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EVALUATION OF STRESSES AND REACTIONS IN RODS WITH PERIODIC – VARIABLE CROSS SECTIONS

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The paper is a continuation of the earlier contributions [2,3]. The microlocal modelling [5,6,7,8] of the problem of constrained torsion of rods with the periodic variable cross–sections and the notion of internal constraints [1,4] were applied. The aim of the paper is to evaluate the stresses and the reaction forces due to the internal modelling contraints for some special solutions.

1. Stresses and reaction forces

In the paper [3] the microlocal modelling [5,6,7,8] of the problem of constrained torsion of straight linear–elastic axial symmetric rod has been used. The radius of the cross–section is ε – periodic and given by the formula:

$$R(X_3) = R_0\Big(1 + \delta\cos\frac{2\pi X_3}{\varepsilon}\Big), \tag{1.1}$$

where $R_0 = \text{const}$, $\delta = \text{const}$, $\varepsilon \ll l$ (l – lenght of the rod).

Using the microlocal approximation we are looking for the approximate solution given by (1.7) in [3], assuming the shape functions $h^a(\cdot)$ in the form:

$$h^a(X_3) = \frac{\varepsilon}{l}\sin\frac{a2\pi X_3}{\varepsilon}, \tag{1.2}$$

where $a = 1, 2, ..., n$.

For the rod loaded like in fig.1, assuming $m = \text{const}$, $q = \text{const}$, we have obtained in [3] the generalized coordinates $\Theta(X_3)$, $\xi(X_3)$, $\eta(X_3)$, $\Big(\psi(X_3) \equiv 0,$ $\varphi(X_3) \equiv 0\Big)$, which describe the displacement vector (cf. (1.1) in [3], (1.2) in [3]) by means of

$$\begin{aligned} u_1 &= -\Theta(X_3)X_2, \\ u_2 &= \Theta(X_3)X_1, \\ u_3 &= \Phi(X_1, X_2)\zeta(X_3) + \eta(X_3), \end{aligned} \tag{1.3}$$

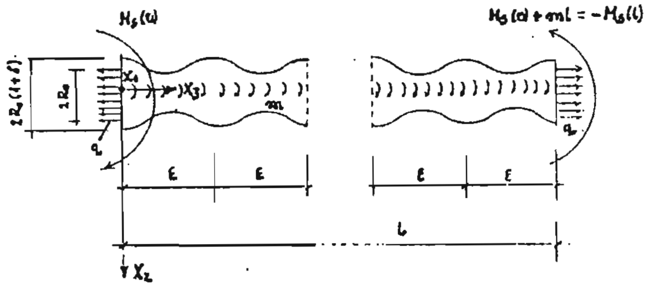

Fig. 1.

where $\Phi(X_1, X_2) = X_1^2 + X_2^2$.

Introducing the following constant characteristics of the mean cross–section of the rod

$$
\begin{aligned}
\bar{F} &= \pi R_0^2, \\
\bar{J}_0 &= \frac{\pi R_0^4}{2}, \\
\bar{J}_k &= 4\bar{J}_0, \\
\bar{J} &= \frac{\pi R_0^6}{3},
\end{aligned}
\tag{1.4}
$$

in accordance with (1.12) in [3] and (2.10) in [3] for $n = 1$ and $\delta = 0.1$ we have ((2.12) in [3])

$$
\begin{aligned}
&\Theta \sim \Theta_0 = -\frac{m}{\mu J_{01}^{\text{eff}}}\frac{X_3^2}{2} - \frac{M_s(0)}{\mu J_{01}^{\text{eff}}}X_3 + C, \\
&\Theta_{,3} \sim \Theta_{0,3} + \Theta_1 h^1{}_{,3} = \Big[-\frac{mX_3}{\mu J_{01}^{\text{eff}}} - \frac{M_s(0)}{\mu J_{01}^{\text{eff}}}\Big]\Big(1 - 0.38562\cos\frac{2\pi X_3}{\varepsilon}\Big), \\
&\zeta \sim \zeta_0 = \frac{q\bar{J}_k}{(\lambda+2\mu)J_1^{\text{eff}}}0.04855\frac{1}{\gamma_1}\frac{\operatorname{sh}\gamma_1(X_3 - \frac{l}{2})}{\operatorname{ch}\frac{\gamma_1 l}{2}}, \\
&\zeta_{,3} \sim \zeta_{0,3} + \zeta_1 h^1{}_{,3} = \frac{q\bar{J}_k}{(\lambda+2\mu)J_1^{\text{eff}}}\Big[\frac{\operatorname{ch}\gamma_1(X_3 - \frac{l}{2})}{\operatorname{ch}\frac{\gamma_1 l}{2}}\cdot \\
&\qquad \cdot\Big(0.04855 - 0.02618\cos\frac{2\pi X_3}{\varepsilon}\Big) - 0.15786\cos\frac{2\pi X_3}{\varepsilon}\Big], \\
&\eta \sim \eta_0 = \frac{q\bar{F}}{(\lambda+2\mu)F_1^{\text{eff}}}\Big[-0.14285\frac{1}{\gamma_1}\frac{\operatorname{sh}\gamma_1(X_3 - \frac{l}{2})}{\operatorname{ch}\frac{\gamma_1 l}{2}} + X_3\Big] + D, \\
&\eta_{,3} \sim \eta_{0,3} + \eta_1 h^1{}_{,3} = \frac{q\bar{F}}{(\lambda+2\mu)F_1^{\text{eff}}}\Big[\frac{\operatorname{ch}\gamma_1(X_3 - \frac{l}{2})}{\operatorname{ch}\frac{\gamma_1 l}{2}}\cdot
\end{aligned}
\tag{1.5}
$$

