FREE VIBRATIONS OF LAYERED BEAM WITH
NON-RECTANGULAR CROSS-SECTION COMPOSED OF
VISCOELASTIC LAYERS

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The paper deals with two methods of calculating of the eigenfrequencies and
the logarithmic decrement for two-layer T-beam composed of viscoelastic
stiffness-comparable layers. One of the methods is developed within the linear
theory of (visco)elasticity by assuming continuity conditions of forces (instead
of stresses) between adjoining layers. Second method is based on Kirchhoff
hypothesis and Rayleigh method. A comparison of the methods has been
presented and simple, useful relationships for calculation of the logarithmic
decrement have been established.

1. Introduction

Layered beams of non-rectangular cross-section composed of stiffness-com-
parable layers have been applied both in civil and machine engineering and because
of this they have been investigated in various aspects. In this short review a few
papers are mentioned in order to introduce a reader to the problems considered by
researchers. It is noteworthy that all the papers discussed here have been de-
voted to static problems. The author has noticed that dynamic problems of the
structures had not in generall been investigated. So, Goodman and Popov (1968)
developed a theory enabling prediction of three-layer wood beam displacements
caused by a static load, assuming interlayer slip or mechanical connections of la-
yers by means of nails or complete connection between layers by means of glue.
Although all considerations of the authors have been restricted to the beam of rec-
tangular cross-section the theory presented there can be extended by taking into
account non-rectangular cross-sections. The equation of equilibrium (16) given by
the authors is the same as the equation (1a) presented by Itani and Brito (1978)
for two-layer T-beam. Problems considered in the latter paper are similar to those
found in Goodman and Popov's work. Ansourian and Roederick (1978) have gi-
given a theory enabling us calculation of static deflections of two-layer structure
composed of concrete plate and steel beam assuming both an interlayer slip and residual stresses within the steel, non-rectangular beam. Troitsky and Zieliński (1989) have analysed a static behaviour of the same structure, however assuming additionally that it is prestressed by means of different tendons. Polensek and Kažic (1991) have introduced a procedure for analyzing I-beams and geometrically similar structures under bending and compressing static loads. Mori et al. (1971) have presented FEM procedure and necessary FORTRAN programs for evaluating stress concentrations of metal-FRP bonded joint where the metal member is of T-cross-section.

In this paper two methods of calculating both the eigenfrequencies and the logarithmic decrement of two-layer T-beam consisting of stiffness-comparable viscoelastic layers have been developed. One of the method has been derived within the linear theory of (visco)elasticity by applying of forces (instead of stresses) continuity conditions between layers. Formulation of the eigenvalue problem within the first method has been developed assuming both isotropic and fibrous layers. The second method is simplified since the Kirchhoff hypothesis of flat cross-sections and the Rayleigh method have been applied. The simplified approach enabled to derive a simple formula for calculating the logarithmic decrement of the structure. Both methods have been compared with respect to their accuracy and the results of comparison have been presented and discussed. According to the author's knowledge the first method is a new one regarding formulation of the problem, however the second one is only an extension of the Baumgarten and Pearce's (1971) approach given for two-layer beam of rectangular cross-section.

2. A new formulation and solution to the eigenvalue problem in the case of isotropic layers

Below we consider an eigenvalue problem of layered simply supported beam of non-rectangular cross-section as for instance shown in Figure 1. The beam is composed of any number of viscoelastic, either isotropic or anisotropic (fibrous), layers and vibrates freely.

In order to formulate the boundary value problem the following kinematic assumptions are applied

\[ u_{z_j} = -g_j(z) \frac{dW(x)}{dx} \exp(i\omega_m t) \quad u_{y_j} \neq 0 \]

\[ u_{z_j} = f_j(z)W(x) \exp(i\omega_m t) \]  

(2.1)

where \( i^2 = -1, \quad j = 1, 2, 3, ... \) denotes a number of the layer, variable \( z \) is the coordinate overlapping the beam deflection, symbol \( t \) and \( \omega_m \) stand for time and
Fig. 1. An exemplary non-rectangular cross-section of the layered beam considered in the paper

for eigenfrequency of $m$th mode of vibration, respectively. Functions $g_j(z), f_j(z)$ are unknown, however

$$ W(x) = W_m \sin \left( m \pi \frac{z}{L} \right) \quad m = 1, 2, 3, \ldots \quad (2.2) $$

where $L$ is the length of beam.

