ON THE GENERALIZED EULER EQUATIONS

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In this paper the Euler equations applied to the motion of bodies placed in the frames of reference subjected to acceleration are presented. Positions of bodies are defined with respect to these frames by \( n \) generalized coordinates. Therefore, it is possible to establish an equivalence between the generalized Euler equations, the generalized Kane’s equations and the generalized Lagrange’s equations valid for accelerated frame of reference. General formulae for the generalized inertial forces together with the way of their application to general cases are given. Application to a special case of a rigid body is added. This problem was considered by Lure [2] in a completely different way; however the physical interpretation given in this work allows us to formulate general and complete expressions for inertial and gyroscopic forces and to establish the differences between them.

1. Introduction

The generalized Euler equations, representing the motion of a body or a system of bodies, relative the frame of reference \( S_1 \), which it is also subjected to an arbitrary movement defined by the velocity of it origin and by the rotation vector \( \omega_{10} \) with respect to the inertial frame of reference, have already been presented in the previous paper (cf Passos Morgado (1993)). These equations are

\[
\left( \frac{dP_r}{dt} \right)_{S_1} = F - M \ddot{R}_0 - \left[ M \omega_{10} \times \left( \frac{dR_c}{dt} \right)_{S_1} + \omega_{10} \times P_r \right] + \left( \frac{d\omega_{10}}{dt} \right)_{S_1} \times M r_c - \omega_{10} \times (\omega_{10} \times M r_c) \tag{1.1}
\]
\[
\left( \frac{dH_r}{dt} \right)_{S_1} = M_0 - M r_c \times \dot{R}_0 - \left[ \left( \frac{dI_0}{dt} \right)_{S_1} \omega_{10} + bm\omega_{10} \times H_r \right] + \left( I_0 \frac{d\omega_{10}}{dt} \right)_{S_1} - \omega_{10} \times I_0 \omega_{10}
\]

Let us assume the motion of a rigid body in \( S_1 \). In the most general case, its motion depends on \( n \) generalized coordinates and time. Therefore, the linear momentum will be defined by \( P_r = P_r(t, q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n) \), the kinetic moment by \( H_r = H_r(t, q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n) \), the vector of mass center positions in \( S_1 \) by \( r_c = r_c(t, q_1, ..., q_n) \), and the inertial tensor with respect to the origin of the system \( S_1 \) by \( I_0 = I_0(t, q_1, ..., q_n) \) origin.

The rotation vector \( \omega_{10} \) is independent of the generalized coordinates and so the velocity of the \( S_1 \) system origin.

The meaning of each term in the above expressions has already been carefully explained by Passos Morgado (1993b).

2. Equivalence of generalized Euler, Kane’s and Lagrange’s equations

However, what we believe deserves further explanation is that it is possible to pass from Eqs (1.1) and (1.2) to the generalized Kane’s equations for accelerated systems (cf Passos Morgado (1993a)) and to the generalized Lagrange’s equations directly, enabling the Author to show their equivalency. Let us suppose a body being a member of the holonomic system \( S_1 \) with \( n \) degrees of freedom. It is possible to define an instantaneous rotation vector representing the body motion as a function of the \( n \) generalized coordinates and the \( n \) generalized velocities (cf Kane (1968))

\[
\omega_{21} = \omega_{q_1} \dot{q}_1 + ... + \omega_{q_n} \dot{q}_n + \omega_t
\]

where the vectors \( \omega_{q_r} \) and \( \omega_t \) are functions of generalized coordinates and time.

Let \( r \) be the position vector of a point \( P \) of the body in the coordinate system \( S_1 \) and let \( r_{0r} \) be the position vector of the reference frame \( S_2 \) origin attached to the body.