$$\cdot\Big(-0.14285 + 0.05227\cos\frac{2\pi X_3}{\varepsilon}\Big) + 1 + 0.30786\cos\frac{2\pi X_3}{\varepsilon}\Big],$$

where C, D are some arbitrary constans (possibly equal to zero) and

$$\begin{aligned} J_{01}^{\mathrm{eff}} &= 0.95228\bar{J}_0, \\ J_1^{\mathrm{eff}} &= 0.90939\bar{J}, \\ F_1^{\mathrm{eff}} &= 0.93742\bar{F}, \\ \gamma_1^{\mathrm{eff}} &= \frac{5.21384}{R_0}\sqrt{\frac{\mu}{\lambda+2\mu}}. \end{aligned} \tag{1.6}$$

Analogously, for $n = 2$ and $\delta = 0.1$ we have (cf. (2.19) in [3]).

Hence

$$\begin{aligned}
\Theta &\sim \Theta_0 = -\frac{m}{\mu J_{02}^{\mathrm{eff}}}\frac{X_3^2}{2} - \frac{M_s(0)X_3}{\mu J_{02}^{\mathrm{eff}}} + C, \\
\Theta_{,3} &\sim \Theta_{0,3} + \Theta_1 h^1{}_{,3} + \Theta_2 h^2{}_{,3} = \Big[-\frac{mX_3}{\mu J_{02}^{\mathrm{eff}}} - \frac{M_s(0)}{\mu J_{02}^{\mathrm{eff}}}\Big]\cdot \\
&\cdot\ \Big(1 - 0.39496\cos\frac{2\pi X_3}{\varepsilon} + 0.04828\cos\frac{4\pi X_3}{\varepsilon}\Big), \\
\zeta &\sim \zeta_0 = \frac{q\bar{J}_k}{(\lambda+2\mu)J_2^{\mathrm{eff}}}0.05748\frac{1}{\gamma_2}\frac{\mathrm{sh}\gamma_2(X_3-\frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}}, \\
\zeta_{,3} &\sim \zeta_{0,3} + \zeta_1 h^1{}_{,3} + \zeta_2 h^2{}_{,3} = \frac{q\bar{J}_k}{(\lambda+2\mu)J_2^{\mathrm{eff}}}\cdot \\
&\cdot\ \Big[\frac{\mathrm{ch}\gamma_2(X_3-\frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}}\Big(0.05748 - 0.03326\cos\frac{2\pi X_3}{\varepsilon} + 0.00541\cdot \\
&\cdot\ \cos\frac{4\pi X_3}{\varepsilon}\Big) - 0.17992\cos\frac{2\pi X_3}{\varepsilon} + 0.04767\cos\frac{4\pi X_3}{\varepsilon}\Big], \\
\eta &\sim \eta_0 = \frac{q\bar{F}}{(\lambda+2\mu)F_2^{\mathrm{eff}}}\Big[\frac{-0.1682}{\gamma_2}\frac{\mathrm{sh}\gamma_2(X_3-\frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}} + X_3\Big] + D, \\
\eta_{,3} &\sim \eta_{0,3} + \eta_1 h^1{}_{,3} + \eta_2 h^2{}_{,3} = \frac{q\bar{F}}{(\lambda+2\mu)F_2^{\mathrm{eff}}}\cdot \\
&\cdot\ \Big[\frac{\mathrm{ch}\gamma_2(X_3-\frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}}\Big(-0.16820 + 0.06634\cos\frac{2\pi X_3}{\varepsilon} - 0.007552\cdot \\
&\cdot\ \cos\frac{4\pi X_3}{\varepsilon}\Big) + 1 + 0.36190\cos\frac{2\pi X_3}{\varepsilon} - 0.07712\cos\frac{4\pi X_3}{\varepsilon}\Big],
\end{aligned} \tag{1.7}$$

where

$$\begin{aligned} J_{02}^{\mathrm{eff}} &= 0.95118\bar{J}_0, \\ J_2^{\mathrm{eff}} &= 0.90120\bar{J}, \end{aligned}$$

$$
\begin{aligned}
F_2^{\mathrm{eff}} &= 0.92787\tilde{F}, \\
\gamma_2 &= \frac{5.23747}{R_0}\sqrt{\frac{\mu}{\lambda+2\mu}}.
\end{aligned}
\tag{1.8}
$$

We are considering the rod represented in fig.1 assuming the following data:

$$
\begin{aligned}
&M_s(0) = 0, \qquad m = \mathrm{const}, \qquad q = \mathrm{const}, \\
&R_0 = 5\ \mathrm{cm}, \\
&\delta = 0.1, \\
&l = 100\ \mathrm{cm}, \\
&\varepsilon = \frac{l}{25} = 4\ \mathrm{cm}, \\
&\mu = 78.846\ \mathrm{GPa}, \\
&\lambda = 118.269\ \mathrm{GPa},
\end{aligned}
\tag{1.9}
$$

where μ and λ are the Lame' modulae.

In figs. 2,5,6 the diagrams of functions $\Theta_{,3}$, $\zeta_{,3}$, $\eta_{,3}$, for $n = 1$ are presented – the influence of the microlocal parameters Θ_1, ζ_1, η_1 on the forementioned functions is evident. Comparing fig.2 and fig.3 which show $\Theta_{,3}$, for $n = 1$ and for $n = 2$, respectively, we can find that the microlocal parameter Θ_2 has a certain influence on the values of function $\Theta_{,3}$, for $n = 2$ but the oscillation created by Θ_2 can be neglected.

