tag{2.1}$$

where u_2^0 is the displacement of the material points lying on the beam axis, ω_1 is the angle of bending, γ_1 is yhe shear angle, and χ_2 is the function of the beam cross-section warping caused by shearing. We suppose that χ_2 is the bending function, and is the solution of the equation

$$\Delta\chi_2 = 2k_2 \frac{GA}{EJ_{11}} x_2 \qquad (x_1, x_2) \in A \tag{2.2}$$

with the boundary condition

$$\frac{\partial\chi_2}{\partial n} = \Big(-\nu k_2 \frac{GA}{EJ_{11}} x_1 x_2\Big) n_1 + \Big(1 - \nu k_2 \frac{GA}{EJ_{11}} \frac{x_2^2 - x_1^2}{2}\Big) n_2 \qquad (x_1, x_2) \in \partial A \tag{2.3}$$

where γ, G, E, are the material constants, A and J_{11} are the area and the second moment of inertia, respectively, of the beam cross-section, n_1 and n_2 are the components of the unit vector normal to the boundary ∂A of the area A, and k_2 is the number, which will be defined later. Taking χ_2 as a solution to the boundary problem of the elastostatics, we will restrict ourselves to long waves. Ewing (1990) accepted some similar assumptions concerning the warping of cross-sections in shearing.

The kinematic model, given above can be presented in the matrix form

$$\boldsymbol{u} = \mathbf{U}\boldsymbol{q} \tag{2.4}$$

where

$$q = [u_2^0, \omega_1, \gamma_1] \tag{2.5}$$

is the vector of generalised displacements and

$$\mathsf{U} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & x_2 & \chi_2 \end{bmatrix} \tag{2.6}$$

is the matrix defining the beam model.

The internal forces appear in the beam as a result of its deformation. In one-dimensional models the kinetic hipothesis is also adopted assuming that the state of tension in the beam cross-section is defined only by components τ_{13}, τ_{23}, τ_{33} of the stress tensor. On the assumption of homogeneity, isotropy and elasticity of the beam material in our case we have the constitutive equations

$$\tau_{\alpha 3} = 2G\varepsilon_{\alpha 3} \qquad \tau_{33} = E\varepsilon_{33} \qquad (\alpha = 1,2) \tag{2.7}$$

where $\varepsilon_{\alpha 3}$ and ε_{33} are the components of deformation tensor.

As soon as the models have precisely been stated it is possible to start definig the internal forces and loads, and to derive appropriate equations of the beam motion.

3. Equations of motion and internal forces

Equations of motion and definitions of internal forces and loads are established with the use of general equations and relationships for slender bodies described by one-dimensional models of a continuous body (cf Janecki, 1977, 1981). The equations of motion are of the form

$$\frac{\partial \boldsymbol{H}}{\partial x} - \boldsymbol{Q}^* + \boldsymbol{h} = \boldsymbol{0} \tag{3.1}$$

The vectors of internal forces, on the asumption of small deformations (the deformation gradient $\mathsf{F} = \mathbf{1}$) look like

$$\boldsymbol{H} = \int\limits_A (\mathsf{T}\boldsymbol{e}_3)\mathsf{U}\, dA \qquad \boldsymbol{Q}^* = \int\limits_A \operatorname*{tr}_{(1,3)}(\mathsf{T}\nabla\mathsf{U})\, dA \tag{3.2}$$

The vector of the body loads caused by the motion

$$\boldsymbol{h} = -\varrho \int\limits_A \mathsf{U}^\top \boldsymbol{u}\, dA \tag{3.3}$$

where $\mathbf{T}$ - stress tensor, e_3 - unit vector tangential to the beam axis and ϱ - specific density of the beam material, tr - operator of the matrix trace.

For the kinematic model (2.4) described by Eqs (2.5) and (2.6) on the basis of (3.1), we obtain the scalar equations of the beam equilibrium

$$Q_2' + q_2 = 0 \qquad M_1' - Q_2 + m_1 = 0 \qquad H_1' - G_2^* + h_1 = 0 \tag{3.4}$$

The internal forces appearing in the above equations, according to the definition (3.2) are

$$\begin{aligned} Q_2 &= \int\limits_A \tau_{23}\, dA & M_1 &= \int\limits_A x_2\tau_{33}\, dA \\ H_1 &= \int\limits_A \chi_2\tau_{33}\, dA & G_2^* &= \int\limits_A \frac{\partial \chi_2}{\partial x_k}\tau_{k3}\, dA \end{aligned} \tag{3.5}$$

where Q_2 - transverse shear force, M_1 - bending moment, H_1 - shearing moment and G_2^* - shearing force, which appear when the constrained warping of a bent beam is taken into account. The internal force G_2^* in the case of prismatic beam. can be written in the form

$$G_2^* = Q_2 - G_2 \tag{3.6}$$

where

$$G_2 = \int\limits_A \Big(\delta_{2\beta} - \frac{\partial \chi_2}{\partial x_\beta}\Big)\tau_{3\beta}\, dA \tag{3.7}$$

is the transverse shearing force caused by constrained warping of the beam cross-sections.

The body loads, according to Eq (3.3) are

$$\begin{aligned} q_2 &= -\varrho \int\limits_A \ddot{u}_2\, dA & m_1 &= -\varrho \int\limits_A x_2\ddot{u}_3\, dA \\ h_1 &= -\varrho \int\limits_A \chi_2\ddot{u}_3\, dA \end{aligned} \tag{3.8}$$

The derived equations of motion cover also constrained warping of the beam cross-sections, appearing both in twisting and bending. $H_1 = h_1 = 0$ when the constrained warping of the beam cross-sections in bending is neglected. Then $G_2^* = 0$ from Eq (3.4) for the shearing moment, whereas from Eq (3.6) it follows that $G_2 = Q_2$.