The velocity of point \( P \), as seen by an observer belonging to \( S_1 \) is

\[
v_P = \left( \frac{d\dot{r}}{dt} \right)_{S_1} = \left( \frac{d\dot{r}}{dt} \right)_{S_2} + \omega_{21} \times r
\]
where, by differentiation by parts with respect \( \dot{q} \), we obtain

\[
\frac{\partial}{\partial \dot{q}} \left( \frac{d\mathbf{r}}{dt} \right) s_1 = \frac{\partial}{\partial \dot{q}} \left( \frac{d\mathbf{r}}{dt} \right) s_2 + \frac{\partial \omega_{21}}{\partial \dot{q}} \times \mathbf{r} \tag{2.3}
\]

or, having in mind that \( \mathbf{r} = \mathbf{r}_0' + \mathbf{r}' \)

\[
\left( \frac{d\mathbf{r}}{dt} \right) s_2 = \left( \frac{d\mathbf{r}_0'}{dt} \right) s_2 \tag{2.4}
\]

we have

\[
\frac{\partial}{\partial \dot{q}} \left( \frac{d\mathbf{r}}{dt} \right) s_1 = \frac{\partial}{\partial \dot{q}} \left( \frac{d\mathbf{r}_0'}{dt} \right) s_2 + \frac{\partial \omega_{21}}{\partial \dot{q}} \times \mathbf{r} \tag{2.5}
\]

To pass directly from Eqs (1.1) and (1.2) to the generalized Kane's equations let us multiply Eq (1.1) by \( \frac{\partial}{\partial \dot{q}} \left( \frac{d\mathbf{r}_0'}{dt} \right) s_2 \) and multiply Eq (1.2) by \( \frac{\partial \omega_{21}}{\partial \dot{q}} \) given by Eq (2.5).

Therefore we obtain

\[
\left( \frac{dP}{dt} \right) s_1 \frac{\partial}{\partial \dot{q}} \left( \frac{d\mathbf{r}_0'}{dt} \right) s_2 = \mathbf{F} \frac{\partial}{\partial \dot{q}} \left( \frac{d\mathbf{r}_0'}{dt} \right) s_2 - M\ddot{R}_0 \frac{\partial}{\partial \dot{q}} \left( \frac{d\mathbf{r}_0'}{dt} \right) s_2 +
\]

\[
- \left[ M\omega_{10} \times \left( \frac{d\mathbf{r}_c}{dt} \right) s_1 + \omega_{10} \times \mathbf{P}_r \right] \frac{\partial}{\partial \dot{q}} \left( \frac{d\mathbf{r}_0'}{dt} \right) s_2 + \]

\[
- \left( \frac{d\omega_{10}}{dt} \right) s_1 \times M\mathbf{r}_c \frac{\partial}{\partial \dot{q}} \left( \frac{d\mathbf{r}_0'}{dt} \right) s_2 - \omega_{10} \times (\omega_{10} \times M\mathbf{r}_c) \frac{\partial}{\partial \dot{q}} \left( \frac{d\mathbf{r}_0'}{dt} \right) s_2 \tag{2.6}
\]

for the first one, and

\[
\left( \frac{dH}{dt} \right) s_1 \frac{\partial \omega_{21}}{\partial \dot{q}} = M_0 \frac{\partial \omega_{21}}{\partial \dot{q}} - M\mathbf{r}_c \times \ddot{R}_0 \frac{\partial \omega_{21}}{\partial \dot{q}} +
\]

\[
- \left[ \left( \frac{dI_0}{dt} \right) s_1 \omega_{10} + \omega_{10} \times \mathbf{H}_r \right] \frac{\partial \omega_{21}}{\partial \dot{q}} = I_0 \left( \frac{d\omega_{10}}{dt} \right) s_1 \frac{\partial \omega_{21}}{\partial \dot{q}} - \omega_{10} \times I_0 \omega_{10} \frac{\partial \omega_{21}}{\partial \dot{q}} \tag{2.7}
\]

for the second equation.

