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# COUPLED LONGITUDINAL AND BENDING FORCED VIBRATION OF TIMOSHENKO CANTILEVER BEAM WITH A CLOSING CRACK

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The paper presents the method of modelling of the coupled longitudinal and bending forced vibrations of the Timoshenko cantilever beam with a transverse, one-edge, non-propagating, closing crack. Two models of closing crack are examined in the paper. The beam is modelled by the finite element method. The equation of motion is formulated using the harmonic balance method. The influence of various parameters (crack location and its depth, axial force frequency, closing crack model etc.) upon the steady state response of the beam free end is analyzed. The possibility of crack detection based on the analysis of higher harmonics variations in the frequency spectrum is examined.

## 1. Introduction

Vibration testing is being recognized as an effective and fast method for detecting cracks in structural elements (cf Adams and Cawley, 1979, 1985; Stubbs, 1985). In order to detect cracks by vibration method the study of the changes of the structural dynamic behavior due to damages is required. The results of theoretical and experimental studies suggest that crack location and its depth can be detected from the changes in the first few natural frequencies (cf Ju and Mimovich, 1988), the modes of vibrations (cf Pandey et al., 1991) and the amplitudes of forced vibrations (cf Akgun and Ju, 1990), respectively. The effect of coupling longitudinal and bending vibrations (cf Papadopoulos and Dimarogonas, 1988) or torsional and bending vibrations (cf Papadopoulos and Dimarogonas, 1987) is also included to identification systems.

From the point of view, of the accuracy of numerical calculations a proper model of crack behavior during vibrations is needed. In general two models, which describe the behavior of crack during vibrations, are applied. The first model assumes that crack is open during vibration. This model is frequently applied and widely described in literature (cf Wauer, 1990). In this case the equation of motion is linear. The second model take into account the contact phenomenon at crack interfaces. This approach describes better the real behavior of crack but it is not frequency applied to the dynamic analysis of cracked structures, due to the fact (among the others) that the structure vibrations are governed by the nonlinear equation or by the linear equation with the time dependent coefficients.

Gudmudson (1983) experimentally investigated the influence of the one-edge, closing crack upon eigenfrequencies of the cantilever beam. He pointed out that the decrease in the natural frequencies caused by the closing crack is much lower than the decrease due to the open crack. Gudmudson's results were theoretically confirmed by Ibrahim et al. (1987). In their work the crack was modelled by bilinear spring. Using the bond-graph technique they analyzed the first five modes of the cracked cantilever beam and concluded that the frequency drop due to the closing crack is always smaller than the one computed applying the open crack model. Zastrau (1985) used the finite element method to study the forced vibrations of the simply supported beam with the multiple closing cracks. The time histories of vibrations proved nonlinear behavior of the structure with the crack. Actis and Dimarogonas (1989) applied the finite element method to determination of the changes in the spectrum of forced vibrations of a simply supported beam with the one-edge closing crack. The similar approach was proposed by Qian et al. (1990). They concluded that the difference between the displacement responses of the beam without and with the crack,respectively, is reduced when the effect of the closing crack is considered. Ostachowicz and Krawczuk (1990) analyzed the influence of the closing crack upon the steady state response of the cracked cantilever beam. They applied the finite element method and introduced the special point finite element within the contact area.

Several papers concerned vibrations of beams with closing cracks were published in 1992. Schen and Chu (1992) analyzed vibrations of the simply supported beam with the so called "breathing crack". The bilinear equation of motion for each mode of vibration was formulated by Galerkin procedure. The results of numerical studies have shown the feasibility of using the spectrum pattern to detect cracks. Friswell and Penny (1992) proposed the equivalent one-degree-of-freedom nonlinear model of the cracked beam vibrating in its first mode. Abraham and Brandon (1992) applied the piece-wise linear approach to the "breathing crack" modelling. They analyzed the influence of local effects such as dry friction and impact at the crack interfaces upon the amplitudes of forced vibrations. Collins et al. (1991) used the pair of self-balancing forces and the piece-wise stiffness to model the contact phenomenon in the crack under longitudinal vibrations. They obtained the governing equation of motion and studied dynamic responses of the cantilever beam employing the two-term Galerkin approximation. Krawczuk and Ostachowicz (1992), (1993) analyzed the influence of the closing crack upon the areas of parametric vibrations and the dynamic stability of cracked columns. The crack was modeled by the spring with periodically, time-varying stiffness.

From the presented review of the papers it results that the problem of coupled vibration of the beam with closing crack has not been analyzed up till now. For this reason in the present paper the coupled longitudinal and bending vibrations due to the one-edge, transverse, non-propagating, closing crack located at the cantilever Timoshenko beam are investigated. The beam is modelled using the finite element method. The equation of motion is formulated according to the harmonic balance method. The possibility of crack detection based on the analysis of higher harmonics variations in the frequency spectrum is examined.

## 2. Discrete model of analyzed cantilever Timoshenko beam with closing crack

A discrete model of the analyzed beam with the transverse, one-edge, non-propagating, closing crack is shown in Fig.1.

The analyzed structure is discretized using the finite element method. Undamaged parts of the beam are modelled by the beam finite elements of Timoshenko type with two nodes and three degrees of freedom at the node (cf Przemieniecki 1968). A special Timoshenko beam finite element with the crack is substituted for cracked part of the beam (cf Krawczuk, 1992). The cracked element has the same degrees of freedom number as the non-cracked one, Fig.2. In the case, when the crack depth is equal to zero all characteristic matrices of the cracked element have the same form as in the case of the non-cracked element proposed by Przemieniecki (1968).