The assumptions (2.1) are not exact in respect of requirements of the set of equations of motion of the theory of elasticity however, it is shown in the paper, that they are exact enough considering correctness of computational results thus practical accuracy of the formulation of problem considered here. Within any layer of the beam occurs plane stress so upon the basis of Hooke’s law we obtain

$$ (\sigma_{yy})_j \equiv (\sigma_{22})_j = (\sigma_{yz})_j \equiv (\sigma_{23})_j = (\sigma_{xy})_j \equiv (\sigma_{12})_j = 0 \quad (2.3) $$

$$ (\varepsilon_{yy})_j \equiv (\varepsilon_{22})_j = -\frac{\nu_j}{1-\nu_j}[(\varepsilon_{xx})_j + (\varepsilon_{zz})_j] \quad (2.4) $$

On the ground of Eq. (2.4) one can notice that within the vibrating beam the strains $(\varepsilon_{yy})_j$ are not equal to zero however the assumptions (2.1) satisfy Hooke’s law provided that $(\varepsilon_{yy})_j$ does not depend on the space variable $y$. By taking into account the definition of infinitesimal strain one can write in this case

$$ u_{yz} = C(x, z, t)y + D(x, z, t) \quad (2.5) $$

where $C(x, z, t)$ may be easily evaluated, however $D(x, z, t)$ will be equal to zero when the beam vibrates transversely without twisting and $xz$ plane of the Cartesian coordinate system divides the beam into two parts of equal width.
For an isotropic material the constitutive equation can be written in the well known form

\[(\sigma_{kl})_j = 2\mu_j(\varepsilon_{kl})_j + \delta_{kl}\lambda_j(\varepsilon_{rr})_j \quad k, l, r = 1, 2, 3 \quad (2.6)\]

however equations of motion of the jth layer being in plane stress are, within the linear theory of elasticity, as follows

\[\mu_j \nabla^2 u_{xj} + \mu_j \kappa_j \left( \frac{\partial^2 u_{xj}}{\partial x^2} + \frac{\partial^2 u_{xj}}{\partial z \partial x} \right) - \rho_j \frac{\partial^2 u_{xj}}{\partial t^2} = 0\]

\[\mu_j \nabla^2 u_{zz} + \mu_j \kappa_j \left( \frac{\partial^2 u_{zz}}{\partial z^2} + \frac{\partial^2 u_{zz}}{\partial z \partial x} \right) - \rho_j \frac{\partial^2 u_{zz}}{\partial t^2} = 0\]  
\[\frac{\partial^2 u}{\partial t^2} = 0\]  

where \(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\), \(\mu_j\), \(\lambda_j\) are the Lame constants and \(\rho_j\) is the mass density of jth layer. The factor \(\kappa_j\) is defined as follows

\[\kappa_j = \frac{1 + \nu_j}{1 - \nu_j} \quad (2.8)\]

where \(\nu_j\) denotes the Poisson ratio. It is obvious that during sinusoidal vibration of the beam, the third equation of motion is not satisfied. This equation is not taken into account in further considerations because it has been assumed that displacements \(u_{yz}\) and accelerations \(\frac{\partial^2 u_{yz}}{\partial t^2}\) of the particles within transversely vibrating beam are rather small in comparison with displacements and accelerations in directions \(x\) and \(z\), respectively. By using the expressions (2.1) and (2.6) one can transform Eq (2.5) to the following form

\[-\mu_j \frac{d^2 g_j}{dz^2} + [(\lambda_j' + 2\mu_j)\alpha_m^2 - \rho_j\omega_m^2]g_j + (\lambda_j' + \mu_j) \frac{df_j}{dz} = 0\]  
\[(\lambda_j' + 2\mu_j) \frac{d^2 f_j}{dz^2} - (\mu_j\alpha_m^2 - \rho_j\omega_m^2)f_j + \alpha_m^2(\lambda_j' + \mu_j) \frac{dg_j}{dz} = 0\]  
\[\alpha_m = m \frac{\pi}{L} \quad \lambda_j' = 2\mu_j \frac{\nu_j}{1 - \nu_j} = \lambda_j \frac{1 - 2\nu_j}{1 - \nu_j} \quad (2.10)\]

After solving the set of equations of motion one obtains functions \(f_j(z), g_j(z)\). A form of the functions in the case of elastic layer depends on quantitative relationships between geometrical and material parameters appearing in Eqs (2.9). The problem mentioned has been discussed by Levinson (1985) for elastic, isotropic plate thus it is not discussed here. In the case of viscoelastic layer both the parameters \(\mu_j\), \(\lambda_j'\), \(\nu_j\) and the functions \(f_j(z), g_j(z)\) are complex. Taking into
account the correspondence principle one can write the functions for isotropic layer in the following form

\[
f_j(z) = X_{1j} \cosh(z \beta_{1j}) + X_{2j} \sinh(z \beta_{1j}) + X_{3j} \cosh(z \beta_{2j}) + X_{4j} \sinh(z \beta_{2j})
\]

\[
g_j(z) = X'_{1j} \cosh(z \beta_{1j}) + X'_{2j} \sinh(z \beta_{1j}) + X'_{3j} \cosh(z \beta_{2j}) + X'_{4j} \sinh(z \beta_{2j})
\]

(2.11)

where

\[
\beta_{1j}^2 = \alpha_m^2 - \frac{\rho_j \omega_m^2}{\mu_j}
\]

\[
\beta_{2j}^2 = \alpha_m^2 - \frac{\rho_j \omega_m^2}{\lambda_j + 2\mu_j}
\]

(2.12)

The vector \(X'_j\) is dependent on vector \(X_j\) thus there are only five unknown values in Eqs (2.11) i.e., the vector \(X_j\) and the natural frequency \(\omega_m\). For any layer denoted by subscript \(n \neq j\) we have another unknown vector \(X_n\). Thus for \(m\)th vibration mode of a beam consisting of \(p\) layers one obtains \(4p + 1\) unknown parameters while one of them is the eigenfrequency \(\omega_m\).