In fig.2 the diagram of function

$$
\frac{\Theta_{,3_{\mathrm{for}\ n=1}} - \Theta_{,3_{\mathrm{for}\ n=2}}}{\Theta_{,3_{\mathrm{for}\ n=1}}} \cdot 100\%,
$$

is shown; the corresponding values are oscillating between + 4.72 % and −6.47 %.

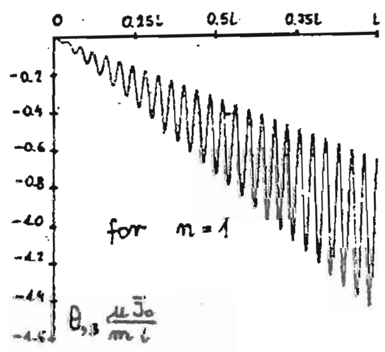

Fig. 2.

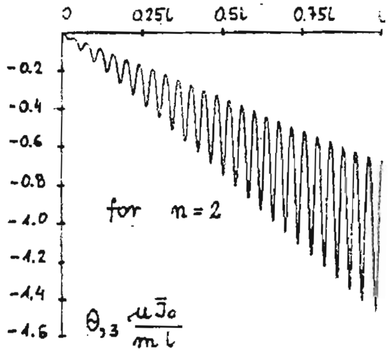

Fig. 3.

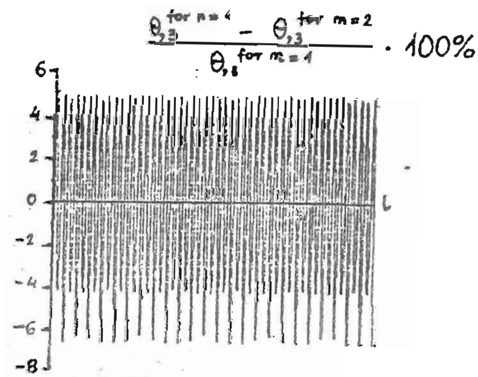

Fig. 4.

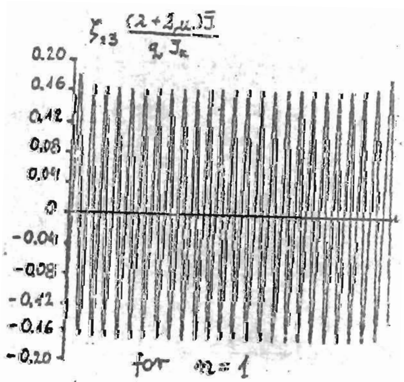

Fig. 5.

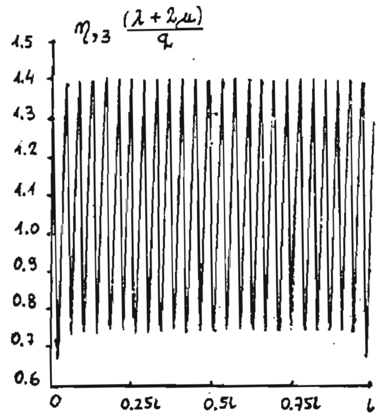

Fig. 6.

We will calculate below the stresses and the reaction forces due to the internal constraints for the rod loaded in the way shown in fig.1, assuming m = const, q = const, $\delta = 0.1$ and $n = 2$. Using (1.6) of [3] and formula (1.7) given above for homogeneous isotropic materials the following stress components can be obtained

$$
\begin{aligned}
T^{11} &= T^{22} = \lambda(\Phi\zeta_{,3} + \eta_{,3}) \sim \frac{q\lambda}{\lambda + 2\mu}\Big\{\frac{X_1^2 + X_2^2}{R_0^2}\Big[\frac{\mathrm{ch}\gamma_2(X_3 - \frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}} \cdot \\
&\cdot \Big(0.38269 - 0.22144\cos\frac{2\pi X_3}{\varepsilon} + 0.03602\cos\frac{4\pi X_3}{\varepsilon}\Big) + \\
&- 1.19787\cos\frac{2\pi X_3}{\varepsilon} + 0.31738\cos\frac{4\pi X_3}{\varepsilon}\Big] + \frac{\mathrm{ch}\gamma_2(X_3 - \frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}} \cdot \\
&\cdot \Big(-0.18128 + 0.07150\cos\frac{2\pi X_3}{\varepsilon} - 0.00814\cos\frac{4\pi X_3}{\varepsilon}\Big) + 1.07774 + \\
&+ 0.39003\cos\frac{2\pi X_3}{\varepsilon} - 0.08312\cos\frac{4\pi X_3}{\varepsilon}\Big\}, \\
T^{13} &= \mu(-\Theta_{,3}X_2 + \zeta\Phi_{,1}) \sim \Big[\frac{mX_3}{J_{02}^{\mathrm{eff}}} + \frac{M_s(0)}{J_{02}^{\mathrm{eff}}}\Big] \cdot \\
&\cdot \Big(1 - 0.39496\cos\frac{2\pi X_3}{\varepsilon} + 0.04828\cos\frac{4\pi X_3}{\varepsilon}\Big)X_2 + \\
&+ \frac{\mu q \bar{J}_k}{(\lambda + 2\mu)J_2^{\mathrm{eff}}} 0.11496 \frac{1}{\gamma_2}\frac{\mathrm{sh}\gamma_2(X_3 - \frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}} X_1, \\
T^{23} &= \mu(\Theta_{,3}X_1 + \zeta\Phi_{,2}) \sim \Big[-\frac{mX_3}{J_{02}^{\mathrm{eff}}} - \frac{M_s(0)}{J_{02}^{\mathrm{eff}}}\Big] \cdot \qquad (1.10) \\
&\cdot \Big(1 - 0.39496\cos\frac{2\pi X_3}{\varepsilon} + 0.04828\cos\frac{4\pi X_3}{\varepsilon}\Big)X_1 +
\end{aligned}
$$