We define, the generalized force corresponding to the coordinates \( q \), by

\[
\mathbf{F} \frac{\partial}{\partial \dot{q}} \left( \frac{d\mathbf{r}_0'}{dt} \right) s_2 + M_0 \frac{\partial \omega_{21}}{\partial \dot{q}} = Q \tag{2.8}
\]

the generalized inertial force of translation by

\[
M\ddot{R}_0 \frac{\partial}{\partial \dot{q}} \left( \frac{d\mathbf{r}_0'}{dt} \right) s_2 + M\mathbf{r}_c \times \ddot{R}_0 \frac{\partial \omega_{21}}{\partial \dot{q}} = M\ddot{R}_0 \frac{\partial \mathbf{r}_c}{\partial \dot{q}} \tag{2.9}
\]
the generalized Coriolis inertial force by
\[
\left[ M \omega_{10} \times \left( \frac{d\mathbf{r}_c}{dt} \right)_{S_1} + \omega_{10} \times \mathbf{P}_r \right] \frac{\partial}{\partial \dot{q}} \left( \frac{d\mathbf{r}_{0'}}{dt} \right)_{S_2} + \\
\left( \frac{dI_0}{dt} \right)_{S_1} \omega_{10} + \omega_{10} \times \mathbf{H}_r \right] \frac{\partial \omega_{21}}{\partial \dot{q}} = \omega_{10} \left[ \frac{d}{dt} \left( \frac{\partial \mathbf{H}_r}{\partial \dot{q}} \right)_{S_1} - \frac{\partial \mathbf{H}_r}{\partial \dot{q}} \right]
\]
(2.10)

the generalized inertial force of rotation by
\[
\left[ \left( \frac{d\omega_{10}}{dt} \right)_{S_1} \times M\mathbf{r}_c \right] \frac{\partial}{\partial \dot{q}} \left( \frac{d\mathbf{r}_{0'}}{dt} \right)_{S_2} + I_0 \omega_{10} \frac{\partial \omega_{21}}{\partial \dot{q}} = \omega_{10} \frac{\partial \mathbf{H}_r}{\partial \dot{q}}
\]
(2.11)

the generalized centrifugal inertial force by
\[
\omega_{10} \times (\omega_{10} \times M\mathbf{r}_c) \frac{\partial}{\partial \dot{q}} \left( \frac{d\mathbf{r}_{0'}}{dt} \right)_{S_2} + \omega_{10} \times I_0 \omega_{10} \frac{\partial \omega_{21}}{\partial \dot{q}} = \frac{1}{2} \omega_{10} \frac{\partial I_0}{\partial \dot{q}} \omega_{10}
\]
(2.12)

Adding Eqs (2.6) and (2.7) and substituting for the values of the generalized inertial forces the formulae given above we finally obtained
\[
\left( \frac{d\mathbf{P}_r}{dt} \right)_{S_1} \frac{\partial}{\partial \dot{q}} \left( \frac{d\mathbf{r}_{0'}}{dt} \right)_{S_2} + \left( \frac{d\mathbf{H}_r}{dt} \right)_{S_1} \frac{\partial \omega_{21}}{\partial \dot{q}} = Q - M \left[ \left( \frac{d\mathbf{v}_0}{dt} \right)_{S_1} + \omega_{10} \times \mathbf{v}_0 \right] \frac{\partial \mathbf{r}_c}{\partial \dot{q}} + \\
+ \frac{1}{2} \omega_{10} \frac{\partial I_0}{\partial \dot{q}} \omega_{10} - \omega_{10} \frac{\partial \mathbf{H}_r}{\partial \dot{q}} - \omega_{10} \left[ \frac{d}{dt} \left( \frac{\partial \mathbf{H}_r}{\partial \dot{q}} \right)_{S_1} - \frac{\partial \mathbf{H}_r}{\partial \dot{q}} \right]
\]
(2.13)

The equations given above are fully compatible with the generalized Euler equations for accelerated systems. It is worthy to note that Eq (2.13) is written in generalized coordinates and so it must be equivalent to both, the generalized Kane’s equations and to the generalized Lagrange’s equations.

