REFLECTION AND REFRACTION OF SHOCK WAVES IN NEO-HOOKEAN MATERIAL

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The semi-inverse method is used to examine the reflection-refraction problem for a finite amplitude oblique shock wave propagating in an unbounded medium consisting of two joined half-spaces, filled with two different neo-Hookean elastic materials. The incident shock is assumed to be a transverse, horizontally polarised, plane shock. In such a case the solution pattern can be assumed in a form of single reflected and refracted shock wave, which relative strengths follow the similar pattern and values as that of infinitesimal SH waves. The complex values of the amplitude in the linear theory (total reflection, grazing incidence) give occasion to existence of Stoneley interfacial waves but in nonlinear theory the likeness does not exist, and such solutions suggest rather the change of solution pattern.

1. Introduction

Wright in his paper [1] on reflection of an oblique finite elastic shock wave at a plane boundary of a nonlinear elastic solid presented a semi-inverse method, based on strictly mechanical considerations, of finding the reflected waves. In this method a reflection pattern is assumed: the reflected waves form a family of plane simple waves (or shocks) centered on a moving line of contact between the incident wave and the boundary. Each reflected wave connects a fixed state ahead of the wave with a one parameter family of states behind the wave. In anisotropic solids there are three possible families of reflected waves so that a sequence of such waves connects the state behind the incident shock with a three parameter family of states adjacent to the boundary. In general, there are three independent boundary conditions from which the parameter specifying the reflected waves can be determined. The assumed pattern reduces the reflection problem to an initial-boundary value problem for a system of ordinary differential equation governing
the variation of the deformation gradient and velocity fields in the regions of simple waves. Its solution determines the wedge shaped regions of simple waves and the distribution and strengths of the wavelets within each wave. If the assumed reflection pattern fails the admissibility test, it is modified to include shocks as well; for shocks, the reflection problem is then reduced to solving a system of algebraic equations for the direction of propagation and strength of the reflected shocks.

The reflection-refraction problem for a shock incident on a plane interface joining two elastic solids is exactly analogous, but now there will be three reflected waves and three refracted waves, so there will be six parameters to be found from six continuity conditions for the displacements and stresses at the interface. In some cases of a particular material property there may be less than six reflected and refracted waves.

We will apply the semi-inverse method to examine the reflection-refraction problem for a plane shock wave propagating in an unbounded medium consisting of two joined half-spaces, filled with two different nonlinear elastic materials. The medium initially is unstrained and at rest. Bearing in mind the complexity of the analysis of finite amplitude waves in nonlinear solids we select the simplest nontrivial case for which a detailed discussions is conducted. We assumethat both materials are homogeneous, incompressible and isotropic, and both are characterised by the neo-Hookean strain energy function but with different material constants. Since incompressible solids transmit transverse waves only, the incident shock is assumed to be a transverse shock; we assume further that this shock is horizontally polarised. Such waves have the displacement components in one direction only. In such a case the reflection-refraction solution pattern can be assumed in a form of singlesimple reflected wave and a single simple refracted wave both centered on the line of incidence at the interface.

Section 2 contains a summary of necessary theory and derivation of the propagation condition for shocks and simple waves in incompressible materials. In sections 3 and 4 geometric and analytic description of the incident shock and the solution pattern are given. The analysis of the solution for shock reflection-refraction in neo-Hookean elastic material are presented in section 5. The results are illustrated graphically.