$$
\begin{aligned}
&+\quad \frac{\mu q \bar{J}_k}{(\lambda+2\mu)J_2^{\mathrm{eff}}}0.11496\frac{1}{\gamma_2}\frac{\mathrm{sh}\gamma_2(X_3-\frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}}X_2,\\
T^{33} &= (\lambda+2\mu)(\Phi\zeta_{,3}+\eta_{,3}) \sim q\Big\{\frac{X_1^2+X_2^2}{R_0^2}\Big[\frac{\mathrm{ch}\gamma_2(X_3-\frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}}\cdot\\
&\cdot\quad \Big(0.38269-0.22144\cos\frac{2\pi X_3}{\varepsilon}+0.03602\cos\frac{4\pi X_3}{\varepsilon}\Big)+\\
&-\quad 1.19787\cos\frac{2\pi X_3}{\varepsilon}+0.31738\cos\frac{4\pi X_3}{\varepsilon}\Big]+\frac{\mathrm{ch}\gamma_2(X_3-\frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}}\cdot\\
&\cdot\quad \Big(-0.18128+0.07150\cos\frac{2\pi X_3}{\varepsilon}-0.00814\cos\frac{4\pi X_3}{\varepsilon}\Big)+\\
&+\quad 1.07774+0.39003\cos\frac{2\pi X_3}{\varepsilon}-0.08312\cos\frac{4\pi X_3}{\varepsilon}\Big\}.
\end{aligned}
$$

The theory of the constrained torsion of rods given above has been obtained on the basis of the constrained continuum mechanics [1]. Because on the motion of the rod the physical constraints have not been imposed we confined ourselves to the modelling constraints (1.3) only, the obtained reaction forces are exclusively due to the internal modelling constraints.

The second derivatives of the generalized coordinates are calculated analogously as in [3] (cf. (1.12) in [3]), assuming the components comprising factor $h^a(X_3)$ to be negligible:

$$
\begin{aligned}
\Theta_{,33} &\sim \Theta_{0,33}+2\Theta_{a,3}h^a{}_{,3}+\Theta_a h^a{}_{,33},\\
\zeta_{,33} &\sim \zeta_{0,33}+2\zeta_{a,3}h^a{}_{,3}+\zeta_a h^a{}_{,33},\\
\eta_{,33} &\sim \eta_{0,33}+2\eta_{a,3}h^a{}_{,3}+\eta_a h^a{}_{,33}.
\end{aligned}
\tag{1.11}
$$

After some calculations we have

$$
\begin{aligned}
\Theta_{,33} &\sim -\frac{m}{\mu J_{02}^{\mathrm{eff}}}\Big(1-0.78992\cos\frac{2\pi X_3}{\varepsilon}+0.09656\cos\frac{4\pi X_3}{\varepsilon}\Big)+\\
&+\quad \frac{\pi}{\varepsilon}\Big(-\frac{mX_3}{\mu J_{02}^{\mathrm{eff}}}-\frac{M_s(0)}{\mu J_{02}^{\mathrm{eff}}}\Big)\Big(0.78992\sin\frac{2\pi X_3}{\varepsilon}+\\
&-\quad 0.19312\sin\frac{4\pi X_3}{\varepsilon}\Big),\\
\zeta_{,33} &\sim \frac{q\bar{J}_k}{(\lambda+2\mu)J_2^{\mathrm{eff}}}\Big[\gamma_2\frac{\mathrm{sh}\gamma_2(X_3-\frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}}\Big(0.05748-0.06652\cdot\\
&\cdot\quad \cos\frac{2\pi X_3}{\varepsilon}+0.01082\cos\frac{4\pi X_3}{\varepsilon}\Big)+\frac{\pi}{\varepsilon}\frac{\mathrm{ch}\gamma_2(X_3-\frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}}\cdot\\
&\cdot\quad \Big(0.06652\sin\frac{2\pi X_3}{\varepsilon}-0.02165\sin\frac{4\pi X_3}{\varepsilon}\Big)+\frac{\pi}{\varepsilon}\cdot
\end{aligned}
\tag{1.12}
$$

$$\cdot \quad \left(0.35984 \sin \frac{2\pi X_3}{\varepsilon} - 0.19069 \sin \frac{4\pi X_3}{\varepsilon}\right)\Big],$$

$$\begin{aligned}
\eta_{,33} \quad &\sim \quad \frac{q\bar{F}}{(\lambda+2\mu)F_2^{\text{eff}}}\Big[\gamma_2 \frac{\operatorname{sh}\gamma_2(X_3-\frac{l}{2})}{\operatorname{ch}\frac{\gamma_2 l}{2}}\Big(-0.16820 + 0.13268 \cdot \\
&\cdot \quad \cos \frac{2\pi X_3}{\varepsilon} - 0.01511 \cos \frac{4\pi X_3}{\varepsilon}\Big) + \frac{\pi}{\varepsilon}\frac{\operatorname{ch}\gamma_2(X_3-\frac{l}{2})}{\operatorname{ch}\frac{\gamma_2 l}{2}} \cdot \\
&\cdot \quad \Big(-0.13268 \sin \frac{2\pi X_3}{\varepsilon} + 0.03021 \sin \frac{4\pi X_3}{\varepsilon}\Big) + \\
&+ \quad \frac{\pi}{\varepsilon}\Big(-0.72380 \sin \frac{2\pi X_3}{\varepsilon} + 0.30848 \sin \frac{4\pi X_3}{\varepsilon}\Big)\Big].
\end{aligned}$$