To prove the aforementioned thesis, let us note that \( \mathbf{P}_r \) and \( \mathbf{H}_r \) are defined by
\[
\mathbf{P}_r = M \left( \frac{d\mathbf{r}_{0'}}{dt} \right)_{S_1} + \omega_{21} \times M\mathbf{r}_c'
\]
(2.14)
\[
\mathbf{H}_r = \mathbf{r}_{0'} \times \mathbf{P}_r + M\mathbf{r}_c' \times \left( \frac{d\mathbf{r}_{0'}}{dt} \right)_{S_1} + I_0 \omega_{21}
\]
(2.15)
and, since
\[
\left( \frac{d\mathbf{r}_{0'}}{dt} \right)_{S_1} = \left( \frac{d\mathbf{r}_{0'}}{dt} \right)_{S_2} + \omega_{21} \times \mathbf{r}_{0'}
\]
(2.16)
we obtain the relationship
\[
\frac{\partial}{\partial \dot{q}} \left( \frac{d\mathbf{r}_{0'}}{dt} \right)_{S_1} = \frac{\partial}{\partial \dot{q}} \left( \frac{d\mathbf{r}_{0'}}{dt} \right)_{S_2} + \frac{\partial \omega_{21}}{\partial \dot{q}} \times \mathbf{r}_{0'}
\]
(2.17)
Eq (2.17) can be substituted into Eq (2.13) giving

$$
\left( \frac{dP_r}{dt} \right) s_1 \left[ \frac{\partial}{\partial q} \left( \frac{d\mathbf{r}_0'}{dt} \right) \right] s_2 - \frac{\partial \omega_{21}}{\partial q} \times \mathbf{r}_0' + \left( \frac{dH_r}{dt} \right) s_1 \frac{\partial \omega_{21}}{\partial q} =
$$

$$
= Q - M \left[ \left( \frac{d\mathbf{v}_0}{dt} \right) s_1 + \omega_{10} \times \mathbf{v}_0 \right] \frac{\partial \mathbf{r}_c}{\partial q} + \frac{1}{2} \omega_{10} \frac{\partial I_0}{\partial q} \omega_{10} - \omega_{10} \omega_{10} \left[ \frac{d}{dt} \left( \frac{\partial H_r}{\partial q} \right) \right] s_1 - \frac{\partial H_r}{\partial q}
$$

(2.18)

By simplifying this mathematical expression and making the substitution

$$
\frac{\partial \mathbf{v}_0'}{\partial q} = \left( \frac{\partial \mathbf{r}_0'}{dt} \right) s_1,
$$

we finally obtain

$$
\left[ \frac{d}{dt} \left( \omega_{21} \times M \mathbf{r}_c' \right) \right] s_1 + M a_0' \frac{\partial \mathbf{v}_0'}{\partial q} + \left[ \frac{d}{dt} \left( I_0 \omega_{21} \right) \right] s_1 + M \mathbf{r}_c' \times a_0' \frac{\partial \omega_{21}}{\partial q} =
$$

$$
= Q - M \tilde{R}_0 \frac{\partial \mathbf{r}_c}{\partial q} - \omega_{10} \frac{\partial H_r}{\partial q} + \frac{1}{2} \omega_{10} \frac{\partial I_0}{\partial q} \omega_{10} - \omega_{10} \omega_{10} \left[ \frac{d}{dt} \left( \frac{\partial H_r}{\partial q} \right) \right] s_1 - \frac{\partial H_r}{\partial q}
$$

(2.19)

Eq (2.19) are precisely the Kane’s equations for accelerated systems, which proves the equivalency between Euler equations for mechanical systems and the generalized Kane’s equations. The equivalency between the generalized Euler equations and the generalized Lagrange’s equations is now straightforward (cf Passos Morgado (1993a)). To complete the discussion we present the general formulae for the forces of inertia and the moments of the referred forces in the Euler equations and the formulae for the generalized forces employed in the Lagrange’s and Kane’s equations.