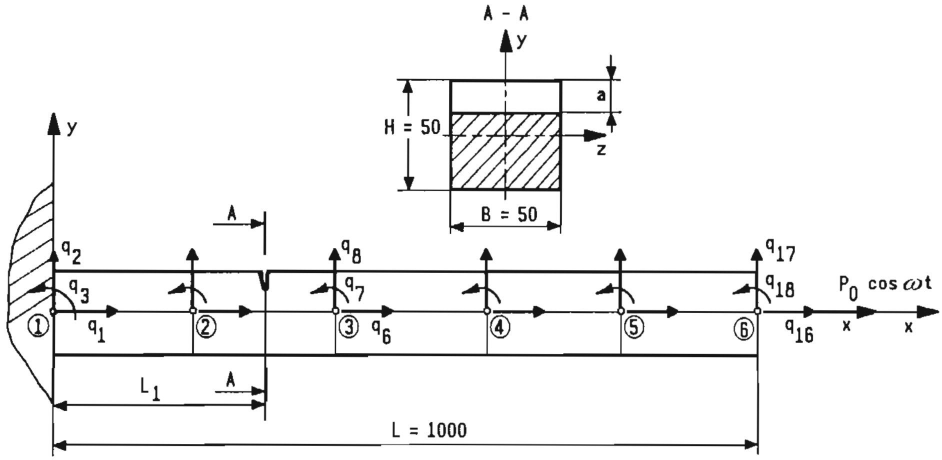


Fig. 1. Dimensions and the discretization of the cantilever beam with the closing crack analyzed in the paper

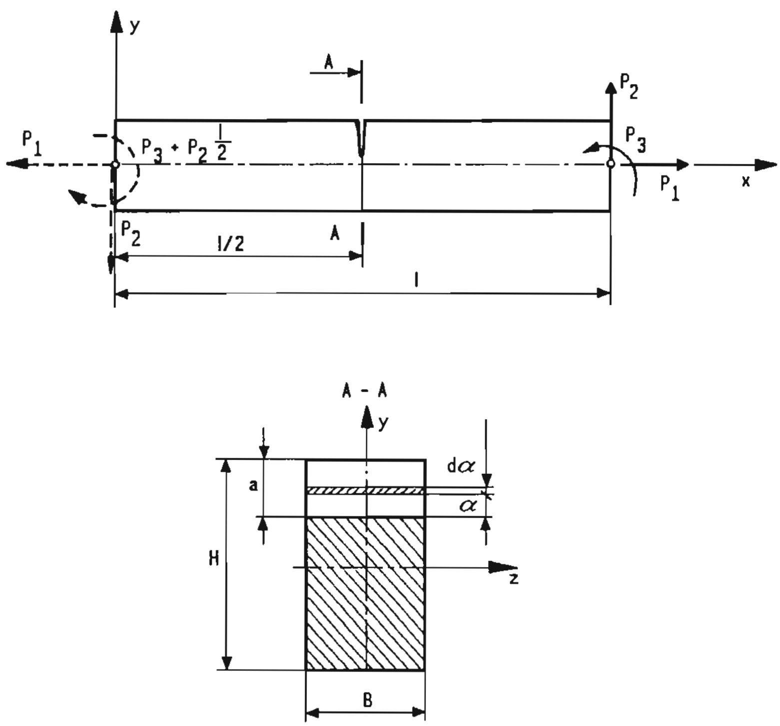


Fig. 2. Beam finite element with the one-edge, transverse, nonpropagating crack

## 2.1. Stiffness matrix of the cracked element

The stiffness matrix of the cracked beam finite element $\mathbf{K}_e$ can be written in the following form

$$\mathbf{K}_e = \mathbf{T}^\top(\mathbf{C}^0 + \mathbf{C}^1)^{-1}\mathbf{T} \tag{2.1}$$

where

$\mathbf{C}^0$ – flexibility matrix of the non-cracked element

$\mathbf{C}^1$ – flexibility matrix of the element due to crack

$\mathbf{T}$ – transformation matrix of the system of dependent nodal forces into the system of independent ones

the upper index $\top$ denotes transposition of the matrix, whereas the upper index $-1$ denotes its inverse.

The flexibility matrices $\mathbf{C}^0$ and $\mathbf{C}^1$ are calculated according to the method described by Krawczuk (1992). The flexibility matrix of the non-cracked element $\mathbf{C}^0$ can be expressed in the following form (cf Krawczuk, 1992)

$$\mathbf{C}^0 = \begin{bmatrix} \dfrac{l}{AE} & 0 & 0 \\ 0 & \dfrac{l^3}{3EI} + \dfrac{\beta l}{GA} & \dfrac{l^2}{2EI} \\ 0 & \dfrac{l^2}{2EI} & \dfrac{l}{EI} \end{bmatrix} \tag{2.2}$$

where

$l$ – length of the element

$A$ – area of the element cross section

$I$ – geometrical moment of inertia of the element cross-section

$E$ – Young modulus

$G$ – shear modulus

$\beta$ – shear coefficient of the element cross-section (cf Cowper, 1966).

For the rectangular cross-section

$$\beta = \frac{12 + 11\nu}{10(1 - \nu^2)}$$

where $\nu$ denotes the Poisson ratio.

The final form of the element flexibility matrix due to crack $\mathbf{C}^1$ is given by the following relation (cf Krawczuk, 1992)

$$\mathbf{C}^1 = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} \tag{2.3}$$

where

$$c_{11} = \frac{2\pi(1-\nu^2)}{BE} \int\limits_0^{\bar{\alpha}_k} \bar{\alpha} F_1^2(\bar{\alpha})\, d\bar{\alpha} \tag{2.4}$$

$$c_{12} = \frac{6\pi(1-\nu^2)l}{BHE} \int\limits_0^{\bar{\alpha}_k} \bar{\alpha} F_1^2(\bar{\alpha})\, d\bar{\alpha} \tag{2.5}$$

$$c_{13} = \frac{12\pi(1-\nu^2)}{BHE} \int\limits_0^{\bar{\alpha}_k} \bar{\alpha} F_1^2(\bar{\alpha})\, d\bar{\alpha} \tag{2.6}$$

$$c_{22} = \frac{18\pi(1-\nu^2)l^2}{BH^2E} \int\limits_0^{\bar{\alpha}_k} \bar{\alpha} F_1^2(\bar{\alpha})\, d\bar{\alpha} + \frac{2\beta^2\pi(1-\nu^2)}{BE} \int\limits_0^{\bar{\alpha}_k} \bar{\alpha} F_2^2(\bar{\alpha})\, d\bar{\alpha} \tag{2.7}$$

$$c_{23} = \frac{36\pi(1-\nu^2)}{BH^2E} \int\limits_0^{\bar{\alpha}_k} \bar{\alpha} F_1^2(\bar{\alpha})\, d\bar{\alpha} \tag{2.8}$$

$$c_{33} = \frac{72\pi(1-\nu^2)}{BH^2E} \int\limits_0^{\bar{\alpha}_k} \bar{\alpha} F_1^2(\bar{\alpha})\, d\bar{\alpha} \tag{2.9}$$

and $\bar{\alpha} = \alpha/H$ and $\bar{\alpha}_k = a/H$ – see Fig.2, $B$, $H$, $l$ are the dimensions of the element, $F_1(\bar{\alpha})$ and $F_2(\bar{\alpha})$ are the correction functions which take into account finite dimensions of the element.