Putting (1.10) and (1.12) into formula (1.3) given in [3], we obtain the reaction forces due to the internal constrains.

$$\begin{aligned}
\rho r_1 \quad &= \quad -T^{11}{}_{,1} - T^{13}{}_{,3} = \mu\Theta_{,33} X_2 - 2(\lambda+\mu)\zeta_{,3} X_1 \sim \\
&\sim \quad \Big\{-\frac{m}{J_{02}^{\text{eff}}}\Big(1 - 0.78992 \cos \frac{2\pi X_3}{\varepsilon} + 0.09656 \cos \frac{4\pi X_3}{\varepsilon}\Big) + \\
&+ \quad \frac{\pi}{\varepsilon}\Big(-\frac{mX_3}{J_{02}^{\text{eff}}} - \frac{M_s(0)}{J_{02}^{\text{eff}}}\Big)\Big(0.78992 \sin \frac{2\pi X_3}{\varepsilon} + \\
&- \quad 0.19312 \sin \frac{4\pi X_3}{\varepsilon}\Big)\Big\} X_2 - \frac{q\bar{J}_k}{J_2^{\text{eff}}}\frac{(\lambda+\mu)}{(\lambda+2\mu)}\Big[\frac{\operatorname{ch}\gamma_2(X_3-\frac{l}{2})}{\operatorname{ch}\frac{\gamma_2 l}{2}} \cdot \\
&\cdot \quad \Big(0.11496 - 0.06652 \cos \frac{2\pi X_3}{\varepsilon} + 0.01082 \cos \frac{4\pi X_3}{\varepsilon}\Big) + \\
&- \quad 0.35984 \cos \frac{2\pi X_3}{\varepsilon} + 0.09534 \cos \frac{4\pi X_3}{\varepsilon}\Big] X_1, \\
\rho r_2 \quad &= \quad -T^{22}{}_{,2} - T^{23}{}_{,3} = -\mu\Theta_{,33} X_1 - 2(\lambda+\mu)\zeta_{,3} X_2 \sim \\
&\sim \quad \Big\{\frac{m}{J_{02}^{\text{eff}}}\Big(1 - 0.78992 \cos \frac{2\pi X_3}{\varepsilon} + 0.09656 \cos \frac{4\pi X_3}{\varepsilon}\Big) + \\
&+ \quad \frac{\pi}{\varepsilon}\Big(\frac{mX_3}{J_{02}^{\text{eff}}} + \frac{M_s(0)}{J_{02}^{\text{eff}}}\Big)\Big(0.78992 \sin \frac{2\pi X_3}{\varepsilon} + \\
&- \quad 0.19312 \sin \frac{4\pi X_3}{\varepsilon}\Big)\Big\} X_1 - \frac{q\bar{J}_k}{J_2^{\text{eff}}}\frac{(\lambda+\mu)}{(\lambda+2\mu)}\Big[\frac{\operatorname{ch}\gamma_2(X_3-\frac{l}{2})}{\operatorname{ch}\frac{\gamma_2 l}{2}} \cdot \\
&\cdot \quad \Big(0.11496 - 0.06652 \cos \frac{2\pi X_3}{\varepsilon} + 0.01082 \cos \frac{4\pi X_3}{\varepsilon}\Big) + \\
&- \quad 0.35984 \cos \frac{2\pi X_3}{\varepsilon} + 0.09534 \cos \frac{4\pi X_3}{\varepsilon}\Big] X_2, \qquad (1.13) \\
\rho r_3 \quad &= \quad -T^{31}{}_{,1} - T^{32}{}_{,2} - T^{33}{}_{,3} = -4\mu\zeta - (\lambda+2\mu)[\zeta_{,33}(X_1^2 + X_2^2) + \eta_{,33}] \sim \\
&\sim \quad -\frac{q\mu}{\lambda+2\mu}\frac{1.53076}{R_0^2\gamma_2}\frac{\operatorname{sh}\gamma_2(X_3-\frac{l}{2})}{\operatorname{ch}\frac{\gamma_2 l}{2}} - q\Big\{\Big[\gamma_2 \frac{\operatorname{sh}\gamma_2(X_3-\frac{l}{2})}{\operatorname{ch}\frac{\gamma_2 l}{2}} \cdot
\end{aligned}$$

$$\cdot \left(0.38269 - 0.44288\cos\frac{2\pi X_3}{\varepsilon} + 0.07204\cos\frac{4\pi X_3}{\varepsilon}\right) +$$
$$+ \frac{\pi}{\varepsilon}\frac{\mathrm{ch}\gamma_2(X_3-\frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}}\left(0.44288\sin\frac{2\pi X_3}{\varepsilon} - 0.14414\sin\frac{4\pi X_3}{\varepsilon}\right) +$$
$$+ \frac{\pi}{\varepsilon}\left(2.39574\sin\frac{2\pi X_3}{\varepsilon} - 1.26958\sin\frac{4\pi X_3}{\varepsilon}\right)\Big]\frac{X_1^2+X_2^2}{R_0^2} +$$
$$+ \gamma_2\frac{\mathrm{sh}\gamma_2(X_3-\frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}}\left(-0.18128 + 0.14299\cos\frac{2\pi X_3}{\varepsilon} +\right.$$
$$\left.- 0.01628\cos\frac{4\pi X_3}{\varepsilon}\right) + \frac{\pi}{\varepsilon}\frac{\mathrm{ch}\gamma_2(X_3-\frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}}\cdot$$
$$\cdot \left(-0.14299\sin\frac{2\pi X_3}{\varepsilon} + 0.03256\sin\frac{4\pi X_3}{\varepsilon}\right) +$$
$$+ \frac{\pi}{\varepsilon}\left(-0.78007\sin\frac{2\pi X_3}{\varepsilon} + 0.33246\sin\frac{4\pi X_3}{\varepsilon}\right)\Big\},$$

for $X_1, X_2 \in F(X_3)$, $X_3 \in (0,l)$.