When the components of the displacement vector (2.1) are determined, it is also possible to define the components of the strain tensor

$$
\begin{aligned}
\varepsilon_{13} &= \frac{1}{2}\frac{\partial \chi_2}{\partial x_1}\gamma_1 \qquad\qquad \varepsilon_{33} = \omega_1' x_2 + \gamma_1' \chi_2 \\
\varepsilon_{23} &= \frac{1}{2}\Big[(u_2^0)' + (\omega_1 + \gamma_1)\Big] - \frac{1}{2}\Big(1 - \frac{\partial \chi_2}{\partial x_1}\Big)\gamma_1
\end{aligned}
\tag{3.9}
$$

In Timoshenko's beam theory the geometric relation $\Gamma = (u_2^0)' + \omega_1 + \gamma_1 = 0$ is taken for granted. On the above assumption the relation for ε_{23} is considerably simplified. In general case of constrained bending one can assume that $(u_2^0)'$, ω_1 and γ_1 are independent functions.

Now we introduce the nondimensional variables

$$
\xi = \frac{x}{L} \qquad \tau = \frac{C_E}{L}t \qquad u = \frac{u_2^0}{L}
\tag{3.10}
$$

and the dimensionless parameters

$$
\lambda = L\sqrt{\frac{A}{J_{11}}} \qquad \varepsilon = \frac{J_{\chi_2\chi_2}}{J_{11}} \qquad \eta = \frac{J_{2\chi_2}}{J_{11}}
\tag{3.11}
$$

where L is the length of the beam and. $C_E = \sqrt{E/\varrho}$, and

$$
\begin{aligned}
J_{11} &= \int\limits_A x_2^2\;dA \qquad J_{1\chi_2} = \int\limits_A x_2\chi_2\;dA \qquad J_{\chi_2\chi_2} = \int\limits_A \chi_2^2\;dA \\
\kappa_{22} &= \frac{1}{A}\int\limits_A \Big(1 - \frac{\partial \chi_2}{\partial x_2}\Big)\;dA \qquad k_{22} = \frac{1}{A}\int\limits_A \Big[\Big(\frac{\partial \chi_2}{\partial x_1}\Big)^2 + \Big(1 + \frac{\partial \chi_2}{\partial x_2}\Big)^2\Big]\;dA
\end{aligned}
\tag{3.12}
$$

are the geometrical characteristics of the beam cross-section.

Then relationships for the internal forces (3.5) can be rewritten in the form

$$
\begin{aligned}
Q_2 &= GA\Big\{\Big[\frac{\partial u}{\partial \xi} + (\omega + \gamma)\Big] - \kappa_{22}\gamma\Big\} \\
M_1 &= \frac{EJ_{11}}{L}\frac{\partial}{\partial \xi}(\omega + \eta\gamma) \qquad H_1 = \frac{EJ_{11}}{L}\frac{\partial}{\partial \xi}(\eta\omega + \varepsilon\gamma) \\
G_2^* &= GA\Big\{(1 - \kappa_{22})\Big[\frac{\partial u}{\partial \xi} + (\omega + \gamma)\Big] + (k_{22} - \kappa_{22})\gamma\Big\}
\end{aligned}
\tag{3.13}
$$

and the body loads (3.8) in the form

$$q_2 = -\varrho A \frac{C_E^2}{L} \frac{\partial^2 u}{\partial \tau^2} \qquad m_1 = -\varrho J_{11} \frac{C_E^2}{L^2} \frac{\partial^2 u}{\partial \tau^2}(\omega + \eta\gamma)$$
$$h_1 = -\varrho J_{11} \frac{C_E^2}{L^2} \frac{\partial^2 u}{\partial \tau^2}(\eta\omega + \varepsilon\gamma) \tag{3.14}$$

In order to interpret Eqs (3.13) and (3.14) for the moments, the following averaged quantities will be introduced

$$\Phi = \frac{1}{J_{11}} \int\limits_A x_2 u_3 \, dA \qquad \Psi = \frac{1}{J_{11}} \int\limits_A \chi_2 u_3 \, dA \tag{3.15}$$

Making use of (2.1) we have

$$\Phi = \omega + \eta\gamma \qquad \Psi = \eta\omega + \varepsilon\gamma \tag{3.16}$$

The first quantity is identical to the mean angle of rotation derived by Cowper (1966). In Timoshenko's theory, the warping constraint of the beam cross-sections is neglected, then $H_1 = 0$, $h_1 = 0$ and it is assumed that the geometric relation $\Gamma = u' + \omega + \gamma = 0$ between the slope of the deflection line and the angles of bending and shearing is fulfilled. Moreover, from the condition $G_2^* = 0$ it follows that, additionaly, there has to be $k_{22} = \kappa_{22}$.

Making use of Eqs (3.4), (3.13) and (3.14), it is possible to derive the equation describing the dimensionless, transverse displacement of the vibrating beam

$$\widehat{\varepsilon}\frac{E}{G}\Big[\frac{\partial^2}{\partial \xi^2}\Box^2 u - \frac{E}{G}\frac{\partial^2}{\partial \tau^2}\Box^2 u + \lambda^2 \frac{\partial^2 u}{\partial \tau^2}\Box u\Big] +$$
$$-\lambda^2\Big[\frac{\partial^2}{\partial \xi^2}\Box u - \frac{E}{\widehat{\kappa}G}\frac{\partial^2}{\partial \tau^2}\Box u + \lambda^2 \frac{\partial^2 u}{\partial \tau^2}\Big] = 0 \tag{3.17}$$

where

$$\Box = \frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \tau^2}$$

The introduced dimensionless parameters

$$\widehat{\varepsilon} = \frac{\varepsilon - \eta^2}{k_{22} - \kappa_{22}^2} \qquad \frac{1}{\widehat{\kappa}} = 1 + \frac{(1 - \kappa_{22} - \eta)^2}{k_{22} - \kappa_{22}^2} \tag{3.18}$$

characterize the constrained and free shearing, respectively. The first part Eq (3.17) describes the influence of the constrained warping of cross-sections, and the second is the Timoshenko equation with the coefficient of shear equal to $\widehat{\kappa}$. It should be noted that the equation of the beam bending vibration, taking into account the constraint of the distortions of cross-sections, is a differential equation of the sixth order instead of the fourth, as it occurred in the case of the Timoshenko equation. Here the situation is analoqous to the case of torsion. The beam torsional vibration with free warping of cross-sections, is described by a differential equation of the second order, while the vibration with constrained warping is described by an equation of the fourth order (cf Vlassov, 1964).