After substitution of the functions (2.11) into expressions (2.1) one obtains the displacement field within the \(j\)th layer and by using the displacement field functions and constitutive equations one can derive the stress field.

Let us introduce the homogeneous stress boundary conditions on the free surfaces of beam (2.13), the continuity conditions of displacements (2.14) and the continuity conditions of forces (instead of stresses) (2.15), (2.16) between adjoining layers

\[
\left(\sigma_{xx}(x, 0)\right)_1 = \left(\sigma_{xx}(x, h)\right)_p = 0 \quad \left(\sigma_{xx}(x, 0)\right)_1 = \left(\sigma_{xx}(x, h)\right)_p = 0
\]

\[
\left(u_x(x, h_j)\right)_j = \left(u_x(x, h_j)\right)_{j+1}
\]

(2.13)

\[
\left(u_x(x, h_j)\right)_j = \left(u_x(x, h_j)\right)_{j+1}
\]

(2.14)

\[
\int_{-b_j}^{+b_j+1} \left(\sigma_{xx}(x, h_j)\right)_j dz = \int_{-b_j+1}^{+b_j+1} \left(\sigma_{xx}(x, h_j)\right)_{j+1} dz
\]

(2.15)

\[
\int_{-b_j}^{+b_j} \left(\sigma_{xx}(x, h_j)\right)_j dz = \int_{-b_j+1}^{+b_j+1} \left(\sigma_{xx}(x, h_j)\right)_{j+1} dz
\]

(2.16)

where \(h = \sum_i^p h_i\), while \(h_j, b_j\) denote the thickness and the half of \(j\)th layer width, respectively. It is noted that displacement \(u_y\) does not appear in the continuity conditions (2.14) and the equations of continuity of forces (instead of stresses) (2.15), (2.16) are not exact in respect of requirements stated in the theory of elasticity. However the simplifications introduced here have enabled the author to solve correctly the problem considered in the present paper.
Taking into account the relationships (2.6) and the Eqs (2.11) \(\div\) (2.16) one can transform the eigenvalue problem to the form of algebraic, homogeneous, matrix equation
\[
AX = 0
\]  
(2.17)
The square matrix \(A\) is of size \(4p \times 4p\) where \(p\) denotes the number of layers of the beam. An \(\omega_m\) is obtained from the equation
\[
det A = 0
\]  
(2.18)
If any layer of the beam is viscoelastic then \(\omega_m\) consists of real and imaginary parts, respectively
\[
\omega_m = \omega_m^R + i\omega_m^F \quad i^2 = -1
\]  
(2.19)
and the periodic logarithmic decrement is defined as follows
\[
\delta_T = 2\pi \frac{\omega_m^F}{\omega_m^R}
\]  
(2.20)
After calculating \(\delta_T\) one can obtain both the loss factor \(\eta\) and the damping capacity \(\Psi\) according to the formulas given by Karczmarzyk (1989)
\[
\eta = \frac{\delta_T}{\pi} \quad \Psi = 1 - \exp(-2\delta_T) \cong 2\delta_T
\]  
(2.21)

The Eq (2.17) can easily be obtained for a beam consisting of any number of stripes using a computer. Computation of eigenfrequencies is however somewhat more difficult. The matrix elements depend on hyperbolic and trigonometric functions of eigenfrequency i.e.
\[
A_{kl} = A_{kl} \left( \sin(\omega_m, ...), \cos(\omega_m, ...), \sinh(\omega_m, ...), \cosh(\omega_m, ...), ... \right)
\]  
(2.22)
and because of this, Eq (2.18) is the transcendental one. Therefore it cannot be transformed to the following form
\[
det \left( B_1 - \omega_m B_2^2 \right) = 0
\]  
(2.23)
where \(B_1, B_2\) are given matrices.