To this end we must determined, in a general case, the values of

$$
\frac{\partial H_r}{\partial q}, \quad \frac{\partial H_r}{\partial q}, \quad \frac{\partial I_0}{\partial q}
$$

Noting that

$$
H_r = M \mathbf{r}_c' \times \left( \frac{d\mathbf{r}_0'}{dt} \right) s_1 + \mathbf{r}_0' \times (\omega_{21} \times M \mathbf{r}_c') + I_0' \omega_{21}
$$

(2.20)

we have immediately

$$
\frac{\partial H_r}{\partial q} = M \frac{\partial \mathbf{r}_c'}{\partial q} \times \frac{d\mathbf{r}_0'}{dt} s_1 + \mathbf{r}_0' \times \left( \frac{\partial \omega_{21}}{\partial q} \times M \mathbf{r}_c' \right) + I_0' \frac{\partial \omega_{21}}{\partial q}
$$

(2.21)

and

$$
\frac{\partial H_r}{\partial q} = M \frac{\partial \mathbf{r}_c'}{\partial q} \times \left( \frac{d\mathbf{r}_0'}{dt} \right) s_1 + M \mathbf{r}_c' \times \frac{\partial}{\partial q} \left( \frac{d\mathbf{r}_0'}{dt} \right) s_1 + \frac{d\mathbf{r}_0'}{dq} \times (\omega_{21} \times M \mathbf{r}_c') +
$$

$$
+ \mathbf{r}_0' \times \left( \frac{\partial \omega_{21}}{\partial q} \times M \mathbf{r}_c' \right) + \mathbf{r}_0' \times (\omega_{21} \times M \frac{\partial \mathbf{r}_c'}{\partial q}) + \frac{dI_0'}{dq} \omega_{21} + \frac{I_0'}{\partial q} \omega_{21}
$$

(2.22)
The partial derivative of the moment of inertia about the point 0, can be related to the moment of inertia about point 0', attached to the body and origin of the frame of reference S2, by the following equation

$$I_0\omega_{10} = Mr_0' \times (\omega_{10} \times r_0') + r_0' \times (\omega_{10} \times Mr'_0) + Mr'_0 \times (\omega_{10} \times r_0') + I_0\omega_{10}$$

(2.23)

for the generalized centrifugal inertial forces, we have

$$\frac{1}{2} \omega_{10} \frac{\partial I_0}{\partial q} \omega_{10} = M \left(\omega_{10} \times \frac{\partial r_0'}{\partial q}\right) (\omega_{10} \times r_0') + M \left(\omega_{10} \times \frac{\partial r_0'}{\partial q}\right) (\omega_{10} \times r'_c) +$$

$$+ M \left(\omega_{10} \times \frac{\partial r'_c}{\partial q}\right) (\omega_{10} \times r_0') + \frac{1}{2} \omega_{10} \frac{\partial I_0}{\partial q} \omega_{10}$$

(2.24)

of the vectors $\omega_{qr} = \omega_{qr}(q_1, ..., q_n t)$ with respect to time have to be calculated. If $c$ is any vector fixed to the body, then

$$\left(\frac{dc}{dt}\right)_{S_1} = \left(\frac{dc}{dt}\right)_{S_2} + \omega_{21} \times c = \omega_{21} \times c$$

(2.25)

On the other hand, when $c = c(q_1, ..., q_n t)$ is the vector attached to the body dependent on the generalized coordinates and on time, we have

$$\left(\frac{dc}{dt}\right)_{S_1} = \frac{\partial c}{\partial q_1} \dot{q}_1 + \frac{\partial c}{\partial q_2} \dot{q}_2 + ... + \frac{\partial c}{\partial q_n} \dot{q}_n + \frac{\partial c}{\partial t}$$

(2.26)

Upon substitution of Eq (2.1) into (2.25) we have

$$\left(\frac{dc}{dt}\right)_{S_1} = (\omega_{q_1} \times c) \dot{q}_1 + (\omega_{q_2} \times c) \dot{q}_2 + ... + (\omega_{q_n} \times c) \dot{q}_n + \omega_t \times c$$

(2.27)

Now by comparing Eqs (2.26) and (2.27) we can conclude that

$$\frac{\partial c}{\partial q_r} = \omega_{qr} \times c \quad \quad \frac{\partial c}{\partial t} = \omega_t \times c$$

(2.28)

The vectors $\omega_{qr}$ can be treated as operators which produce the partial derivatives when multiplied vectorially by a vector fixed to the body.