Using Eqs (3.4), (3.13) and (3.14) it is possible to establish the following relations for the moments of internal forces

$$
\begin{aligned}
M_1 &= \frac{EJ_{11}}{L}\Big[\Big(-\frac{\partial^2 u}{\partial \xi^2}+\frac{E}{\widehat{\kappa}G}\frac{\partial^2 u}{\partial \tau^2}\Big)+\widehat{\varepsilon}\frac{E}{G}\Big(\frac{\partial^2 u}{\partial \tau^2}+\frac{1}{\lambda^2}\Box\Box^* u\Big)\Big] \\
H_1 &= \frac{EJ_{11}}{L}\Big\{\eta\Big[\Big(-\frac{\partial^2 u}{\partial \xi^2}+\frac{E}{\widehat{\kappa}G}\frac{\partial^2 u}{\partial \tau^2}\Big)+\widehat{\varepsilon}\frac{E}{G}\Big(\frac{\partial^2 u}{\partial \tau^2}+\frac{1}{\lambda^2}\Box\Box^* u\Big)\Big]+ \\
&\quad - \widehat{\varepsilon}\frac{k_{22}-\kappa_{22}^2}{1-\kappa_{22}-\eta}\frac{E}{G}\Big[\Big(\frac{1}{\widehat{\kappa}}-1\Big)\frac{\partial^2 u}{\partial \tau^2}+\widehat{\varepsilon}\Big(\frac{\partial^2 u}{\partial \tau^2}+\frac{1}{\lambda^2}\Box\Box^* u\Big)\Big]\Big\}
\end{aligned}
\tag{3.19}
$$

We have also

$$
\frac{\partial \Gamma}{\partial \xi} = -\frac{\kappa_{22}}{1-\kappa_{22}-\eta}\frac{E}{G}\Big[\Big(\frac{1}{\widehat{\kappa}}-\frac{1-\eta}{\kappa_{22}}+\widehat{\varepsilon}\Big)\frac{\partial^2 u}{\partial \tau^2}+\frac{\widehat{\varepsilon}}{\lambda^2}\Box\Box^* u\Big]
\tag{3.20}
$$

where

$$
\Box^* = \frac{\partial^2}{\partial \xi^2} - \frac{E}{G}\frac{\partial^2}{\partial \tau^2}
$$

If we apply $\eta = 0$, $\varepsilon = 0$ and $k_{22} = \kappa_{22}$, the obtained relation for M_1 is the same as the relation resulting from Timoshenko's theory of a beam with the shear coefficient equal to κ^*.

4. Warping function and shear parameters

Coefficients $\widehat{\varepsilon}$ and $\widehat{\kappa}$ and parameters k_{22} and κ_{22} can be calculated by first solving the boundary value problem (2.2) and (2.3) for the warping function of χ for an arbitrary, compact cross-section of the beam. Function χ depends upon the number k, which has yet to be defined, appearing in the boundary condition (2.3). It can be determined from an additional condition to be discussed later.

For the sake of determining the warping function of a beam with a narrow, rectangular cross section and height h, we have the equation

$$\frac{d^2\chi}{d\xi^2} = \frac{3}{2}\frac{hk}{1+\nu}\xi \qquad \xi \in (-1,1) \tag{4.1}$$

with the boundary condition

$$\frac{d^2\chi}{d\xi^2} = \frac{1}{2}h\Big(1 - \frac{3}{4}\frac{\nu k}{1+\nu}\Big) \qquad \xi = \pm 1 \tag{4.2}$$

where $\xi = 2x_2/h$. Then

$$\chi = \frac{h}{4(1+\nu)}\Big[k\xi^2 + (2-3k) + \frac{1}{2}(4-3k)\nu\Big]\xi \tag{4.3}$$

and

$$\begin{aligned}
\widehat{\kappa} &= \frac{5}{6} & \widehat{\varepsilon} &= \frac{3}{35}\\
\eta &= 1 - \frac{1 + 5\Big(1 + \frac{3}{4}\nu\Big)}{5(1+\nu)}k & \varepsilon &= \frac{3k^2}{175(1+\nu)^2} + \eta^2\\
\kappa_{22} &= \frac{1 + \frac{3}{4}\nu}{1+\nu}k & k_{22} &= \frac{1 + 5\Big(1 + \frac{3}{4}\nu\Big)^2}{5(1+\nu)^2}k^2
\end{aligned} \tag{4.4}$$

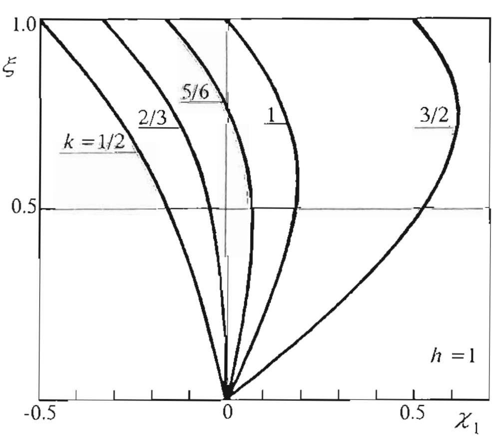

Fig. 2. Warping function $\chi(k)$, Janecki (1977)

It should be noticed that $\widehat{\kappa}$ and $\widehat{\varepsilon}$ do not depend upon the so far undetermined parameter k and the Possion ratio ν. On the other hand, the shape of the cross-section warping function depends upon k and ν. This function for $\nu = 0$ and various k (cf Janecki, 1977) is shown in Fig.2. The image of the presented warpings approximately agrees with those for a bar of circular cross-section, obtained on the basis of the Pochhammer-Chree theory, with various ratios of the cross-sectional radius to the length of the wave (cf Abramson et al., 1958). From a comparison of the warping functions, given in Fig.2 and in Abramson et al. (1958), it can be concluded that the number $k = 2/3$ used by Timoshenko, is limiting value separating the warpings corresponding to long waves from the warpings relating to relatively short waves.