To obtain the solution to eigenproblem (2.18) the following procedure has been proposed (cf Karczmarzyk, 1989, 1991; Karczmarzyk and Osiński, 1990) – at first a function \(F(\omega_m) = \det A\) is derived, then an eigenfrequency of undamped (elastic) beam is estimated, finally a complex eigenfrequency of the damped (viscoelastic) system is computed from the equation \(F(\omega_m) = 0\). All steps of this procedure were followed by the author by means of IBM personal computer. The third step only was taken by using standard subroutine for evaluating roots of an algebraic,
nonlinear, complex equation according to the Muller method with deflation. The
eigenfrequency of undamped system was useful as an approximative value of the
complex solution (eigenfrequency) i.e., as one of input parameters required by the
standard subroutine. It was verified that the Fortran code necessary for the calcu-
lations has to be prepared in double precision. The formulation of the eigenvalue
problem and the way of solution to it given in the present section are acknowl-
ledged by the author as a new exact method of vibration analysis of the layered
viscoelastic beams. In section 4 an alternative simplified method is presented.

We notice finally that formulation of the problem considered here will be of
the same form when the plane strain within layered structure is assumed. In such
a case instead of the parameter $\lambda'$ the Lame constant $\lambda$ should be placed in Eqs
(2.9), (2.12) (Karczmarzyk, Osiński, 1992).

3. Formulation of the problem in the case of fibrous layers

It has been assumed in further considerations that fibres of each layer are
parallel to the longitudinal axis of the beam. Such arrangement of the fibres is
most desirable considering bending stiffness of beam. Let us assume additionally
that material properties of any layer are isotropic within each cross-section of
the layer. If the two assumptions are fulfilled we will have so-called hexagonally
anisotropic layer.

The constitutive equation of fibrous, hexagonally anisotropic, viscoelastic ma-
terial can be written in the form

$$\sigma_j = D_j \varepsilon_j$$

(3.1)

where $\sigma_j$ denotes a stress vector, $\varepsilon_j$ is a strain vector and $D_j$ is a stiffness matrix
of $j$th layer. In the case when the plane stress is considered the matrices are as
follows

$$\sigma_j \equiv \{\sigma_{xx}, \sigma_{zz}, \sigma_{xz}\}_j$$

$$D_j \equiv \begin{bmatrix} b_j & a_j & 0 \\ a_j & q_j & 0 \\ 0 & 0 & 2\mu_j' \end{bmatrix}$$

(3.2)

$$\varepsilon_j \equiv \{\varepsilon_{xx}, \varepsilon_{zz}, \varepsilon_{xz}\}_j$$

where

$$a_j = \frac{E_j' \nu_j'}{w_j} \quad b_j = \frac{E_j' E_j'}{w_j E_j} \quad q_j = \frac{E_j'}{w_j}$$

(3.3)
\[ w_j = \frac{E'_j}{E_j} - \nu'_j \nu'_j \]  

(3.4)

while \( \mu'_j \) denotes the complex Kirchhoff modulus of \( j \)th fibrous viscoelastic layer in \( x-z \) plane, \( E'_j \) is the Young modulus for direction of fibers, \( E_j \) is the Young modulus within \( y-z \) plane perpendicular to direction of fibers (i.e., plane of isotropy), \( \nu'_j \) denotes the Poisson ratio characterizing an abridgement within the plane of isotropy when direction of force fits in with direction of fibers of viscoelastic hexagonally anisotropic material (cf Ambaryumyan, 1987).

By using the formulas (3.1) ÷ (3.4) and the expressions (2.1), (2.2) one can transform the linear elasticity equations of motion

\[ \sigma_{kl,k} = \rho \frac{\partial^2 u_l}{\partial t^2} \]  

(3.5)

to the following form

\[ -\mu'_j \frac{d^2 g_j}{dz^2} + (b_j \alpha^2_m - \rho_j \omega^2_m) g_j + (a_j + \mu'_j) \frac{df_j}{dz} = 0 \] 

(3.6)

\[ q_j \frac{d^2 f_j}{dz^2} - (\mu'_j \alpha^2_m - \rho_j \omega^2_m) f_j + \alpha^2_m (a_j + \mu'_j) \frac{dg_j}{dz} = 0 \]

where \( \alpha_m = m \pi / L \) is defined as in the previous section. It can be noticed that the equations of motion (3.6) are of the same type as the equations (2.9) thus formulation of the boundary value problem in this case is of the same form as the previous one described in section 2. Therefore further considerations concerning the formulation of the aforementioned for a beam composed of anisotropic layers have not been conducted here. We notice that formulation of the problem considered here will be of the same form when the plane strain is assumed within layered structure. In such a case, however, the elements \( a_j, b_j, q_j \) of the matrix \( D \), occurring in Eqs (3.6), are defined by expressions different from Eq (3.3), (3.4) (cf Karczmarzyk, 1989).