The boundary reaction forces due to the internal constraints can be obtained after putting (1.10) into formula (1.4) that was given in [3]. On the lateral boundary surface of the rod (for $X_1, X_2 \in \partial F(X_3)$, $X_3 \in (0,l)$) we have

$$s_1 = T^{11}n_1 + T^{13}n_3 - p_1 \sim \frac{\lambda q}{\lambda+2\mu}\Big\{\frac{X_1^2+X_2^2}{R_0^2}\Big[\frac{\mathrm{ch}\gamma_2(X_3-\frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}}\cdot$$
$$\cdot \left(0.38269 - 0.22144\cos\frac{2\pi X_3}{\varepsilon} + 0.03602\cos\frac{4\pi X_3}{\varepsilon}\right) +$$
$$- 1.19787\cos\frac{2\pi X_3}{\varepsilon} + 0.31738\cos\frac{4\pi X_3}{\varepsilon}\Big] + \frac{\mathrm{ch}\gamma_2(X_3-\frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}}\cdot$$
$$\cdot \left(-0.18128 + 0.07150\cos\frac{2\pi X_3}{\varepsilon} - 0.00814\cos\frac{4\pi X_3}{\varepsilon}\right) +$$
$$+ 1.07774 + 0.39003\cos\frac{2\pi X_3}{\varepsilon} - 0.08312\cos\frac{4\pi X_3}{\varepsilon}\Big\}\cdot$$
$$\cdot X_1 R_0^{-1}\left(1 + 0.2\cos\frac{2\pi X_3}{\varepsilon}\sqrt{1 + 0.04R_0^2\frac{\pi^2}{\varepsilon^2}\sin^2\left(\frac{2\pi X_3}{\varepsilon}\right)}\right)^{-1} +$$
$$+ \Big\{\Big[\frac{mX_3}{J_{02}^{\mathrm{eff}}} + \frac{M_s(0)}{J_{02}^{\mathrm{eff}}}\Big]\left(1 - 0.39496\cos\frac{2\pi X_3}{\varepsilon} +\right.$$
$$\left.+ 0.04828\cos\frac{4\pi X_3}{\varepsilon}\right)X_2 + \frac{2\mu q\tilde{J}_k}{(\lambda+2\mu)J_2^{\mathrm{eff}}}0.05748\frac{1}{\gamma_2}\cdot$$
$$\cdot \frac{\mathrm{sh}\gamma_2(X_3-\frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}}X_1\Big\}\frac{0.2\pi R_0\sin\frac{2\pi X_3}{\varepsilon}}{\varepsilon\sqrt{1+0.04R_0^2\frac{\pi^2}{\varepsilon^2}\sin^2\left(\frac{2\pi X_3}{\varepsilon}\right)}} - p_1,$$

$$
\begin{aligned}
s_2 &= T^{22}n_2 + T^{23}n_3 - p_2 \sim \frac{\lambda q}{\lambda+2\mu}\Big\{\frac{X_1^2+X_2^2}{R_0^2}\Big[\frac{\mathrm{ch}\gamma_2(X_3-\frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}}\cdot \\
&\cdot \Big(0.38269 - 0.22144\cos\frac{2\pi X_3}{\varepsilon} + 0.03602\cos\frac{4\pi X_3}{\varepsilon}\Big) + \\
&- 1.19787\cos\frac{2\pi X_3}{\varepsilon} + 0.31738\cos\frac{4\pi X_3}{\varepsilon}\Big] + \frac{\mathrm{ch}\gamma_2(X_3-\frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}}\cdot \\
&\cdot \Big(-0.18128 + 0.07150\cos\frac{2\pi X_3}{\varepsilon} - 0.00814\cos\frac{4\pi X_3}{\varepsilon}\Big) + \qquad (1.14) \\
&+ 1.07774 + 0.39003\cos\frac{2\pi X_3}{\varepsilon} - 0.08312\cos\frac{4\pi X_3}{\varepsilon}\Big\}\cdot \\
&\cdot X_2R_0^{-1}\left(1+0.2\cos\frac{2\pi X_3}{\varepsilon}\sqrt{1+0.04R_0^2\frac{\pi^2}{\varepsilon^2}\sin^2\Big(\frac{2\pi X_3}{\varepsilon}\Big)}\right)^{-1} + \\
&+ \Big\{\Big[-\frac{mX_3}{J_{02}^{\mathrm{eff}}} - \frac{M_s(0)}{J_{02}^{\mathrm{eff}}}\Big]\Big(1 - 0.39496\cos\frac{2\pi X_3}{\varepsilon} + \\
&+ 0.04828\cos\frac{4\pi X_3}{\varepsilon}\Big)X_1 + \frac{2\mu q\bar{J}_k}{(\lambda+2\mu)J_2^{\mathrm{eff}}}0.05748\frac{1}{\gamma_2}\cdot \\
&\cdot \frac{\mathrm{sh}\gamma_2(X_3-\frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}}X_2\Big\}\frac{0.2\pi R_0\sin\frac{2\pi X_3}{\varepsilon}}{\varepsilon\sqrt{1+0.04R_0^2\frac{\pi^2}{\varepsilon^2}\sin^2(\frac{2\pi X_3}{\varepsilon})}} - p_2, \\
s_3 &= T^{31}n_1 + T^{32}n_2 + T^{33}n_3 \sim \\
&\sim \frac{\mu q}{\lambda+2\mu}\frac{0.76538\Big(1+0.1\cos\frac{2\pi X_3}{\varepsilon}\Big)\mathrm{sh}\gamma_2\Big(X_3-\frac{1}{2}\Big)}{\gamma_2R_0\sqrt{1+0.04R_0^2\frac{\pi^2}{\varepsilon^2}\sin^2(\frac{2\pi X_3}{\varepsilon})}\,\mathrm{ch}\frac{\gamma_2 l}{2}} + \\
&+ q\Big\{\Big(1+0.1\cos\frac{2\pi X_3}{\varepsilon}\Big)^2\Big[\frac{\mathrm{ch}\gamma_2(X_3-\frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}}\cdot \\
&\cdot \Big(0.38269 - 0.22144\cos\frac{2\pi X_3}{\varepsilon} + 0.03602\cos\frac{4\pi X_3}{\varepsilon}\Big) + \\
&- 1.19787\cos\frac{2\pi X_3}{\varepsilon} + 0.31738\cos\frac{4\pi X_3}{\varepsilon}\Big] + \frac{\mathrm{ch}\gamma_2(X_3-\frac{l}{2})}{\mathrm{ch}\frac{\gamma_2 l}{2}}\cdot \\
&\cdot \Big(-0.181275 + 0.07150\cos\frac{2\pi X_3}{\varepsilon} - 0.00814\cos\frac{4\pi X_3}{\varepsilon}\Big) + \\
&+ 1.07774 + 0.39003\cos\frac{2\pi X_3}{\varepsilon} - 0.08312\cos\frac{4\pi X_3}{\varepsilon}\Big\}\cdot \\
&\cdot \frac{0.2\pi R_0\sin\frac{2\pi X_3}{\varepsilon}}{\varepsilon\sqrt{1+0.04R_0^2\frac{\pi^2}{\varepsilon^2}\sin^2(\frac{2\pi X_3}{\varepsilon})}},
\end{aligned}
$$