Noting

$$\frac{\partial}{\partial q_r} \left(\frac{dc}{dt}\right)_{S_1} = \left[\frac{d}{dt} \left(\frac{\partial c}{\partial q_r}\right)\right]_{S_1}$$

(2.29)

we can conclude, basing on the Eqs (2.25) and (2.28), that

$$\frac{\partial}{\partial q_r} (\omega_{21} \times c) = \left[\frac{d}{dt} (\omega_{qr} \times c)\right]_{S_1}$$

(2.30)
and

\[ \frac{\partial \omega_{21}}{\partial q_r} \times c + \omega_{21} \times \frac{\partial c}{\partial q_r} = \left( \frac{d\omega_{q_r}}{dt} \right)_{S_1} \times c + \omega_{q_r} \times \left( \frac{dc}{dt} \right)_{S_1} \] (2.31)

Having in mind Eqs (2.25) and (2.28) we can rewrite Eq (2.31) as follows

\[ \frac{\partial \omega_{21}}{\partial q_r} \times c + \omega_{21} \times (\omega_{q_r} \times c) = \left( \frac{d\omega_{q_r}}{dt} \right)_{S_1} \times c + \omega_{q_r} \times (\omega_{21} \times c) \] (2.32)

and introducing

\[ \omega_{q_r} \times (\omega_{21} \times c) - \omega_{21} \times (\omega_{q_r} \times c) = c \times (\omega_{21} \times \omega_{q_r}) \] (2.33)

we conclude that

\[ \left( \frac{d\omega_{q_r}}{dt} \right)_{S_1} = \frac{\partial \omega_{21}}{\partial q_r} + \omega_{21} \times \omega_{q_r} \] (2.34)

which allows us to determine the derivative of vectors \( \omega_{q_r} \) with respect to time. Also it is worthy to note the relation

\[ \frac{\partial \omega_{21}}{\partial q_r} = \frac{\partial \omega_{q_1}}{\partial q_r} \dot{q}_1 + \frac{\partial \omega_{q_2}}{\partial q_r} \dot{q}_2 + \ldots + \frac{\partial \omega_{q_n}}{\partial q_r} \dot{q}_n + \frac{\partial \omega_i}{\partial q_r} \] (2.35)

Let us now suppose that the origin of the system \( S_2 \) coincides with the origin of system \( S_1 \). In this case \( r_0 = 0 \), (since \( 0' \) and \( 0 \) coincide), and the Eqs (2.21) and (2.22) are reduced to the forms

\[ \frac{\partial H_r}{\partial q} = I_0' \frac{\partial \omega_{21}}{\partial q} \] (2.36)

\[ \frac{\partial H_r}{\partial q} = I_0' + I_0' \frac{\partial \omega_{21}}{\partial q} \] (2.37)

Let us now to find simpler formulae for the generalized inertial forces. For the generalized rotational inertial force we have

\[ \dot{\omega}_{10} \frac{\partial H_r}{\partial \dot{q}} = \dot{\omega}_{10} I_0' \frac{\partial \omega_{21}}{\partial \dot{q}} = \dot{\omega}_{10} I_0' \omega_q = \omega_q I_0' \dot{\omega}_{10} \] (2.38)