4. An alternative simplified method for analysis of free vibrations of viscoelastic layered beams

The method developed in sections 2,3 is accurate since it is derived within the linear theory of (visco)elasticity. The only simplification is introduced in the stress continuity conditions (2.15), (2.16). (In fact instead of the stress continuity it is considered the continuity of forces.) Thus the method reported in sections
2.3 is accurate and quite versatile i.e., it enables us to investigate the layered beams assuming both stiffness-comparable and incomparable adjoining layers and complete material (viscoelastic) characteristics of all layers. However due to the exactitude of the problem formulation one can not obtain a simple formula for evaluation of an eigenfrequency and a logarithmic decrement for layered structures considered here. In this section we present an alternative simplified method of calculating both the eigenfrequency and the logarithmic decrement of T-beam shown in Figure 1 and composed of stiffness-comparable viscoelastic layers.

Upon a basis of formulas given by Polensek and Kazic (1991) one can derive the following relationship

\[ \tilde{\omega} = \frac{h_2}{\xi_t} \frac{1 - \xi_t \xi_b \xi_E}{2} \frac{1 + \xi_t \xi_b \xi_E}{1 + \xi_t \xi_b \xi_E} \]  

(4.1)

where

\[ \xi_t = \frac{h_1}{h_2}, \quad \xi_b = \frac{b_1}{b_2}, \quad \xi_E = \frac{E_1}{E_2} \]  

(4.2)

and \( E_1, E_2 \) are the values of static Young moduli of isotropic layers. The parameter \( \tilde{\omega} \) denotes coordinate of the cross-section neutral axis (point \( C \)) in reference to the interface i.e., the surface of joint of the layers. When the \( \tilde{\omega} \) is greater than zero the point \( C \) lies within the layer 2 however in the opposite case the point \( C \) lies within the layer 1. The relationship (4.1) is a direct extension of the one given by Baumgarten and Pearce (1971) for a two-layer beam of rectangular cross-section. Few other relationships given by the researchers have just been employed by the author to obtain simple formulas for evaluating both the eigenfrequencies and the logarithmic decrement for the two-layer T-beams.

First of all we notice that the formula (25) derived by Baumgarten and Pearce (1971) can be replaced by the following one

\[ \delta_T = \pi \eta_{E_1} \frac{(V_{bc})_{\text{max}}}{T_{\text{max}}} \]  

(4.3)

where \((V_{bc})_{\text{max}}\) is the maximum value of the potential energy of the viscoelastic layer and \(T_{\text{max}}\) denotes the maximum value of the kinetic energy of the two-layer beam. The symbol \( \eta_{E_1} \) is called material loss factor and it is defined as follows

\[ \eta_{E_1} = E_{12}, \quad E_{11} + iE_{12} = E_1 \] \[ i^2 = -1 \]  

(4.4)

while \( E_1 \) in this case is the complex Young modulus of viscoelastic material. By replacing the \( E_{11}, E_{12}, E_1 \) with \( \nu_{11}, \nu_{12}, \nu_1 \), respectively, one obtains the complex Poisson ratio \( \nu_1 \) for the viscoelastic material. Parameters \( E_1, \nu_1 \) are dependent on frequency. In the case of non-slender layered beams one has to take into account both the characteristics to calculate accurately damping parameters (cf Karczmarzyk, Osiński, 1992). When the layered beam is slender it is accurate
enough to include in a computational algorithm the complex Young modulus, thus the loss factor \( \eta_{E1} \).

The formula (4.3) is valid for the two-layer beam when the Kirchhoff hypothesis of flat cross-sections is valid for the beam. It is well known that the hypothesis is satisfied for the first mode of vibration of slender beams (cf Huang, 1961). Thus for such beams we can extend the formula (4.3) in order to include different widths and viscoelasticity of adjoining layers. So one can calculate the logarithmic decrement according to the formula

\[
\delta_T = \pi \frac{\eta_{E1} V_1 + \eta_{E2} V_2}{T_{12}} \tag{4.5}
\]

where

\[
V_j = E_j b_j \frac{\alpha_m^4}{6} |x_j^3 - \bar{x}^3| \quad j = 1, 2 \tag{4.6}
\]

\[
T_{12} = (b_1 h_1 \rho_1 + b_2 h_2 \rho_2) \frac{\omega_m}{2} \tag{4.7}
\]

and \( \alpha_m \) is defined (for a simply supported beam) in section 2, \( E_j \) denotes the real part of the complex Young modulus of \( j \)th viscoelastic layer. By applying the Rayleigh method one obtains finally

\[
\delta_T = \pi \frac{\eta_{E1} V_1 + \eta_{E2} V_2}{V_1 + V_2} \tag{4.8}
\]

The expression (4.8) can be further simplified under additional assumptions i.e.

\[
\text{for} \quad V_1 = V_2 \quad \text{the} \quad \delta_T = \pi \frac{\eta_{E1} + \eta_{E2}}{2} \tag{4.9}
\]

\[
\text{for} \quad \eta_{E1} = \eta_{E2} = \eta_E \quad \text{the} \quad \delta_T = \pi \eta_E \tag{4.10}
\]

We notice that under assumptions mentioned above the viscoelastic damping of two-layer T-beam vibrations does not depend on geometrical and material parameters of the cross-section appearing in the formulas (4.1), (4.2). By comparing the right-hand side of Eq (4.10) and the left-hand side relationship (2.21) one can say: when both layers are characterized by the same material loss factor (i.e., \( \eta_{E1} = \eta_{E2} = \eta_E \)) the loss factor of the two-layer T-beam is equal to the material loss factor. The conclusion also refers to the beam of rectangular cross-section.