For $\nu = 0$ and $k = 2/3$ we obtain $\chi = h\xi^3/6$, $\kappa_{22} = 2/3$, $k_{22} = 8/15$, $\eta = 1/5$. A warping function of this form was assumed by Levinson (1981) on the simultaneous assumption that constraint of this warping does not appear. For the sake of performing this it is necessary, however, that the conditions $\Gamma = u' + \omega + \gamma = 0$ and $\kappa_{22} = k_{22}$ be fulfilled. From the last equality, for the narrow rectangular cross-section, it follows that $k = 5/6$. This means that the condition of free warping in cross-sections under shearing is not satisfied in the theory of Levinson. It should also be noticed that, for $k = 5/6$, we have $\kappa_{22} = k_{22} = 5/6$ and $\eta = 0$. Then, the angle Φ of the mean rotation of the cross-section equals the angle of bending ω.

Conditions $\eta = 0$ and $\kappa_{22} = k_{22}$ can be applied to determination the undefined parameter k characterizing the shape of the warping function χ. For $\nu \neq 0$ from the condition $\eta = 0$ we have

$$(4.5) k = k_C = \frac{5(1+\nu)}{1 + 5\left(1 + \frac{3}{4}\nu\right)} \tag{4.5}$$

Such a shear coefficient value was given by Olson (1958) and Donnell (1976). Whereas, from the equality $\kappa_{22} = k_{22}$ we have

$$k = k_0 = \frac{5(1+\nu)\left(1 + \frac{3}{4}\nu\right)}{1 + 5\left(1 + \frac{3}{4}\nu\right)^2} \tag{4.6}$$

For the function

$$\chi = \frac{h\xi^3}{6\left(1 + \frac{1}{2}\nu\right)} \tag{4.7}$$

tangential to the axis of co-ordinates at the origin of the coordinate system, we obtain the Timoshenko shear coefficient

$$k = k_T = \frac{2(1+\nu)}{1+2\left(1+\frac{3}{4}\nu\right)} \tag{4.8}$$

In the case of a beam with a circular cross-section

$$\chi = \frac{a}{2(1+\nu)}[k\varrho^2 + (2-3k) + 2(1-k)\nu]\varrho \sin\vartheta \tag{4.9}$$

in the polar coordinates (r, ϑ), where $\varrho = r/a$ and a is the radius of the circle. Then,

$$\begin{aligned} \widehat{\kappa} &= \frac{6}{7} & \widehat{\varepsilon} &= \frac{1}{12} \\ \eta &= 1 - \frac{1+6(1+\nu)}{6(1+\nu)}k & \varepsilon &= \frac{k^2}{72(1+\nu)^2} + \eta^2 \\ \kappa_{22} &= k & k_{22} &= \frac{1+6(1+\nu)^2}{6(1+\nu)^2}k^2 \end{aligned} \tag{4.10}$$

From the condition $\eta = 0$ we have

$$k_C = \frac{1+6(1+\nu)}{1+6(1+\nu)} \tag{4.11}$$

This is the coefficient of shear given by Cowper (1966). Instead, from the equality $k_{22} = \kappa_{22}$ it follows

$$k_0 = \frac{6(1+\nu)^2}{1+6(1+\nu)^2} \tag{4.12}$$

Kaneko (1975) reviewd various theoretical shear coefficients and compared them with the experimental results. He recognized that the best coefficients in Timoshenko's theory, are

$$k = \frac{5(1+\nu)}{1+5(1+\nu)} \quad \text{and} \quad k = \frac{6(1+\nu)^2}{1+6(1+\nu)^2 - 2\nu^2} \tag{4.13}$$

with respect to a beam of a narrow rectangular and circular cross-section, respectively. The above expression was formulated by Spenser and Seldin (1942) by comparing the experimental data with the calculation results using the Timoshenko's theory. Stephen (1980), (1981) obtained identical equations comparing results of phase velocities of Timoshenko's and Pochhammer-Chree theories. Identical expressions by a theoretical approach were obtained by Stephen and Levinson (1979), also. It should be pointed out that Eqs (4.13) were determined without giving consideration to the constrained cross-sectional warping.

5. Examples

With the aid of some specified shear characteristics a number of important elementary examples will be solved describing the effect of the constrained warping of the cross-sections.

5.1. The bending of a cantilever beam under a concetrated shear force

Firstly we shall present the solutions for beams with free cross-section warping. This means that we shall consider the classic model of Timoshenko. For this purpose we reject the equation for the shear moment in the set of equations (3.4) and assume the relation $u' + \omega + \gamma = 0$. Then, for the boundary conditions: $u(1) = \omega(1) = 0$ and $Q(0) = Q_0$, $M(0) = M_0$, we have

$$u_T = -\frac{Q_0L^2}{6EJ}(\xi^3 - 3\xi + 2) + \frac{1}{\lambda^2}(\frac{E}{\kappa_{22}G})\frac{Q_0L^2}{EJ}(\xi - 1) \tag{5.1}$$

where Q_0 is the transverse shear force applied to the free end of the beam, L is the length of the beam. For $\nu = 0$ we have $\kappa_{22} = k_{22}$. The indetermined value of k may be determined either from the equality $k_{22} = \kappa_{22}$, or by making some other assumptions.

For example, in case of narrow, rectangular cross section for $k = 5/6$ and $k = 2/3$, $1/5(1+\nu)(Q_0L^2/EJ)(h/L)^2$ and $1/4(1+\nu)(Q_0L^2/EJ)(h/L)^2$ the displacement of a free end of the beam caused by shear, respectively. The latter result coincides with the data obtained by Levinson (1981), which corresponds to the solution relating to the linear theory of elasticity, on the assumption that there is a two-dimensional state of stress.