2. Basic equations

The motion of the continuum is given by $z_i = x_i(X_\alpha, t)$ where $x_i$ and $X_\alpha$ are the Cartesian coordinates of a material particle in the present configuration $B$ and the reference configuration $B_R$ respectively, and the reference configuration is given by $z_i = \delta_{i\alpha} X_\alpha$. The deformation gradient $x_{\alpha i}$, its inverse $X_{\alpha i}$ and the
velocity are defined by

\[ x_{i\alpha} = \frac{\partial x_i}{\partial X_\alpha}, \quad X_{\alpha i} = \frac{\partial X_\alpha}{\partial x_i}, \quad x_i = u_i = \frac{\partial x_i}{\partial t}. \quad (2.1) \]

It is assumed that the material is homogeneous, elastic and incompressible. The incompressibility constraint requires

\[ J = \det(x_{i\alpha}) = 1. \quad (2.2) \]

The Piola-Kirchhoff stress tensor for such material is

\[ T_{i\alpha} = \rho R \frac{\partial \sigma}{\partial x_{i\alpha}} + p X_{\alpha i}, \quad (2.3) \]

where \( \sigma \) denotes internal energy per unit mass in \( B_R \), \( \rho = \rho R \) is the density and \( p(X_\alpha) \) is a pressure field to be determined in each problem.

If the stress and velocity fields are differentiable, then the equations expressing balance of momentum and moment of momentum are

\[ T_{i\alpha,\alpha} = \rho u_i, \quad x_{i\alpha} T_{j\alpha} = x_{j\alpha} T_{i\alpha}. \quad (2.4) \]

If the functions \( x_i = x_i(X_\alpha, t) \) are continuous everywhere but have discontinuous first derivatives on some propagating surface \( S(X_\alpha, t) = 0 \), the equations (2.4) must be replaced by the jump conditions on this surface

\[ a) \quad [T_{i\alpha}]N_\alpha = -\rho V[u_i], \quad (2.5) \]

\[ b) \quad [x_{i\alpha}] = a_i N_\alpha, \quad [u_i] = -a_i V. \]

Such a surface is called a shock wave. The vector \( N_\alpha \) is a material unit normal to the wave, \( V \) is the speed of propagation along \( N_\alpha \) and \( a_i \) is the amplitude vector of the jump. The double square brackets indicate the jump in the quantity enclosed across \( S \); thus

\[ [ ] = ( )^B - ( )^F, \]

where the letters \( F \) and \( B \) refer to the limit values taken in front and rear sides of \( S \) respectively.

Eliminating the velocity jump \([u_i]\) from eqs.(2.5) we obtain the equation for the shock speed, the following propagation condition

\[ V^2 = \frac{1}{\rho m^2} [T_{i\alpha}] a_i N_\alpha, \quad (2.6) \]

where \( m = |a| \) is the shock strength.
It is known (cf.[6]) that the constraint of incompressibility (2.2) restricts the directions of propagation to transversal directions only. This means that shocks in incompressible continua are transverse waves.

Simple waves [1] are defined to be regions of space time in which all field quantities are continuous functions of a single parameter, say \( \lambda = \psi(X_{\alpha}, t) \). Regions of constant \( \lambda \) are propagating surfaces, called wavelets, with unit normal and normal velocity in \( B_R \) given by

\[
N_{\alpha}(\lambda) = \frac{\psi_{,\alpha}}{|\nabla \psi|}, \quad U(\lambda) = -\frac{\dot{\psi}}{|\nabla \psi|}.
\]  
(2.7)

The equation of motion (2.4) and the compatibility condition in the region of simple wave are

\[
\frac{\partial T_{i\alpha}}{\partial x_{j\beta}} x'_{j\beta} \psi_{,\alpha} = \rho u'_{i} \dot{\psi}, \quad x'_{j\beta} \dot{\psi} = u'_{j} \psi_{,\beta},
\]  
(2.8)

where the prime indicates differentiation with respect to \( \lambda \). If \( \psi \neq 0 \) eqs.(2.8) can be rewritten to obtain the propagation condition for simple waves (cf.[1]) and the compatibility condition in the form

\[
(Q_{ij} - \rho U^2 \delta_{ij}) u_j = 0, \quad U x_{j\beta} + u_j N_{\beta} = 0,
\]  
(2.9)

where

\[
Q_{ij} = \frac{\partial T_{i\alpha}}{\partial x_{j\beta}} N_{\alpha} N_{\beta}
\]  
(2.10)

is the acoustic tensor. For simple waves to propagate it is necessary that the eigenvalues of \( Q_{ij} \) are real monotone functions of the wave parameter (cf.[1]).