Forms of the correction functions $F_1(\bar{\alpha})$ and $F_2(\bar{\alpha})$ are given by the following formulas (cf Papadopoulos and Dimarogonas, 1988)

$$F_1(\bar{\alpha}) = \sqrt{\frac{\tan\lambda}{\lambda}}\, \frac{0.752 + 2.02\Big(\frac{\alpha}{H}\Big) + 0.37(1-\sin\lambda)^3}{\cos\lambda} \tag{2.10}$$

$$F_2(\bar{\alpha}) = \frac{1.122 - 0.561\Big(\frac{\alpha}{H}\Big) + 0.085\Big(\frac{\alpha}{H}\Big)^2 + 0.18\Big(\frac{\alpha}{H}\Big)^3}{\sqrt{1-\frac{\alpha}{H}}} \tag{2.11}$$

where $\lambda = \pi\alpha/2H$.

In formulas (2.4) $\div$ (2.9) two types of integrals appear. The changes of these integrals (called non-dimensional flexibilities) related to the crack depth are shown in Fig.3.

In the opposite to the form of the flexibility matrix $\mathbf{C}^0$, the components $c_{12}$ and $c_{13}$ of the additional flexibility matrix $\mathbf{C}^1$, are not equal to zero. For this reason the coupling of longitudinal and bending vibrations resulting from crack can be expected in the spectrum of vibrations.

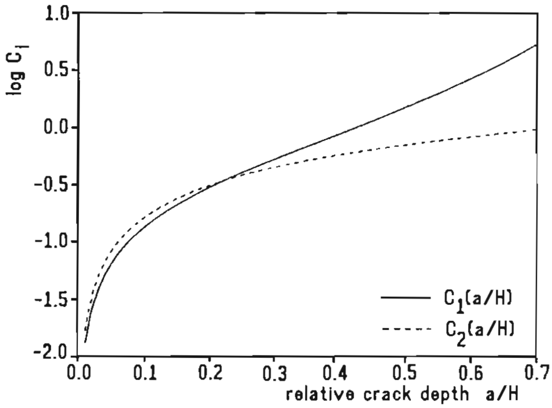


Fig. 3. Variations of the nondimensional flexibilities as the function of the relative depth of the crack; $C_1(a/H) = \int_0^{\bar{\alpha}_k} \bar{\alpha}\, F_1^2\,(\bar{\alpha})\, d\bar{\alpha}$, $C_2(a/H) = \int_0^{\bar{\alpha}_k} \bar{\alpha}\, F_2^2\,(\bar{\alpha})\, d\bar{\alpha}$

The transformation matrix $\mathbf{T}$ is obtained from the statical equilibrium conditions of the element. In the case of presented element this matrix has the form

$$\mathbf{T}^\top = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -l & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.12}$$

Substituting relations (2.2) (2.3) and (2.12) into Eq (2.1) the stiffness matrix of the element with crack $\mathbf{K}_e$ can be determined. When the flexibility matrix of the element due to crack $\mathbf{C}^1$ is equal to zero, the stiffness matrix of the element $\mathbf{K}_e$ has the same form as in the case of the non-cracked element proposed by Przemieniecki (1968).

## 2.2. Inertia matrix of the cracked element

The inertia matrix $\mathbf{M}_e$ of the cracked element is assumed in the same form as the inertia matrix of the non-cracked one (cf Krawczuk, 1992). For the presented element the inertia matrix $\mathbf{M}_e$ has the following form (cf Przemieniecki, 1968)

$$
\mathbf{M}_e = \frac{\rho A l}{(1+\Phi)^2}\begin{bmatrix} m_{11} & 0 & 0 & m_{14} & 0 & 0 \\ 0 & m_{22} & m_{23} & 0 & m_{25} & m_{26} \\ 0 & m_{23} & m_{33} & 0 & m_{35} & m_{36} \\ m_{14} & 0 & 0 & m_{44} & 0 & 0 \\ 0 & m_{25} & m_{35} & 0 & m_{55} & m_{56} \\ 0 & m_{26} & m_{36} & 0 & m_{56} & m_{66} \end{bmatrix} \tag{2.13}
$$

where

$$
m_{11} = m_{44} = \frac{(1+\Phi)^2}{3} \qquad m_{14} = \frac{(1+\Phi)^2}{6} \tag{2.14}
$$

$$
m_{22} = m_{55} = \Big(\frac{13}{35} + \frac{7}{10}\Phi + \frac{1}{3}\Phi^2\Big) + \frac{6}{5}\Big(\frac{r}{l}\Big)^2 \tag{2.15}
$$

$$
m_{33} = m_{66} = \Big(\frac{1}{105} + \frac{1}{60}\Phi + \frac{1}{120}\Phi^2\Big)l^2 + \Big(\frac{r}{l}\Big)^2\Big(\frac{2}{15} + \frac{1}{6}\Phi + \frac{1}{3}\Phi^2\Big)^2 l^2 \tag{2.16}
$$

$$
m_{23} = -m_{56} = \Big(\frac{11}{210} + \frac{11}{120}\Phi + \frac{1}{24}\Phi^2\Big)l + \Big(\frac{r}{l}\Big)^2\Big(\frac{1}{10} - \frac{1}{2}\Phi\Big)l \tag{2.17}
$$

$$
m_{35} = -m_{26} = \Big(\frac{13}{420} + \frac{3}{40}\Phi + \frac{1}{24}\Phi^2\Big)l + \Big(\frac{r}{l}\Big)^2\Big(-\frac{1}{10} + \frac{1}{2}\Phi\Big)l \tag{2.18}
$$

$$
m_{25} = \Big(\frac{9}{70} + \frac{3}{10}\Phi + \frac{1}{6}\Phi^2\Big) - \frac{6}{5}\Big(\frac{r}{l}\Big)^2 \tag{2.19}
$$

$$
m_{36} = -\Big(\frac{1}{140} + \frac{1}{60}\Phi + \frac{1}{60}\Phi^2\Big)l^2 + \Big(\frac{r}{l}\Big)^2\Big(-\frac{1}{30} - \frac{1}{6}\Phi + \frac{1}{6}\Phi^2\Big)l^2 \tag{2.20}
$$