Now we shall take the constrained warping of cross-sections into account. For this purpose we shall make use of the equlibrium equations (3.4) and the constitutive relations. For the boundary conditions (cf Bickford, 1982)

$$u(1) = u'(1) = \omega(1) = 0 \qquad Q(0) = Q_0 \qquad M(0) = H(0) = 0 \tag{5.2}$$

we have, in the general case,

$$u = \widehat{u}_T + \Big[-\frac{\eta\widehat{\kappa}}{1-\kappa_{22}}(\xi - 1) + \Big(1 - \frac{\eta\widehat{\kappa}}{1-\kappa_{22}}\Big)\frac{\sinh\beta - \sinh\beta\xi}{\beta\cosh\beta}\Big]\frac{1}{\lambda^2}\frac{E}{\widehat{\kappa}G}\frac{Q_0L^2}{EJ} \tag{5.3}$$

where $\widehat{u}_T$ is the solution (5.1) resulting from Timoshenko's theory with the coefficients of shear $\kappa_{22} = \widehat{\kappa}$ and $\beta^2 = G\lambda^2/E\widehat{\varepsilon}$. We also obtain

$$\Gamma = \frac{\partial u}{\partial \xi} + \omega + \gamma = \Big[\frac{\widehat{\kappa} - \kappa_{22} - \eta\widehat{\kappa}}{1 - \kappa_{22} - \eta} + \frac{\kappa_{22}}{1 - \kappa_{22}}\Big(1 + \eta\frac{1 - \widehat{\kappa}}{1 - \kappa_{22} - \eta}\Big)\frac{\cosh\beta\xi}{\cosh\beta}\Big]\frac{Q_0}{\widehat{\kappa}GA} \tag{5.4}$$

The influence of the warping constraint of the cross-sections on the magnitudes given above, depends upon the coefficients $\widehat{\kappa}$ and $\widehat{\varepsilon}$ which characterize the beam cross-sections, its slenderness ratio λ, and the number k which describes the warping function χ (Fig.2). For the boundary conditions (cf Ewing, 1990)

$$u(1) = \omega(1) = \gamma(1) = 0 \qquad Q(0) = Q_0 \qquad M(0) = H(0) = 0 \tag{5.5}$$

there is, in general case

$$u = \widehat{u}_T + \frac{1 - \widehat{\kappa}}{\widehat{\kappa}}\frac{\sinh\beta - \sinh\beta\xi}{\beta\cosh\beta}\frac{Q_0}{GA} \tag{5.6}$$

and

$$\Gamma = \Big[\frac{k_{22} - \kappa_{22}(1 - \eta)}{k_{22} - \kappa_{22}^2} + \kappa_{22}\frac{1 - \kappa_{22} - \eta}{k_{22} - \kappa_{22}^2}\frac{\cosh\beta\xi}{\cosh\beta}\Big]\frac{Q_0}{GA} \tag{5.7}$$

We shall now search the departure from the geometric relation $\Gamma = u' + \omega + \gamma = 0$. For a beam with a narrow rectangular cross-section, according to Eqs (5.2) and (5.5), we have

$$\Gamma = \frac{1}{1 - k}\frac{\cosh\beta\xi}{\cosh\beta}\frac{Q_0}{GA} \qquad \text{and} \qquad \Gamma = \frac{\cosh\beta\xi}{\cosh\beta}\frac{Q_0}{GA} \tag{5.8}$$

Hence, it is evident that in the case of slender beams (for large β) Γ is nearly equal to zero, except in the closest vicinity of the beam fixing ($\xi = 1$). Similar results are obtained for beams for other conditions of fixing their ends, and under different loads.

5.2. Waves in an elastic beam of circular cross-section

The right criterion for correctness of any approximate beam theory lies in comparison of the computation results with those obtained from exact theory. For this purpose we shall investigate the dispersion of the waves propagating in an elastic beam of infinite length and of circular cross-section. The obtained results will be compared with those resulting from the Pochhammer-Chree theory. For this reason we shall look for the solution of Eq (3.17) for transverse displacement in the form of

$$u = U \exp\Big[\mathrm{i}\frac{2\pi}{\Lambda}(x - ct)\Big] \tag{5.9}$$

where c is the phase velocity, and Λ is the wavelength. Then the equation of dispersion may be written in the form

$$\widehat{\varepsilon}\frac{E}{G}(\gamma^2-1)\Big[\frac{E}{G}\gamma^4-\Big(1+\frac{E}{G}+\sigma^2\Big)\gamma^2+1\Big]-\sigma^2\Big[\frac{E}{\widehat{\kappa}G}\gamma^4-\Big(1+\frac{E}{\widehat{\kappa}G}+\sigma^2\Big)\gamma^2+1\Big]=0 \tag{5.10}$$

where $\gamma = c/c_E$, $\sigma = 1/[\pi(a/\Lambda)]$, $c_E^2 = E/\varrho$ and $\widehat{\kappa} = 6/7$, $\widehat{\varepsilon} = 1/12$ while a is the radius of the circular cross-section. The last part of the Eq (5.10) in the square bracket is the equation of dispersion according to Timoshenko's theory (cf Abramson, 1957). The first part of this equation, on the contrary, decides upon the influence of the constraint of the cross-section warping caused by the shear, and determines an additional third branch of the wave dispersion curves. It is easy to be convinced about this, if $\widehat{\kappa} = 1$ is assumed. Then the additional dispersion branch is approximately defined by the formula

$$\gamma_3 = \sqrt{1 + \frac{G\sigma^2}{E\widehat{\varepsilon}}} \tag{5.11}$$

The relationship between γ and Λ for the free and constrained warping of the beam circular cross-sections is shown in Table 1.