3. Incident shock

Consider an unbounded medium consisting of two elastic incompressible, isotropic half-spaces of different material properties, joined rigidly along the plane \( x_2 = 0 \), and initially unstrained and at rest. Suppose that a plane, horizontally polarised transverse shock wave of strength \( m_0 \) propagates in the half-space \( x_2 > 0 \) with speed \( V_0 \), and approaches the interface \( x_2 = 0 \) at an angle \( \Theta_0 \) (fig.1). Thus, this propagating discontinuity surface belongs to a one-parameter family of parallel planes, with normals

\[
N_0 = (\sin \Theta_0, -\cos \Theta_0, 0), \quad 0 < \Theta_0 < \frac{\pi}{2}.
\]  
(3.1)

Such waves have displacement components in the \( x_3 \) - direction only.
Fig. 1. Incident shock and assumed reflection and refraction pattern

Since the amplitude vector \( \mathbf{a} \) is parallel to the \( x_3 \) axis and the medium in front of the incident shock is unstrained and at rest, the jumps (2.5) become now

\[
\begin{align*}
\text{a)} \quad [x_{31}] &= (x_{31})^B = m_0 \sin \Theta_0, \quad [x_{32}] = (x_{32})^B = -m_0 \cos \Theta_0, \\
\text{b)} \quad [u_3] &= (u_3)^B = -m_0 V_0,
\end{align*}
\]  

(3.2)

where \( m_0 = a \) is the shock strength. Eq.(2.6) for the shock speed is

\[ V_0^2 = \frac{1}{\rho m^2} ([T_{31}] \sin \Theta_0 - [T_{32}] \cos \Theta_0). \]

(3.3)

The state behind the propagating shock (region 1 in fig.1) is now completely specified by the angle of incidence \( \Theta_0 \) and the shock strength \( m_0 \). Eqs.(3.2) determine the deformation gradient and its inverse

\[
(x_{i0})^B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ v_1 & v_2 & 1 \end{bmatrix}, \quad (X_{ni}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 10 \\ -v_1 & -v_2 & 0 \end{bmatrix}
\]

(3.4)

and particle velocity

\[ \mathbf{u} = (0, 0, u) \]

(3.5)

in this state. We denoted here \( v_1 = (x_{31})^B, \ v_2 = (x_{32})^B, \ u = (u_3)^B. \)
For isotropic incompressible materials the internal energy $\sigma$ is a function of $I_1$, $I_2$ and $S$, where

$$I_1 = B_{ii}, \quad I_2 = \frac{1}{2}(B_{ii}B_{jj} - B_{ij}B_{ij})$$

are the invariants of the left Cauchy-Green strain tensor $B_{ij}$, and $S$ is the entropy (per unit volume in $B_R$).

The components of the stress tensor $T_{ia}$ required in this paper, are now

$$
\begin{align*}
T_{11} &= 2\rho(\sigma_1 + \sigma_2(2 + v_1^2)) + p, \quad T_{13} = -2\rho\sigma_2 v_1 - pv_1, \\
T_{22} &= 2\rho(\sigma_1 + \sigma_2(2 + v_1^2)) + p, \quad T_{31} = 2\rho(\sigma_1 + \sigma_2)v_1, \\
T_{33} &= 2\rho(\sigma_1 + \sigma_2) + p, \quad T_{23} = -2\rho\sigma_2 v_2 - \dot{p}v_2, \quad (3.6) \\
T_{12} &= T_{21} = -2\rho\sigma_2 v_1 v_2, \quad T_{32} = 2\rho(\sigma_1 + \sigma_2)v_2,
\end{align*}
$$

where

$$\sigma_1 = \frac{\partial \sigma}{\partial I_1}, \quad \sigma_2 = \frac{\partial \sigma}{\partial I_2}, \quad I_1 = I_2 = 3 + v_1^2 + v_2^2.$$ 