$$
\Phi = 2\beta\Big(\frac{H}{l}\Big)^2 \qquad r = \frac{H}{2} \tag{2.21}
$$

## 3. Equation of motion of the model

In general case, the equation of forced vibrations of the cracked beam has a nonlinear character (cf Gash, 1993). As long as, the amplitudes of vibrations are small in comparison to the static deflections due to a certain loading component of the beam (dead loads, own weight etc.) the nonlinear equation of motion can be transformed to the linear, periodically time-varying equation. Taking into account the above assumption, the discretization process leads to the following equation of motion

$$
\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + [\mathbf{K} - \Delta\mathbf{K}f(t)]\mathbf{q} = \mathbf{P}(t) \tag{3.1}
$$

where

| | | |
|---|---|---|
| $\mathbf{M}$ | – | global inertia matrix |
| $\mathbf{C}$ | – | global damping matrix |
| $\mathbf{K}$ | – | global stiffness matrix of the non-cracked model |
| $\Delta\mathbf{K}$ | – | changes in the global stiffness matrix caused by crack |
| $f(t)$ | – | crack function which describes the process of crack closing |
| $\mathbf{P}(t)$ | – | global column matrix of forces acting on the structure |
| $\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}$ | – | column matrices of generalized accelerations, velocities and displacements,respectively, of the discrete model nodes. |

The above equation of motion constitute the system of linear periodically time-varying differential equations of the second order and it can be solved using the harmonic balance method. When the harmonic balance method is applied, the crack function $f(t)$ and the solution form of the Eq (3.1) $\mathbf{q}$, should be expressed as the Fourier series.

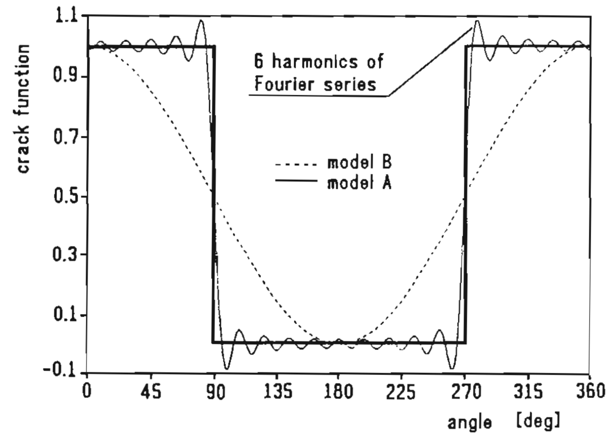


Fig. 4. The closing behaviour of the "breathing crack"-model A; Maye's model-model B

Because one of the aims of the present paper is to compare the steady state responses of the beam for different models of closing crack, two crack functions are analyzed. The first one is called "breathing crack" (cf Abraham and Brandon, 1992; Shen and Chu, 1992). In this case crack can be fully open or fully closed, Fig.4. Other states are excluded from the analysis. The "breathing crack" model can be presented as the Fourier series in the form

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\cos(\omega t) - \frac{2}{3\pi}\cos(3\omega t) + \frac{2}{5\pi}\cos(5\omega t)... \tag{3.2}$$

where $\omega$ is the frequency of the axial force acting on the beam. The second model examined in the paper was proposed by Mayes and Davies (1976) for deep cracks (Fig.4). In this case the crack function has the form

$$f(t) = \frac{1}{2}[1 + \cos(\omega t)] \tag{3.3}$$

For this model crack can be fully open or closed and also partially open or closed.

The solution $\mathbf{q}$ of the Eq (3.1) can be expressed as the truncated Fourier series

$$\mathbf{q} = \sum_{j=1}^{j=R} [\mathbf{a}_j \sin(j\omega t) + \mathbf{b}_j \cos(j\omega t)] \tag{3.4}$$

where $\mathbf{a}_j$, $\mathbf{b}_j$ are the column matrices of the constant variables.

From the point of view of the accuracy of the assumed solution $\mathbf{q}$, it is of great importance to define the proper number of harmonics. When the number of harmonics is too small the accuracy of the solution can be insufficient. On the other hand, when the number of harmonics is too large the time and the cost of numerical calculations increase. The solution of this problem, in the case of cracked rotating shafts, was given by Gash (1993), who pointed out that only three first harmonics have a significant influence on the solution of the equation of motion form. In the presented paper the four first harmonics are take into consideration.

By substituting Eq (3.4) into Eq (3.1) and equating the corresponding coefficients of $\sin(j\omega t)$ and $\cos(j\omega t)$, Eq (3.1) can be transformed into the set of linear homogeneous equations in terms of $\mathbf{a}_j$ and $\mathbf{b}_j$. When the crack function is given by Eq (3.2) the equation of motion has the form

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{P} \end{bmatrix} \tag{3.5}$$

where

$$\mathbf{A}_{11} = \begin{bmatrix} \mathbf{K}' - 16\omega^2\mathbf{M} & -\frac{4}{7\pi}\Delta\mathbf{K} & \mathbf{0} & \frac{4}{15\pi}\Delta\mathbf{K} \\ -\frac{4}{7\pi}\Delta\mathbf{K} & \mathbf{K}' - 9\omega^2\mathbf{M} & -\frac{2}{5\pi}\Delta\mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\frac{2}{5\pi}\Delta\mathbf{K} & \mathbf{K}' - 4\omega^2\mathbf{M} & -\frac{2}{3\pi}\Delta\mathbf{K} \\ \frac{4}{15\pi}\Delta\mathbf{K} & \mathbf{0} & -\frac{2}{3\pi}\Delta\mathbf{K} & \mathbf{K}' - \omega^2\mathbf{M} \end{bmatrix} \tag{3.6}$$

$$\mathbf{A}_{22} = \begin{bmatrix} \mathbf{K}' - \omega^2\mathbf{M} & -\frac{1}{3\pi}\Delta\mathbf{K} & \mathbf{0} & \frac{1}{15\pi}\Delta\mathbf{K} \\ -\frac{1}{3\pi}\Delta\mathbf{K} & \mathbf{K}' - 4\omega^2\mathbf{M} & -\frac{3}{5\pi}\Delta\mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\frac{3}{5\pi}\Delta\mathbf{K} & \mathbf{K}' - 9\omega^2\mathbf{M} & -\frac{3}{7\pi}\Delta\mathbf{K} \\ \frac{1}{15\pi}\Delta\mathbf{K} & \mathbf{0} & -\frac{3}{7\pi}\Delta\mathbf{K} & \mathbf{K}' - 16\omega^2\mathbf{M} \end{bmatrix} \tag{3.7}$$