Table 1. Comparison of the phase velocity $\gamma = c/c_E$ for various a/Λ for a circular cross-section of a beam with free and constrained warping ($\nu = 0.25$)

a/Λ	Free warping		Constrained warping		
	γ_1	γ_2	γ_1	γ_2	γ_3
0	0	∞	0	∞	∞
0.1	0.26882	2.1780	0.26889	2.1680	7.6589
0.2	0.40959	1.4295	0.41042	1.4243	3.9320
0.3	0.47802	1.2248	0.48061	1.2214	2.7310
0.4	0.51446	1.1381	0.51959	1.1357	2.1575
0.5	0.53563	1.0931	0.54377	1.0913	1.8316
0.6	0.54883	1.0668	0.56014	1.0654	1.6268
0.7	0.55752	1.0502	0.57200	1.0491	1.4892
0.8	0.56351	1.0390	0.58101	1.0381	1.3922
0.9	0.56780	1.0312	0.58810	1.0305	1.3213
1.0	0.57096	1.0254	0.59382	1.0249	1.2678

The constraint of the cross-section warping causes an insignificant increase in the phase velocity value c, in the first branch γ_1, and a decrease in the second γ_2 which is qualitatively consistent with the results of the Abramson (1957).

5.3. The free vibration of a simply supported beam

The analysis of this problem will enable us to investigate the influence of the cross-section warping constraint upon the frequencies of higher modes of free vibration. We look after the solution of Eq (3.17), satisfying boundary conditions

$$u = U \sin \alpha_n \xi \cos p\tau \qquad \alpha_n = \pi n \qquad (n = 1, 2, \ldots) \tag{5.12}$$

Then the equation of vibration is of the form

$$\begin{aligned} &\widehat{\varepsilon}\frac{E}{G}(x^2 - \beta_n^2)\Big[\frac{E}{G}x^4 - \beta_n^2\Big(1 + \frac{E}{G} + \beta_n^2\Big)x^2 + \beta_n^4\Big] + \\ &-\beta_n^4\Big[\frac{E}{\widehat{\kappa}G}x^4 - \beta_n^2\Big(1 + \frac{E}{\widehat{\kappa}G} + \beta_n^2\Big)x^2 + \beta_n^4\Big] = 0 \end{aligned} \tag{5.13}$$

whereby the following notation is introduced: $x = p/p_E$, $p_E = \alpha_n^2/\lambda$, $\beta_n = \lambda/\alpha_n$ and p_E is the circular nondimensional frequency of vibration acording to the Euler-Bernoulli model of the beam, λ is the slenderness ratio of the beam. The second square bracket in Eq (5.13), when compared to zero, is the Timoshenko equation. The results of calculation of the frequency ratios $x_T^{(\alpha)}$ and $x^{(i)}$; $(\alpha = 1, 2;\ i = 1, 2, 3)$ according to Timoshenko and the model of constrained warping of a beam with a circular cross-section, are shown in Table 2. A comparison of the results received by the use of Timoshenko's model and those obtained using the constrained warping model, explains that the constraint causes an increase in the frequency of the first branch and a frequency decrease of the second branch. The differences between them are greater, the smaller the beam slenderness ratio is and the higher the mode of vibration.

Table 2. Comparison of $x = p/p_E$ for various λ with a simply supported circular cross-sectional beam with free and constrained warping ($\nu = 0.25$)

Mode n	Nondim. frequency	Slenderness ratio λ 8	12	16	20
1	$x_T^{(1)}$	0.8009	0.8919	0.9335	0.9556
	$x_T^{(2)}$	4.7513	9.5799	16.2664	24.8307
	$x^{(1)}$	0.8011	0.8919	0.9335	0.9556
	$x^{(2)}$	4.7200	9.5321	16.1819	24.6996
	$x^{(3)}$	15.4772	34.5444	61.2372	95.5566
2	$x_T^{(1)}$	0.5723	0.7139	0.8009	0.8556
	$x_T^{(2)}$	1.6584	2.9916	4.7413	6.6145
	$x^{(1)}$	0.5745	0.7149	0.8011	0.8559
	$x^{(2)}$	1.5256	2.9793	4.7199	6.8997
	$x^{(3)}$	4.0323	8.8027	15.4772	24.0579

Making use of the results of calculating $x^{(\alpha)}(n,\lambda)$, $(\alpha = 1,2)$ obtained for the beam model with constrained warping, we can determine the shear coefficients $\kappa_T^{(\alpha)}$), $(\alpha = 1,2)$ which should appear in the classic equation of Timoshenko. Then

$$\frac{1}{\widehat{\kappa}_T^{(\alpha)}} = \frac{G/E}{(\alpha_n x^{(\alpha)}/\lambda)^2}\Big[1 - \frac{(x^{(\alpha)})^2}{(\alpha_n x^{(\alpha)}/\lambda)^2}\Big] \tag{5.14}$$

Table 3. Shear coefficient $\kappa_T^{(\alpha)}(n,\lambda)$ for a simply supported circular cross-section beam

λ	Mode n	Frequency branch $\alpha = 1$	$\alpha = 2$	λ	Mode n	Frequency branch $\alpha = 1$	$\alpha = 2$
8	1	0.86045	0.84676	16	1	0.85794	0.84749
	2	0.86849	0.84537		2	0.86045	0.84676
	3	0.87904	0.84447		3	0.86406	0.84601
	4	0.89076	0.84392		4	0.86849	0.84537

Hence, it is evident that the coefficients of shear, defined in this way, depend upon the number of the frequency branch $(\alpha = 1,2)$, the mode of beam vibration (n), as well as, upon the beam slenderness ratio. The results of these calculations are given in Table 3. From Table 3, it is clear that the coefficients of shear of the first branch $\widehat{\kappa} < \kappa_T^{(1)} < 1$, the coefficients of the second branch $\kappa_T^{(\alpha)} < \widehat{\kappa} = 6/7$ for $n = 1,2,3,4$.