5. Numerical results and discussion

In order to verify and compare the methods developed for calculating of both the eigenfrequencies and the logarithmic decrement the author has made calculations for three types of beams. Material and geometrical (cross-sectional) parameters of the beams have been given in Table 1 while computational results
have been shown in Tables 2-6. We note that the beam 1 is homogeneous i.e., the material parameters of both layers are the same. However to verify an influence of the viscoelasticity of the thin layer (flange) on the logarithmic decrement the calculations have been made assuming for the thick layer (web) at first $\eta_{E2} = 0$ and then $\eta_{E2} = 0.1$. The beams 2,3 consist of two stiffness-comparable layers of different materials. Values of the logarithmic decrement in Table 3 and those in Table 5 with subscripts "12" have been given to prove the right-hand side relationship (4.10) and to show that $\delta_T$ is independent of the length of beam when the left-hand side Eq (4.10) is assumed. Parameter $\varepsilon_2$ is defined as follows

$$\varepsilon_2 = \frac{(\delta_{TA})_2 - (\delta_{TB})_2}{(\delta_{TB})_2} \times 100$$

(5.1)

Table 1. Material and geometrical (cross-sectional) parameters of the beams investigated in the paper. Symbols $b_j$, $h_j$ have been shown in Fig.1, while symbols $E_{j1}$, $\nu_j$, $\rho_j$ denote the Young modulus, the Poisson ratio and the mass density, respectively – for $j = 1, 2$.

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<th></th>
<th>$2b_1$</th>
<th>$2b_2$</th>
<th>$h_1$</th>
<th>$h_2$</th>
<th>$E_{11}$</th>
<th>$E_{21}$</th>
<th>$\nu_1$</th>
<th>$\nu_2$</th>
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<td>35</td>
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<td>35</td>
<td>105</td>
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<td>1.6×10$^{10}$</td>
<td>0.25</td>
<td>0.30</td>
<td>7860</td>
<td>1750</td>
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<tr>
<td>beam 3</td>
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<td>35</td>
<td>10</td>
<td>140</td>
<td>207×10$^9$</td>
<td>1.6×10$^{10}$</td>
<td>0.25</td>
<td>0.30</td>
<td>7860</td>
<td>1750</td>
</tr>
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Table 2. Eigenfrequencies and logarithmic decrements for the 1st mode of vibration of beam 1. Subscript $A$ denotes values calculated according to the method developed in section 2, while subscript $B$ denotes values obtained according to the method developed in section 4. Values with subscript 1 are obtained for $\eta_{E1} = 0.1$, $\eta_{E2} = 0$ while those with subscripts 12 are calculated for $\eta_{E1} = \eta_{E2} = 0.1$.

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<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\omega_A$ [rad/s]</td>
<td>2349.3</td>
<td>1066.1</td>
<td>604.4</td>
<td>388.2</td>
<td>270.2</td>
<td>182.9</td>
</tr>
<tr>
<td>$\omega_B$ [rad/s]</td>
<td>2442.6</td>
<td>1085.6</td>
<td>610.7</td>
<td>390.8</td>
<td>271.4</td>
<td>183.3</td>
</tr>
<tr>
<td>$(\delta_{TA})_1$</td>
<td>0.0757</td>
<td>0.0769</td>
<td>0.0774</td>
<td>0.0777</td>
<td>0.0778</td>
<td>0.0779</td>
</tr>
<tr>
<td>$(\delta_{TB})_1$</td>
<td>0.0785</td>
<td>0.0785</td>
<td>0.0785</td>
<td>0.0785</td>
<td>0.0785</td>
<td>0.0785</td>
</tr>
<tr>
<td>$(\delta_{TA})_{12}$</td>
<td>0.31288</td>
<td>0.31288</td>
<td>0.31288</td>
<td>0.31288</td>
<td>0.31288</td>
<td>0.31288</td>
</tr>
<tr>
<td>$(\delta_{TB})_{12}$</td>
<td>0.31414</td>
<td>0.31414</td>
<td>0.31414</td>
<td>0.31414</td>
<td>0.31414</td>
<td>0.31414</td>
</tr>
</tbody>
</table>
Table 3. Eigenfrequencies and logarithmic decrements for the 1st mode of vibration of beam 2. Subscript $A$ denotes values calculated according to the method developed in section 2, while subscript $B$ denotes values obtained according to the method developed in section 4. Values with subscript 12 are calculated for $\eta_{E1} = \eta_{E2} = 0.1$.