$$\mathbf{A}_{12} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -4\omega\mathbf{C} \\ \mathbf{0} & \mathbf{0} & -3\omega\mathbf{C} & \mathbf{0} \\ \mathbf{0} & -2\omega\mathbf{C} & \mathbf{0} & \mathbf{0} \\ -\omega\mathbf{C} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{3.8}$$

$$\mathbf{A}_{21} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \omega\mathbf{C} \\ \mathbf{0} & \mathbf{0} & 2\omega\mathbf{C} & \mathbf{0} \\ \mathbf{0} & 3\omega\mathbf{C} & \mathbf{0} & \mathbf{0} \\ 4\omega\mathbf{C} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{3.9}$$

and $\mathbf{K}' = \mathbf{K} - \frac{1}{2}\Delta\mathbf{K}$ , $\mathbf{a} = \mathrm{col}(a_4, a_3, a_2, a_1)$ , $\mathbf{b} = \mathrm{col}(b_1, b_2, b_3, b_4)$, $\mathbf{P} = \mathrm{col}(0, P_0, 0, 0)$, $P_0$ is the amplitude of the axial force acting on the beam.

For the crack function proposed by Mayes the equation of motion can be expressed as

$$\begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{P} \end{bmatrix} \tag{3.10}$$

where

$$\mathbf{B}_{11} = \begin{bmatrix} \mathbf{K}' - 16\omega^2\mathbf{M} & -\frac{1}{4}\Delta\mathbf{K} & \mathbf{0} & \mathbf{0} \\ -\frac{1}{4}\Delta\mathbf{K} & \mathbf{K}' - 9\omega^2\mathbf{M} & -\frac{1}{4}\Delta\mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\frac{1}{4}\Delta\mathbf{K} & \mathbf{K}' - 4\omega^2\mathbf{M} & -\frac{1}{4}\Delta\mathbf{K} \\ \mathbf{0} & \mathbf{0} & -\frac{1}{4}\Delta\mathbf{K} & \mathbf{K}' - \omega^2\mathbf{M} \end{bmatrix} \tag{3.11}$$

$$\mathbf{B}_{22} = \begin{bmatrix} \mathbf{K}' - \omega^2\mathbf{M} & -\frac{1}{4}\Delta\mathbf{K} & \mathbf{0} & \mathbf{0} \\ -\frac{1}{4}\Delta\mathbf{K} & \mathbf{K}' - 4\omega^2\mathbf{M} & -\frac{1}{4}\Delta\mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\frac{1}{4}\Delta\mathbf{K} & \mathbf{K}' - 9\omega^2\mathbf{M} & -\frac{1}{4}\Delta\mathbf{K} \\ \mathbf{0} & \mathbf{0} & -\frac{1}{4}\Delta\mathbf{K} & \mathbf{K}' - 16\omega^2\mathbf{M} \end{bmatrix} \tag{3.12}$$

$$\mathbf{B}_{12} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -4\omega\mathbf{C} \\ \mathbf{0} & \mathbf{0} & -3\omega\mathbf{C} & \mathbf{0} \\ \mathbf{0} & -2\omega\mathbf{C} & \mathbf{0} & \mathbf{0} \\ -\omega\mathbf{C} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{3.13}$$

$$\mathbf{B}_{21} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \omega\mathbf{C} \\ \mathbf{0} & \mathbf{0} & 2\omega\mathbf{C} & \mathbf{0} \\ \mathbf{0} & 3\omega\mathbf{C} & \mathbf{0} & \mathbf{0} \\ 4\omega\mathbf{C} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{3.14}$$

where

$$\mathbf{K}' = \mathbf{K} - \frac{1}{2}\Delta\mathbf{K}$$

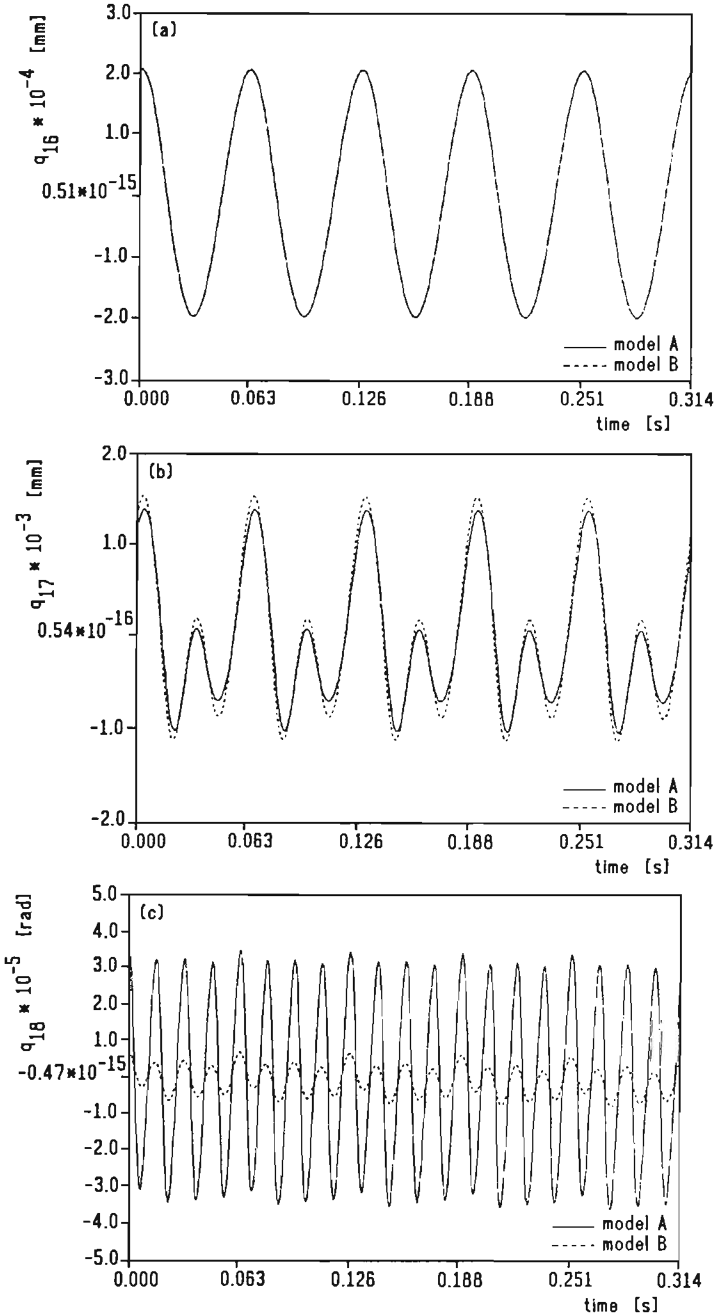


Fig. 5. Vibrations of the cracked beam for two models of the closing crack; $a/H = 0.3$, $L_1/L = 0.1$, $P_0 = 100$ N, $\omega = 100$ rad/s