We now introduce the new denotation $s_n = [Ep/(G\lambda)]^2$. Then, for the thickness shear mode $(n = 0)$ for a simply supported beam. Eq (5.13) may be presented in the form

$$\widehat{\varepsilon} s_0^2 - \left(\widehat{\varepsilon} + \frac{1}{\widehat{\kappa}}\right) s_0 + 1 = 0 \tag{5.15}$$

Hence, for the narrow rectangular cross-section $(\widehat{\kappa} = 5/6,\ \widehat{\varepsilon} = 3/35)$ the smaller root amounts to $(s_0)_1 = 0.822925$. A value like this was obtained by Leung (1990). For a beam with a circular cross-section $(\widehat{\kappa} = 6/7,\ \widehat{\varepsilon} = 1/12)$ is $(s_0)_1 = 0.847933$ and $(s_0)_2 = 14.152067$. We see that $(s_0)_1 < \widehat{\kappa}$ and $(s_0)_2 > 1/\widehat{\varepsilon}$.

It is also worth noticing that $(s_0)_1 = 1$ and $(s_0)_2 = 1/\widehat{\varepsilon}$ if it is assumed that $\widehat{\kappa} = 1$. For Timoshenko's beam model $(\widehat{\varepsilon} = 0)$ we have $s_0 = \widehat{\kappa}$.

5.4. The natural vibration of a cantilever beam

In the above considered problem of a simply supported beam, the frequencies of natural vibration do not depend upon the parameter k, determining the form of the warping function χ. For the example of the cantilever beam vibration, we shall investigate the influence of that parameter upon the frequencies. We shall first do it in the case of beam in which the warping of the cross-section is free (Timoshenko's model). For this, in Eqs (3.4), we neglect the equation of the shearing moment, and assume a'priori the relation $u' + \omega + \gamma = 0$. Then, on the basis of the remaining equations and the constitutive relations (see Eqs (3.13) and (3.14)), we obtain

$$\begin{aligned}
&\frac{\partial \overline{u}}{\partial \xi} = -\omega + \frac{E}{\kappa_{22} G}\overline{Q} &\qquad& \frac{\partial \omega}{\partial \xi} = \overline{M} + \eta \frac{E}{\kappa_{22} G}\frac{\partial^2 \overline{u}}{\partial \tau^2} \\
&\frac{\partial \overline{Q}}{\partial \xi} = \frac{\partial^2 \overline{u}}{\partial \tau^2} && \frac{\partial \overline{M}}{\partial \xi} = \frac{\partial^2 \omega}{\partial \tau^2} + \lambda^2 \overline{Q} - \eta \frac{E}{\kappa_{22} G}\frac{\partial^2 \overline{Q}}{\partial \tau^2}
\end{aligned} \tag{5.16}$$

where the dimensionless magnitudes: $\overline{u} = u/L$, $\overline{Q} = Q/EA$, $\overline{M} = ML/EJ$ have been introduced. The boundary conditions for a cantilever beam are: $\overline{u}(0) = \omega(0) = 0$ and $\overline{Q}(1) = \overline{M}(1) = 0$. The set of equations and conditions allows us to calculate the nondimensional frequencies of natural vibration $\overline{p} = pL/c_E$ with the use of the Runge-Kutta method of numerical integration of ordinary differential equations. Table 4 presents the calculation result of the magnitudes $r = p\lambda = \omega L^2\sqrt{\varrho A/EJ}$ for circular cross-sections of a beam and for coefficients of shear k_T – Timoshenko's, k_C – Cowper's and k_1 determined from condition $\chi(1) = 0$. A comparison of the results of the above mentioned calculations indicates that:

- considering Possion's coefficient ν in the formulae for the coefficient of shear κ_{22}, causes a decrease of the $r = p\lambda$ value;
- an increase of the coefficient k results in an increase of the $r = p\lambda$ value.

Table 4. Comparison of $r = p\lambda$ for various k and ν for a circular cross-sectional cantilever beam with free warping

λ	Mode	Circle ($\nu = 0$)			Circle ($\nu = 0.3$)		
		$k_T = 2/3$ ($\eta = 2/9$)	$k_C = 6/7$ ($\eta = 0$)	$k_1 = 1$ ($\eta = -1/6$)	$k_T = 0.722$ ($\eta = 0.185$)	$k_C = 0.886$ ($\eta = 0$)	$k_1 = 1$ ($\eta = -0.128$)
10	1	3.2250	3.2769	3.3036	3.1884	3.2388	3.2647
	2	14.879	15.335	15.586	14.276	14.661	14.872
	3	33.129	33.913	34.391	31.431	32.025	32.381
20	1	3.4351	3.4503	3.4579	3.4237	3.4388	3.4464
	2	19.328	19.575	19.702	18.979	19.212	19.332
	3	47.862	48.621	49.386	46.386	47.059	47.416

The results of the $r = p\lambda$ calculations, for a narrow rectangular cross-sections, and for different values of the parameter k and different values of the slenderness ratio λ, are shown in Table 5. As before, an increase of k and λ causes an increase of r. Similar calculations were made by Ewing (1990) for $\lambda = 13.856$ and $\lambda = 34.641$ in the case when $k = 5/6$.