<table>
<thead>
<tr>
<th>$L$ [mm]</th>
<th>1000</th>
<th>1500</th>
<th>2000</th>
<th>2500</th>
<th>3000</th>
<th>3650</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_A$ [rad/s]</td>
<td>1000.83</td>
<td>456.25</td>
<td>259.10</td>
<td>166.53</td>
<td>115.93</td>
<td>78.46</td>
</tr>
<tr>
<td>$\omega_B$ [rad/s]</td>
<td>1049.22</td>
<td>466.31</td>
<td>262.30</td>
<td>167.87</td>
<td>116.58</td>
<td>78.76</td>
</tr>
<tr>
<td>$(\delta_{TA})_{12}$</td>
<td>0.31338</td>
<td>0.31338</td>
<td>0.31338</td>
<td>0.31338</td>
<td>0.31338</td>
<td>0.31338</td>
</tr>
<tr>
<td>$(\delta_{TB})_{12}$</td>
<td>0.31416</td>
<td>0.31416</td>
<td>0.31416</td>
<td>0.31416</td>
<td>0.31416</td>
<td>0.31416</td>
</tr>
</tbody>
</table>

Table 4. Eigenfrequencies and logarithmic decrements for the 1st mode of vibration of beam 3. Subscript $A$ denotes values calculated according to the method developed in section 2, while subscript $B$ denotes values obtained according to the method developed in section 4. Decrement values with subscript 2 are calculated for $\eta_{E2} = 0.1$, $\eta_{E1} = 0$.

<table>
<thead>
<tr>
<th>$L$ [mm]</th>
<th>1000</th>
<th>1500</th>
<th>2000</th>
<th>2500</th>
<th>3000</th>
<th>3650</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_A$ [rad/s]</td>
<td>1492.1</td>
<td>693.0</td>
<td>396.3</td>
<td>255.7</td>
<td>178.3</td>
<td>120.8</td>
</tr>
<tr>
<td>$\omega_B$ [rad/s]</td>
<td>1621.0</td>
<td>720.5</td>
<td>405.3</td>
<td>259.4</td>
<td>180.1</td>
<td>121.7</td>
</tr>
<tr>
<td>$(\delta_{TA})_{2}$</td>
<td>0.26212</td>
<td>0.25730</td>
<td>0.25534</td>
<td>0.25438</td>
<td>0.25384</td>
<td>0.25344</td>
</tr>
<tr>
<td>$(\delta_{TB})_{2}$</td>
<td>0.25345</td>
<td>0.25345</td>
<td>0.25345</td>
<td>0.25345</td>
<td>0.25345</td>
<td>0.25345</td>
</tr>
<tr>
<td>$\varepsilon_2$</td>
<td>3.42</td>
<td>1.52</td>
<td>0.75</td>
<td>0.37</td>
<td>0.15</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 5. Eigenfrequencies and logarithmic decrements for the 3rd mode of vibration of beam 3. Subscript $A$ denotes values calculated according to the method developed in section 2, while subscript $B$ denotes values obtained according to the method developed in section 4. Values with subscript 2 are obtained for $\eta_{E1} = 0$, $\eta_{E2} = 0.1$ while those with subscripts 12 are calculated for $\eta_{E1} = \eta_{E2} = 0.1$.

<table>
<thead>
<tr>
<th>$L$ [mm]</th>
<th>1000</th>
<th>1500</th>
<th>2000</th>
<th>2500</th>
<th>3000</th>
<th>3650</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_A$ [rad/s]</td>
<td>9155.7</td>
<td>4974.2</td>
<td>3083.2</td>
<td>2081.1</td>
<td>1492.1</td>
<td>1033.6</td>
</tr>
<tr>
<td>$\omega_B$ [rad/s]</td>
<td>14589.3</td>
<td>6484.1</td>
<td>3647.3</td>
<td>2334.3</td>
<td>1621.0</td>
<td>1095.0</td>
</tr>
<tr>
<td>$(\delta_{TA})_{2}$</td>
<td>0.28813</td>
<td>0.27746</td>
<td>0.27014</td>
<td>0.26530</td>
<td>0.26212</td>
<td>0.25943</td>
</tr>
<tr>
<td>$(\delta_{TB})_{2}$</td>
<td>0.25345</td>
<td>0.25345</td>
<td>0.25345</td>
<td>0.25345</td>
<td>0.25345</td>
<td>0.25345</td>
</tr>
<tr>
<td>$(\delta_{TA})_{12}$</td>
<td>0.31338</td>
<td>0.31338</td>
<td>0.31338</td>
<td>0.31338</td>
<td>0.31338</td>
<td>0.31338</td>
</tr>
<tr>
<td>$(\delta_{TB})_{12}$</td>
<td>0.31416</td>
<td>0.31416</td>
<td>0.31416</td>
<td>0.31416</td>
<td>0.31416</td>
<td>0.31416</td>
</tr>
<tr>
<td>$\varepsilon_2$</td>
<td>13.68</td>
<td>9.47</td>
<td>6.59</td>
<td>4.68</td>
<td>3.42</td>
<td>2.36</td>
</tr>
</tbody>
</table>
Table 6. Eigenfrequencies and logarithmic decrements for the 5th mode of vibration of beam 3. Subscript $A$ denotes values calculated according to the method developed in section 2, while subscript $B$ denotes values obtained according to the method developed in section 4. Values with subscript 2 are calculated for $\eta_{E1} = 0$, $\eta_{E2} = 0.1$.