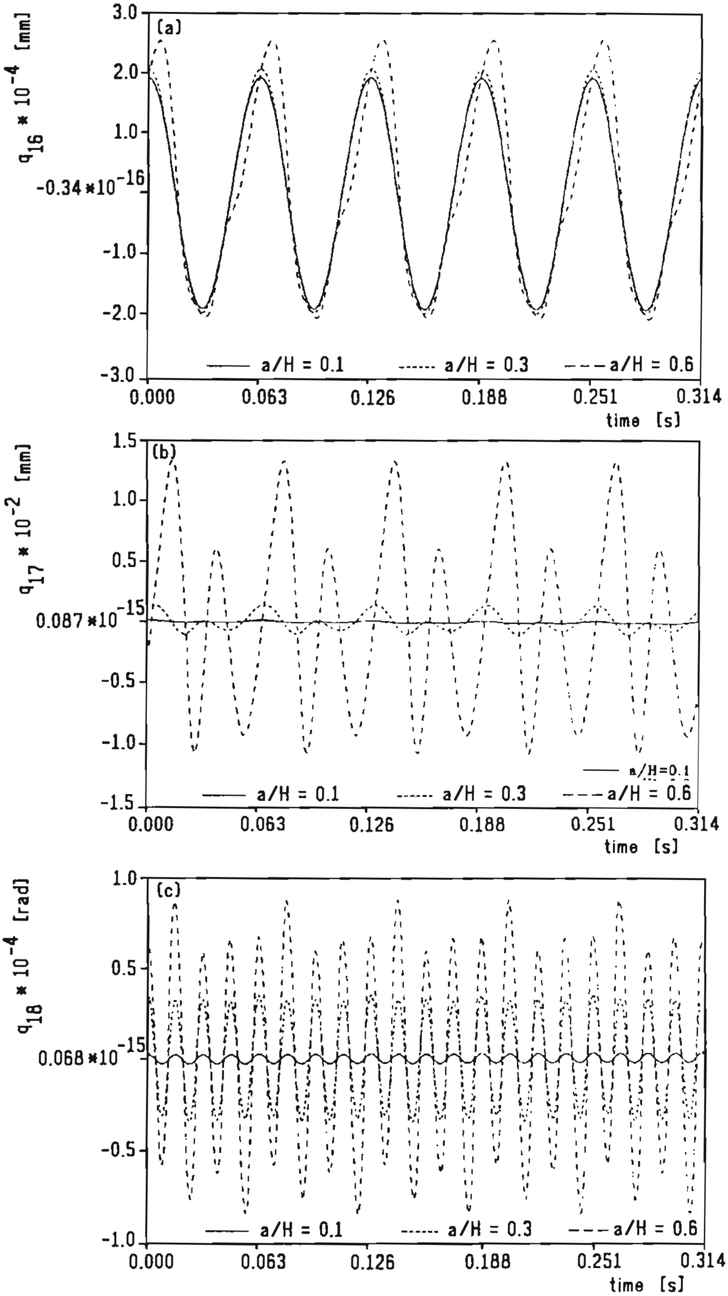


Fig. 6. Vibrations of the beam with the closing crack as the function of the crack depth; $L_1/L = 0.1$, $P_0 = 100$ N, $\omega = 100$ rad/s

## 4. Numerical calculations

Numerical calculations were done for the cantilever beam with the rectangular cross-section. Dimensions and the discrete model of the analyzed beam are shown in Fig.1. The beam is made of steel of the following material properties: Young modulus $E = 2.1 \cdot 10^{11}$ N/m$^2$, shear modulus $G = 8.07 \cdot 10^{10}$ N/m$^2$, mass density $\rho = 7860$ kg/m$^3$ and Poisson ratio $\nu = 0.3$. The free end of the beam is subjected to the axial, harmonic force $(P_0 \cos \omega t)$. The damping matrix $\mathbf{C}$ is calculated as the linear combination of the stiffness and the inertia matrices of the non-cracked element $(\mathbf{C} = \eta \mathbf{M} + \psi \mathbf{K})$. The coefficients $\eta$ and $\psi$ are calculated according to the method described by Rakowski et al. (1984). In the all numerical examples the coefficients $\eta$ and $\psi$ are: $\eta = 22.619, \psi = 0.000052474$.

The results of numerical calculations are presented in Fig.5 ÷ Fig.9. The steady state responses of the beam free end obtained for two models of the closing crack are shown in Fig.5. The first model (called A) corresponds to the "breathing crack", whereas the second model (called B) corresponds to the model proposed by Mayes and Davies (1976). In this example the relative depth of the crack is equal to 0.3 and its location is $L_1/L = 0.1$. The amplitude of the axial force is equal to 100 N with the frequency equal to 100 rad/s. The numerical results presented in Fig.5 clearly show that the character of the beam steady state responses is similar for the both models. Nevertheless the amplitudes of the rotational degrees of freedom $(q_{18})$ for the model A are much greater than those received according to the model B.

The influence of the crack depth upon vibrations of the beam free end is illustrated in Fig.6. The crack is located at the first element $(L_1/L = 0.1)$. The parameters of the axial force are the same as in the first example. The "breathing crack" model is applied. The results obtained indicate that when the crack depth increases the amplitudes of transverse $(q_{17})$ and rotating $(q_{18})$ vibrations also increase, whereas the amplitudes of axial vibrations are almost unchanged.