and on the boundary cross–sections of the rod we have

$$
\begin{aligned}
\hat{s}_1(0) &= -T^{13}(0) - p_1(0) \sim \frac{-M_s(0)}{J_{02}^{\text{eff}}} 0.65332 X_2 + \\
&+ \frac{q\bar{J}_k}{J_2^{\text{eff}}} \frac{\mu}{(\lambda + 2\mu)} \frac{0.11496}{\gamma_2} X_1 - p_1(0), \\
\hat{s}_2(0) &= -T^{23}(0) - p_2(0) \sim \frac{M_s(0)}{J_{02}^{\text{eff}}} 0.65332 X_1 + \\
&+ \frac{q\bar{J}_k}{J_2^{\text{eff}}} \frac{\mu}{(\lambda + 2\mu)} \frac{0.11496}{\gamma_2} X_2 - p_2(0), \\
\hat{s}_3(0) &= -T^{33}(0) - p_3(0) \sim q\Big(\frac{X_1^2 + X_2^2}{R_0^2} 0.68321 - 0.26674\Big), \\
\hat{s}_1(l) &= T^{13}(l) - p_1(l) \sim \Big[\frac{ml}{J_{02}^{\text{eff}}} + \frac{M_s(0)}{J_{02}^{\text{eff}}}\Big] 0.65332 X_2 + \\
&+ \frac{q\bar{J}_k}{J_2^{\text{eff}}} \frac{\mu}{(\lambda + 2\mu)} \frac{0.11496}{\gamma_2} X_1 - p_1(l), \\
\hat{s}_2(l) &= T^{23}(l) - p_2(l) \sim \Big[-\frac{ml}{J_{02}^{\text{eff}}} - \frac{M_s(0)}{J_{02}^{\text{eff}}}\Big] 0.65332 X_1 + \\
&+ \frac{q\bar{J}_k}{J_2^{\text{eff}}} \frac{\mu}{(\lambda + 2\mu)} \frac{0.11496}{\gamma_2} X_2 - p_2(l), \\
\hat{s}_3(l) &= T^{33}(l) - p_3(l) \sim q\Big(-\frac{X_1^2 + X_2^2}{R_0^2} 0.68321 + 0.26674\Big).
\end{aligned}
\tag{1.15}
$$

The α–components of the reaction forces are self – balanced in each cross–section of the rod

$$
\int_{F(X_3)} \rho r_\alpha dF \equiv 0 \qquad \text{for } X_3 \in (0,l),
$$

$$
\int_{\partial F(X_3)} \sqrt{1 + R_{,3}^2} s_\alpha d(\partial F) \equiv 0 \qquad \text{for } X_3 \in (0,l)
$$

and

$$
\int_{F(0)} \hat{s}_\alpha dF \equiv 0, \qquad \int_{F(l)} \hat{s}_\alpha dF \equiv 0.
$$

For the data (1.9) the integrals $\int_{F(X_3)} \rho r_3 dF$ and $\int_{\partial F(X_3)} \sqrt{1 + R_{,3}^2} s_3 d(\partial F)$ are oscillatory functions of X_3, of values which do not exceed $0.34 q\bar{F}$.