Now we shall investigate the influence of the constrained warping of the cross-sections (the influence of $\widehat{\varepsilon}$) and the influence of the form of the warping function (the effect of k) on the frequencies of free vibration of a cantilever beam, making use of the variational method in calculations. By using Hamilton's principle to determine the nondimensional frequency of vibration p, we obtain the equation

$$\det|\mathbf{K} - p^2\mathbf{M}| = 0 \tag{5.17}$$

where

$$\mathbf{K} = \int_0^1 \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \boldsymbol{\psi}_2'\boldsymbol{\psi}_2'^{T} & \eta\boldsymbol{\psi}_2'\boldsymbol{\psi}_3'^{T} \\ 0 & \eta\boldsymbol{\psi}_3'\boldsymbol{\psi}_2'^{T} & \varepsilon\boldsymbol{\psi}_3'\boldsymbol{\psi}_3'^{T} \end{bmatrix} + \right.$$
$$\left. + \frac{G}{E}\lambda^2 \begin{bmatrix} \boldsymbol{\psi}_1'\boldsymbol{\psi}_1'^{T} & \boldsymbol{\psi}_1'\boldsymbol{\psi}_2^{T} & (1-\kappa_{22})\boldsymbol{\psi}_1'\boldsymbol{\psi}_3^{T} \\ \boldsymbol{\psi}_2\boldsymbol{\psi}_1'^{T} & \boldsymbol{\psi}_2\boldsymbol{\psi}_2^{T} & (1-\kappa_{22})\boldsymbol{\psi}_2\boldsymbol{\psi}_3^{T} \\ (1-\kappa_{22})\boldsymbol{\psi}_3\boldsymbol{\psi}_1'^{T} & (1-\kappa_{22})\boldsymbol{\psi}_3\boldsymbol{\psi}_2 & (1-2\kappa_{22}+k_{22})\boldsymbol{\psi}_3\boldsymbol{\psi}_3^{T} \end{bmatrix} \right\} d\xi \tag{5.18}$$

$$\mathbf{M} = \int_0^1 \begin{bmatrix} \lambda^2 \boldsymbol{\psi}_1 \boldsymbol{\psi}_1^\top & 0 & 0 \\ 0 & \boldsymbol{\psi}_2 \boldsymbol{\psi}_2^\top & \eta \boldsymbol{\psi}_2 \boldsymbol{\psi}_3^\top \\ 0 & \eta \boldsymbol{\psi}_3 \boldsymbol{\psi}_2^\top & \varepsilon \boldsymbol{\psi}_3 \boldsymbol{\psi}_3^\top \end{bmatrix} d\xi \tag{5.19}$$

In the matrices of stiffness and mass written above, $\boldsymbol{\psi}_i = (\psi_i^n)^\top$, $(i = 1, 2, 3;$ $n = 1, 2, \ldots)$ are subsequently the vectors of the basic functions of the displacement expansion u, the angle of bending ω and the angle of shearing γ.

Table 5. Comparison of $r = p\lambda$ for various k of a narrow rectangular cross-sectional cantilever beam with free warping

k	Mode n	Slenderness ratio λ, $(\nu = 0)$ 10	13.856	20	34.641
$k_T = 2/3$ $(\eta = 0.2)$	1	3.1689	3.2207	3.4177	3.4822
	2	14.045	16.589	18.837	20.790
	3	30.873	38.066	45.849	54.548
$k_C = 5/6$ $(\eta = 0)$	1	3.2271	3.3548	3.4353	3.4884
	2	14.469	16.971	19.104	20.907
	3	31.503	38.856	46.604	54.989
$k_I = 1$ $(\eta = -0.2)$	1	3.2675	3.3780	3.4471	3.4925
	2	14.788	17.244	19.288	20.986
	3	32.032	39.466	47.146	55.290

Table 6. Comparison of $r = p\lambda$ for various k of a narrow rectangular cross-sectional cantilever beam with constrained warping

k	Mode n	Slenderness ratio λ, $(\nu = 0)$ 10	13.856	20	34.641
$k_T = 2/3$ $(\eta = 0.2)$	1	3.2298	3.3615	3.3446	3.4881
	2	14.560	17.213	19.203	20.906
$k_C = 5/6$ $(\eta = 0)$	1	3.2358	3.3650	3.4354	3.4884
	2	14.746	16.980	19.110	20.910
	3	32.446	38.882	46.633	55.006
$k_I = 1$ $(\eta = -0.2)$	1	3.2298	3.3551	3.4354	3.4884
	2	14.560	17.196	19.093	20.912

In Table 6 the results are presented of the calculations of the $r = p\lambda$ for the subsequent vibration modes, for narrow rectangular cross-sections of a cantilever beam. The basic functions $\boldsymbol{\psi}_i = (\xi_i^n)^\top$ are adopted as simple polynomials satisfying the boundary conditions at the fixed end of the beam.

A comparison of the calculation results in the data shown in Tables 5 and Tables 6, allows us to state that:

- constrained warping of cross-sections results in an increase of the frequencies of vibration when $\eta \geq 0$, ($k \leq 5/6$) and a decrease when $\eta < 0$, ($k > 5/6$);
- for some sufficiently large values of the slenderness ratio λ, the influence of the form of the warping function (i.e. the influence of k) on the frequencies is small.

In conclusion. we may state that the constrained warping, in the case of a cantilever beam with narrow rectangular cross-sections, has little influence on the vibration frequencies.

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Wpływ kształtu i skrępowania spaczania przekrojów poprzecznych na zgięcie belek

Streszczenie

W pracy przedstawiono równania ruchu belki zginanej o dowolnych jednospójnych przekrojach poprzecznych. Uwzględniono skrępowanie deplanacji przekrojów poprzecznych, spowodowanej nierównomiernym ścinaniem. Wprowadzono dodatkowy parametr charakteryzujący kształt funkcji deplanacji przekrojów poprzecznych belki. W podanych równaniach występują dwa niezależne, bezwymiarowe współczynniki ścinania. Jeden z nich charakteryzuje ścinanie skrępowane, a drugi ścinanie swobodne. Pokazano, że współczynniki te są niezależne od kształtu funkcji deplanacji. Równania ruchu i związki konstytutywne w szczególnym przypadku deplanacji swobodnej sprowadzają się do równań i związków występujących w teorii Timoshenko.

W pracy przedstawiono szereg ważnych przykładów ilustrujących teorie zgięcia belek. Wzięto w nich pod uwagę kształt funkcji deplanacji i skrępowanie deplanacji dowolnych, jednospójnych przekrojów poprzecznych. W szczególnym przypadku podano równanie określające wielkość parametrów postaci drgań ścianania poprzecznego belki (thickness shear mode) swobodnie podpartej, posiadającej dowolny przekrój poprzeczny.

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