The steady state responses of the beam free end for various locations of the crack are presented in Fig.7. In this example the relative depth of the crack is equal to 0.3. The amplitude and the frequency of the axial force and also the closing crack model are the same as in the second example. The results indicate that the amplitudes of transverse and rotating vibrations generated by the crack are the biggest for the crack located near the fixed end of the beam.

The influence of the axial force frequency upon the steady state responses

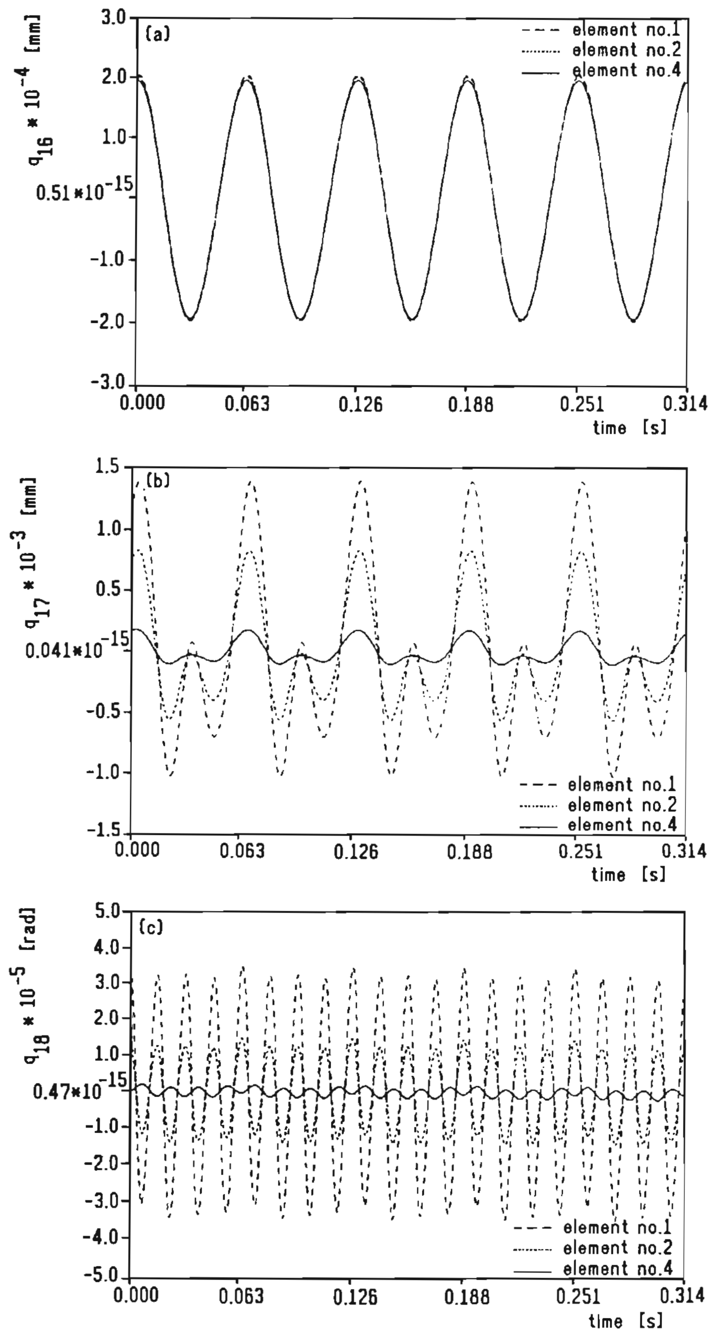


Fig. 7. Vibrations of the beam with the closing crack as the function of the crack location; $a/H = 0.3$, $P_0 = 100$ N, $\omega = 100$ rad/s

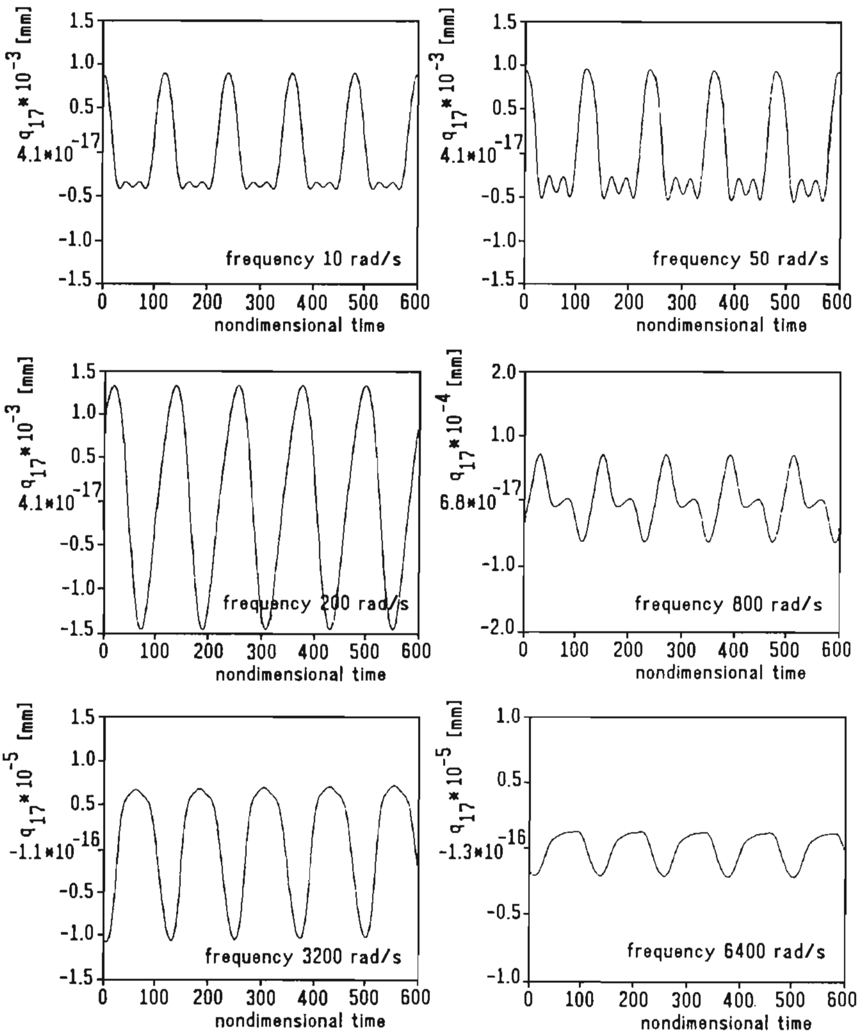


Fig. 8. Vibrations of the beam with the closing crack as the function of the frequency of exciting force; $a/H = 0.3$, $L_1/L = 0.1$, $P_0 = 100$ N