On the boundary cross–sections of the rod we have

$$\int\limits_{F(0)} \hat{s}_3 dF = -0.19114q\bar{F}, \qquad \int\limits_{F(l)} \hat{s}_3 dF = 0.19114q\bar{F},$$

$$\int\limits_{F(0)} (\hat{s}_2 X_1 - \hat{s}_1 X_2) dF = 0, \qquad \int\limits_{F(l)} (\hat{s}_2 X_1 - \hat{s}_1 X_2) dF = -0.00562ml.$$

In the case of the pure torsion of the rod we can calculate the components of the reaction forces due to the internal constraints from (1.11) ÷ (1.13) assuming $q = 0$. Hence $\rho r_3 = 0$, $s_3 = 0$, $\hat{s}_3 = 0$. The remaining components are different from zero but additionaly we have

$$\int\limits_{F(X_3)} \rho r_\alpha dF \equiv 0 \qquad \text{for } X_3 \in (0,l),$$

$$\int\limits_{\partial F(X_3)} \sqrt{1 + R_{,3}^2} s_\alpha d(\partial F) \equiv 0 \qquad \text{for } X_3 \in (0,l),$$

$$\int\limits_{F(0)} \hat{s}_\alpha dF \equiv 0,$$

$$\int\limits_{F(l)} \hat{s}_\alpha dF \equiv 0,$$

$$\int\limits_{F(0)} (\hat{s}_2 X_1 - \hat{s}_1 X_2) dF = 0.00562 M_s(0),$$

$$\int\limits_{F(l)} (\hat{s}_2 X_1 - \hat{s}_1 X_2) dF = -0.00562[ml + M_s(0)].$$

To compare the results let us consider the Saint–Venant problem. The solution of the uncostrained torsion of the rod with the circular and constant cross–section can be obtained from (2.21) in [3], assuming $m = 0$, $q = 0$. In this case

$$\begin{aligned} \Theta &= -\frac{M_s(0)}{\mu J_0} X_3 + C, \\ \zeta &\equiv 0, \\ \eta &\equiv 0, \end{aligned}$$

and there exist only tangential components of the stress – tensor

$$\begin{aligned} T^{13} &= -\mu \Theta_{,3} X_2 = \frac{M_s(0)}{J_0} X_2, \\ T^{23} &= \mu \Theta_{,3} X_1 = -\frac{M_s(0)}{J_0} X_1, \end{aligned}$$

and those of reaction forces at the ends of the rod

$$\begin{aligned}\hat{s}_1 &= T^{13}n_3 - p_1 = \frac{M_s(0)}{\bar{J}_0}n_3X_2 - p_1,\\ \hat{s}_2 &= T^{23}n_3 - p_2 = -\frac{M_s(0)}{\bar{J}_0}n_3X_1 - p_2.\end{aligned}$$

Additionaly, we have for $X_3 = 0$ and $X_3 = l$

$$\int\limits_{F(X_3)} \hat{s}_\alpha dF \equiv 0, \qquad \int\limits_{F(X_3)} (\hat{s}_2X_1 - \hat{s}_1X_2)dF \equiv 0.$$

All a forementioned reaction forces are equal to zero on the extra assumption

$$p_1 = \frac{M_s(0)}{\bar{J}_0}n_3X_2 \quad \text{and} \quad p_2 = -\frac{M_s(0)}{\bar{J}_0}n_3X_1.$$

2. Final remarks

The application of the microlocal modelling approach (based on the concepts of nonstandard analysis [6], outlined in [7,8] to the problem of torsion of rods with periodically variable cross – section permits us to obtain the solution in a simple way. We approximate the system of differential equations with variable ε – periodic coefficients to the system of differential equations with the constant coefficients. The microlocal parameters can be eliminated from this system of equations (with corresponding boundary conditions) and we obtain the system of so called effective equations for the torsional problem of rods with ε – periodic variable cross – section [2].

The effective charakteristics of the homogenous rod (J_0^{eff}, J^{eff}, F^{eff}) take into account the oscillation of the radius of the rod. Notice that the values of the effective characteristics are lawer than those of the corresponding characteristics of the mean cross – section of the rod,

$$\begin{aligned}J_0^{\text{eff}} &\le \bar{J}_0,\\ J^{\text{eff}} &\le \bar{J},\\ F^{\text{eff}} &\le \bar{F},\end{aligned}$$

where the above inequalities hold for $\delta > 0$ and the identities for $\delta = 0$. This result is compatible with our expectation. In formulae which describe Θ_0, Θ_a, ζ_0, ζ_a, η_0, η_a the effective characteristics occur only in the pertinent denominators. (Evidently, in the case of the oscillating radius of cross – section (about the mean value R_0) the additional increments of the functions mentioned above along the segments where $R < R_0$ will be highar, and decreasing along the segments where $R > R_0$). Hence the corresponding displacements will be larger than for the rod with mean cross – section ((2.23) in [3] and (2.24) in [3]).

Observe that the cross–section of ε–periodic rod loaded with the torques (even if $m \neq 0$) remains plane. After the stretching of this rod – the cross–sections are warping; this phenomenon occurs only at the end near the segments of the rod.

The microlocal parameters Θ_a, ζ_a, η_a have very small influence (not exceed 0.5 %) on the displacements, but if we calculate the stresses and reaction forces they play an essential role.

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Streszczenie

Praca jest kontynuacją publikacji [2,3]. Zastosowano metodę modelowania mikrolokalnego [5,6,7,8] do problemu nieswobodnego skręcania prętów o okresowo zmieniających się przekrojach, rozważając zagadnienie w ramach mechaniki ciał z wewnętrznymi więzami [1,4]. Tematem niniejszej pracy jest analiza naprężeń i sił reakcji więzów wewnętrznych dla niektórych przypadków szczególnych.

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