For the generalized centrifugal inertial force we get

\[ \frac{1}{2} \omega_{10} \frac{\partial I_0}{\partial q} \omega_{10} = \frac{1}{2} \omega_{10} \frac{\partial I_0'}{\partial q} \omega_{10} = \frac{1}{2} \omega_{10} \left( \frac{\partial \omega_{21}}{\partial q} \times I_0' - I_0' \times \frac{\partial \omega_{21}}{\partial q} \right) \omega_{10} \] (2.39)
and as the dyad is a symmetrical one, we can write
\[
\frac{1}{2} \omega_{10} \frac{\partial I_0}{\partial q} \omega_{10} = \frac{1}{2} \omega_{10} \left( \frac{\partial \omega_{21}}{\partial q} \times I_0' \right) \omega_{10} - \frac{1}{2} \omega_{10} \left( I_0' \times \frac{\partial \omega_{21}}{\partial q} \right) \omega_{10} =
\]
\[
= \frac{1}{2} \left[ \left( \omega_{10} \times \frac{\partial \omega_{21}}{\partial q} \right) I_0' \omega_{10} - \omega_{10} I_0' \left( \frac{\partial \omega_{21}}{\partial q} \times \omega_{10} \right) \right] =
\]
\[
= \frac{1}{2} \left[ \left( \omega_{10} \times \frac{\partial \omega_{21}}{\partial q} \right) I_0' \omega_{10} + \left( \omega_{10} \times \frac{\partial \omega_{21}}{\partial q} \right) I_0 \omega_{10} \right] =
\]
\[
= \left( \omega_{10} \times \frac{\partial \omega_{21}}{\partial q} \right) I_0' \omega_{10} = \frac{\partial \omega_{21}}{\partial q} \left( \omega_{10} \times I_0 \omega_{10} \right)
\]

(2.40)

For the Coriolis inertial force of we have
\[
\omega_{10} \left[ \frac{d}{dt} \left( \frac{\partial H_r}{\partial q} \right) s_1 - \frac{\partial H_r}{\partial q} \right] = \omega_{10} \left[ \frac{d}{dt} \left( I_0' \frac{\partial \omega_{21}}{\partial q} \right) s_1 - \left( \frac{\partial I_0'}{\partial q} \omega_{21} + I_0' \frac{\partial \omega_{21}}{\partial q} \right) \right] =
\]
\[
= \omega_{10} \left[ \left( \omega_{21} \times I_0' - I_0' \times \omega_{21} \right) \frac{\partial \omega_{21}}{\partial q} + I_0' \left( \frac{\partial \omega_{21}}{\partial q} + \omega_{21} \times \frac{\partial \omega_{21}}{\partial q} \right) +
\]
\[
- \left( \frac{\partial I_0'}{\partial q} \omega_{21} + I_0' \frac{\partial \omega_{21}}{\partial q} \right) \right] = \omega_{10} \left[ \left( \omega_{21} \times I_0' - I_0' \times \omega_{21} \right) \frac{\partial \omega_{21}}{\partial q} + I_0' \left( \frac{\partial \omega_{21}}{\partial q} + \omega_{21} \times \frac{\partial \omega_{21}}{\partial q} \right) +
\]
\[
+ I_0' \left( \omega_{21} \times \frac{\partial \omega_{21}}{\partial q} \right) - \left( \frac{\partial \omega_{21}}{\partial q} \times I_0' - I_0' \times \frac{\partial \omega_{21}}{\partial q} \right) \omega_{21} \right] =
\]
\[
= \omega_{10} \left[ \left( \omega_{21} \times I_0' \right) \frac{\partial \omega_{21}}{\partial q} - \left( I_0' \times \omega_{21} \right) \frac{\partial \omega_{21}}{\partial q} - \left( \frac{\partial \omega_{21}}{\partial q} \times I_0' \right) \omega_{21} \right]
\]

(2.41)

since
\[
\left( I_0' \times \frac{\partial \omega_{21}}{\partial q} \right) \omega_{21} = I_0' \left( \frac{\partial \omega_{21}}{\partial q} \times \omega_{21} \right) = -I_0' \left( \omega_{21} \times \frac{\partial \omega_{21}}{\partial q} \right)
\]