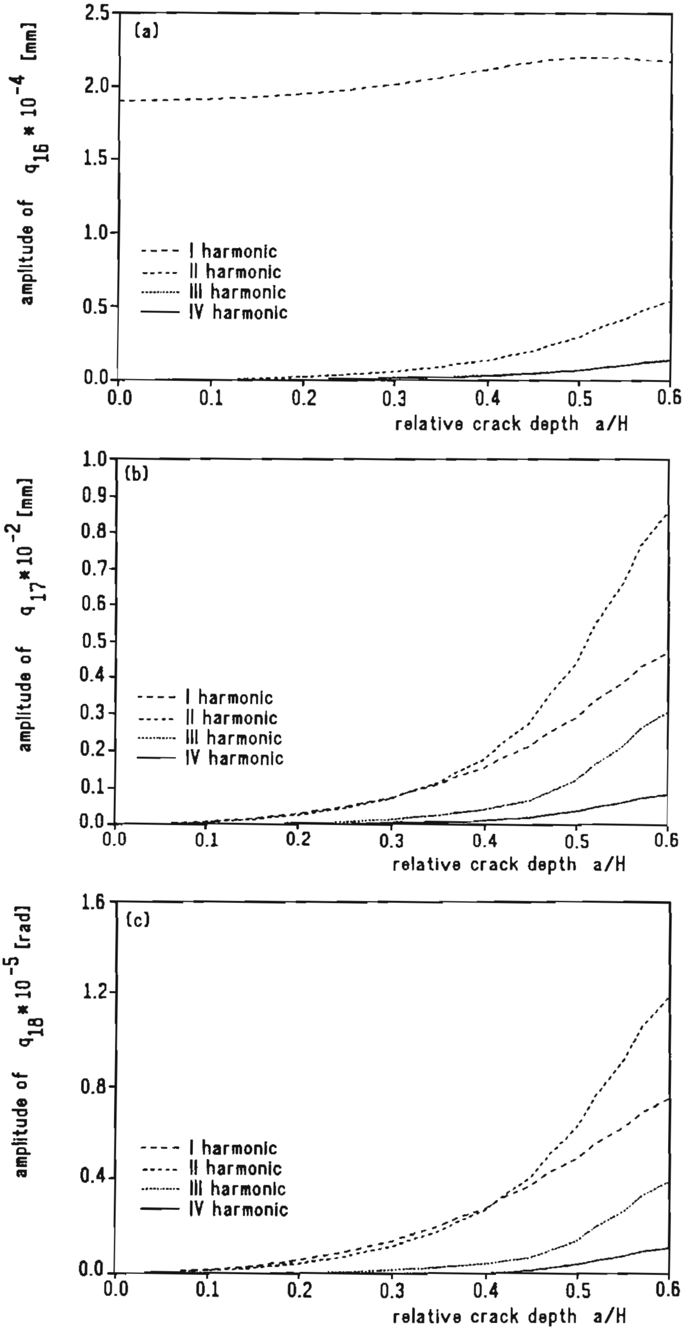


Fig. 9. Amplitudes of higher harmonics as the function of the relative depth of the crack; $a/H = 0.3$, $L_1/L = 0.1$, $P_0 = 100$ N, $\omega = 100$ rad/s

of the beam is presented in Fig.8. The results clearly shown that the character of transverse vibrations generated by the crack strongly depends on the exciting force frequency. When the frequency of axial force lies near the first bending natural frequency of cracked beam (about 246 rad/s) the character of vibrations is similar to the character of vibrations of the non-cracked one.

Fig.9 shows the variation of the four first amplitudes of harmonics for the various depth of the crack. For parameters which are taken into account ($P_0 = 100$ N, $\omega = 100$ rad/s, $a/H = 0.3$ and $L_1/L = 0.1$) it occurs that the amplitudes of the first and the second harmonics, for small cracks $a/H < 0.4$, are the same. The third harmonic appears when the crack depth is greater than 30% of the beam height, whereas the fourth harmonic appears for the relatively deep cracks ($a/H > 0.5$).

## 5. Conclusions

The paper presents the method of modelling of the coupled longitudinal and bending forced vibrations of the cantilever Timoshenko beam with the one-edge, transverse, non-propagating, closing crack.

The results of numerical calculations clearly shown that the amplitude of harmonics in the spectrum of vibration are strongly dependent on the location of crack and its depth and also the frequency of axial force acting on the beam. On the other hand, the character of steady state responses of the beam slightly depends on the model of closing crack.

It is well known that in the case of the non-cracked beams the effect of coupling between longitudinal and bending vibrations does not appear. The crack presence in the beam causes the coupling of these two modes of vibrations. This effect was reported by Papadopoulos and Dimarogonas in the case of beam with open crack (cf Papadopoulos and Dimarogonas, 1988). When closing crack is analyzed this effect appears also, and additionally the higher harmonics are generated in the spectrum of vibrations.

The results of numerical calculations presented in the paper have shown that crack in the vibration beam can be easily detected by analyzing the spectrum response for the known axial forcing function. The high harmonics clearly indicate crack presence in the beam. The number of harmonics and their amplitudes can be correlated with crack depth.

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## Wymuszone drgania wzdłużno-giętne belki wspornikowej typu Timoszenki z zamykającym się pęknięciem

### Streszczenie

W pracy przedstawiono metodę modelowania i analizy wymuszonych wzdłużno-giętnych drgań belki wspornikowej typu Timoszenki z poprzecznym jednostronnym pęknięciem zamykającym się. Przeanalizowano dwa modele zamykania się szczeliny. Belkę modelowano metodą elementów skończonych. Równanie ruchu wyprowadzono w oparciu o metodę bilansu współczynników harmonicznych. Przedstawiono wyniki obliczeń numerycznych ilustrujące wpływ położenia i głębokości pęknięcia a także częstości osiowej siły wymuszającej na drgania ustalone końca belki. Omówiono możliwości detekcji pęknięć w oparciu o analizę wyższych składowych harmonicznych generowanych w widmie drgań w wyniku zamykania się szczeliny.

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