<table>
<thead>
<tr>
<th>$L$ [mm]</th>
<th>1000</th>
<th>1500</th>
<th>2000</th>
<th>2500</th>
<th>3000</th>
<th>3650</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_A$ [rad/s]</td>
<td>17656.8</td>
<td>10590.9</td>
<td>7026.9</td>
<td>4974.2</td>
<td>3688.3</td>
<td>2633.2</td>
</tr>
<tr>
<td>$\omega_B$ [rad/s]</td>
<td>40525.9</td>
<td>18011.5</td>
<td>10131.5</td>
<td>6484.1</td>
<td>4502.9</td>
<td>3042.0</td>
</tr>
<tr>
<td>$(\delta T_A)_2$</td>
<td>0.29584</td>
<td>0.29052</td>
<td>0.28349</td>
<td>0.27746</td>
<td>0.27268</td>
<td>0.26804</td>
</tr>
<tr>
<td>$(\delta T_B)_2$</td>
<td>0.25345</td>
<td>0.25345</td>
<td>0.25345</td>
<td>0.25345</td>
<td>0.25345</td>
<td>0.25345</td>
</tr>
<tr>
<td>$\varepsilon_2$</td>
<td>16.73</td>
<td>14.63</td>
<td>11.85</td>
<td>9.47</td>
<td>7.59</td>
<td>5.76</td>
</tr>
</tbody>
</table>

We notice that the higher is mode of vibration the higher value of the parameter $\varepsilon_2$ occurs. This parameter is strongly dependent on the length of beam. Generally when the web of the two-layer beam is viscoelastic the values of logarithmic decrement calculated according to the method developed in section 2 are higher than those obtained according to the method given in section 4. It is because of the shear deformations omitting in the method described in section 4 while the deformations have been included in the method shown in sections 2 and 3. The difference between the methods is much more perceptible when the eigenfrequencies are compared. For each mode of the beams vibration the eigenfrequencies $\omega_B$ are different and higher than the eigenfrequencies $\omega_A$. The difference is dependent on the mode of vibration and the slenderness of beams. Of course the values $\omega_B$ are not exact for the 1st mode of vibration and false for the higher vibration modes. The latter conclusion, based upon the results given in Tables 2 ÷ 6, fits in well with that given by Huang (1961).

The method given in section 4 can be useful inspite of its general inaccuracy since it is far less complicated than that given in section 2 and the formulas (4.1) ÷ (4.10) are simple and quite accurate when the 1st mode of vibration is investigated.

6. Final conclusion

Two methods for calculating of both the eigenfrequencies and the logarithmic decrement for two-layer T-beam composed of isotropic or fibrous stiffness-comparable layers have been developed in the paper. The method given in section 2 has been obtained within the linear theory of (visco)elasticity, however that one from section 4 is derived by applying the Kirchhoff hypothesis of flat cross-sections and the Rayleigh method. By comparing results obtained according to these methods one can conclude: 1) damping of the 1st mode of vibration of the two-layer
T-beam resulting from the viscoelasticity of layers is accurately predicted by both the methods, 2) both the logarithmic decrement and (especially) the eigenfrequencies of higher modes of vibrations are predicted accurately only by the method derived within the linear theory of (visco)elasticity, 3) validity of the right-hand side expression (4.10), for any mode of vibration, has been confirmed by applying both the methods.

References

5. Huang T.C., 1961, The effect of rotatory inertia and of shear deformation on the frequency and normal mode equations of uniform beams with simple end conditions, Journal of Applied Mechanics, December, 579-584
Drgania swobodne bełk warstwowych o przekroju nie prostokątnym złożonym z warstw lepkosprężystych

Streszczenie

Praca dotyczy dwóch metod obliczania częstości własnych i logarytmicznego dekrementu tłumienia dla dwuwarstwowej belki T-owej złożonej z lepkosprężystych warstw o porównywalnej sztywności. Jedna z tych metod została opracowana w ramach liniowej teorii (lepkosprężystości przy założeni warunków ciągłości sił (zamiast naprężeń) między przylegającymi warstwami. Druga metoda jest oparta na założeniu Kirchhoffa i metodzie Rayleigh’a. Przedstawiono porównanie obu metod i ustalono proste i użyteczne zależności do obliczania dekrementu tłumienia.

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