To simplify Eqs (2.41) let us introduce the relation
\[
\left( \omega_{21} \times I_0' \right) \frac{\partial \omega_{21}}{\partial q} = \omega_{21} \times \left( I_0' \frac{\partial \omega_{21}}{\partial q} \right)
\]

and the left-side product by \( \omega_{10} \) yields
\[
-\omega_{10} \left( \omega_{21} \times I_0' \right) \frac{\partial \omega_{21}}{\partial q} = \omega_{10} \omega_{21} \times \left( I_0' \frac{\partial \omega_{21}}{\partial q} \right) =
\]
\[
= \left( \omega_{10} \times \omega_{21} \right) I_0' \frac{\partial \omega_{21}}{\partial q} = \frac{\partial \omega_{21}}{\partial q} I_0' \left( \omega_{10} \times \omega_{21} \right)
\]

(2.42)

Introducing now the relation
\[
- \left( I_0' \times \omega_{21} \right) \frac{\partial \omega_{21}}{\partial q} = -I_0' \left( \omega_{21} \frac{\partial \omega_{21}}{\partial q} \right)
\]
and the left-side product by \( \omega_{10} \) yields

\[
-\omega_{10}(I_{0'} \times \omega_{21}) \frac{\partial \omega_{21}}{\partial \dot{q}} = -\omega_{10} I_{0'} \left( \omega_{21} \times \frac{\partial \omega_{21}}{\partial \dot{q}} \right) =
\]

(2.43)

\[
= -\omega_{10} I_{0'} \left( \frac{\partial \omega_{21}}{\partial \dot{q}} \times \omega_{21} \right) = \left( \frac{\partial \omega_{21}}{\partial \dot{q}} \right) \omega_{10} \times \omega_{21} = \frac{\partial \omega_{21}}{\partial \dot{q}} \omega_{21} \times I_{0'} \omega_{10}
\]

Another relation to be considered is the following one

\[
-(\frac{\partial \omega_{21}}{\partial \dot{q}} \times I_{0'}) \omega_{21} = -\frac{\partial \omega_{21}}{\partial \dot{q}} (I_{0'} \omega_{21})
\]

in which, again, multiplying it by \( \omega_{10} \), we have

\[
-\omega_{10} \left( \frac{\partial \omega_{21}}{\partial \dot{q}} \times I_{0'} \right) \omega_{21} = -\omega_{10} \frac{\partial \omega_{21}}{\partial \dot{q}} \times (I_{0'} \omega_{21}) =
\]

(2.44)

\[
= -\omega_{10} \frac{\partial \omega_{21}}{\partial \dot{q}} \times I_{0'} \omega_{21} = \frac{\partial \omega_{21}}{\partial \dot{q}} \omega_{10} \times I_{0'} \omega_{21}
\]

Substitution Eqs (2.42) \( \div \) (2.44) into Eq (2.41), we have for the generalized Coriolis inertial forces

\[
\omega_{10} \left[ \frac{d}{dt} \left( \frac{\partial H_F}{\partial \dot{q}} \right) \right]_{\dot{s}} - \frac{\partial H_F}{\partial \dot{q}} =
\]

(2.45)

\[
= \frac{\partial \omega_{21}}{\partial \dot{q}} I_{0'} (\omega_{10} \times \omega_{21}) + \frac{\partial \omega_{21}}{\partial \dot{q}} (\omega_{21} \times I_{0'} \omega_{10} + \omega_{10} \times I_{0'} \omega_{21})
\]

which give us the Coriolis generalized force of inertia for the proposed case.

We must note that the quantity which appears in the first term of the second member of Eq (2.45), being related with the Resal acceleration, is not a gyroscopic force (cf Skalmierski (1979)) but in the reality is a term of the Coriolis generalized force.

### 3. Conclusions

In this paper we have generalized the Euler equations and demonstrated their equivalence to the generalized Kane's and Lagrange's equations for accelerated systems. Also we presented the general formulae for the generalized inertial forces and the way of deriving them in the general case. In another paper it will be presented the distinction between inertia forces and gyroscopic ones, and it will be shown that, in the most general case, there are four, and only four, different types of gyroscopic forces.
References


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