4. Reflection-refraction pattern

When the incident shock wave strikes the interface $x_2 = 0$, part of it is reflected and part transmitted across the interface, in a form of plane reflected and refracted waves centered on the line of contact with the boundary $x_2 = 0$ (point $Q$ in fig.1), and propagating away from the interface. The point $Q$ moves along the boundary ($x_1$-axis) with speed

$$U_h = \frac{V_0}{\sin \Theta_0}. \quad (4.1)$$

It is also assumed that both the reflected wave (region 2 in fig.1) and the refracted wave (region 2) are simple waves connecting the front and rear regions of constant state. Since all waves are centered on $Q$, we have for the reflected wave

$$
\begin{align*}
\text{a)} \quad N(\lambda) &= (\sin \Theta(\lambda), -\cos \Theta(\lambda), 0), \quad \frac{\pi}{2} < \Theta < \pi, \\
\text{b)} \quad U(\lambda) &= U_h \sin \Theta(\lambda),
\end{align*}
$$

and for the refracted wave

$$
\begin{align*}
\text{a)} \quad N(\mu) &= (\sin \hat{\Theta}(\mu), -\cos \hat{\Theta}(\mu), 0), \quad 0 < \hat{\Theta} < \frac{\pi}{2}, \\
\text{b)} \quad U(\mu) &= \frac{U_h}{\sin \hat{\Theta}(\mu)}, \quad (4.3)
\end{align*}
$$
where $\Theta$ and $\hat{\Theta}$ denote the angle of reflection and refraction (ref. fig.1), and $U$, $\lambda$, $\hat{U}$, $\mu$ are the normal speed and parameter of the reflected and refracted wave, respectively. The material region ahead of the refracted wave (region 0) is unstrained and at rest. The deformation gradient $x_{i\alpha}(\mu)$ and the velocity $u(\mu)$ in region 1) assume an analogous to (3.4) and (3.5) form.

It is required that the displacement vector and the stress vector are continuous at the interface $x_2 = 0$. This means that the system of the incident, reflected and refracted waves must satisfy four continuity conditions for the velocity and stress fields in both media at $x_2 = 0$.

\[ u = \hat{u}, \quad T_{i2} = \hat{T}_{i2}, \quad i = 1, 2, 3. \]  \hspace{1cm} (4.4)

Since regions 0 and $\hat{0}$ are unstrained and at rest, conditions (4.4) at the interface joining these two regions are, for the suitably chosen pressures $p_0$ and $\hat{p}_0$, satisfied identically.

The constant values of region 3 and $\hat{3}$ adjacent to the remaining part of the interface are the final values of region 2 and $\hat{2}$ corresponding to the final values $\hat{\lambda}$ and $\hat{\mu}$ of the wave parameters $\lambda$ and $\mu$. Substitution into (4.4) gives a system of four independent equations involving two unknown terminal values of the wave parameters. In general system (4.4) has no solution in $\hat{\lambda}$ and $\hat{\mu}$. A solution may exist, however, if some additional restrictions on $\sigma$ or on the incident shock are imposed.

5. Reflection and refraction in neo-Hookean material

For rubber-like materials under moderate strain, the strain energy function $W$ can be approximated by [7,8]

\[ W(I_1, I_2) = \rho \sigma(I_1, I_2) = C_1 (I_1 - 3) + C_2 (I_2 - 3) + C_3 (\frac{I_1^2}{2} - 9). \]  \hspace{1cm} (5.1)

The components (3.6) of the stress tensor $T_{i\alpha}$ are now

\[ T_{12} = 2C_2 v_1 v_2, \]

\[ T_{32} = c^2 (1 + \eta (v_1^2 + v_2^2)) v_2, \] \hspace{1cm} (5.2)

\[ T_{22} = c^2 (1 + \eta (v_1^2 + v_2^2)) + 2C_2 v_1^2 + p, \]

where

\[ c^2 = 2(C_1 + C_2 + 6C_3), \quad \eta = \frac{4C_3}{c^2}. \]

The other field quantity required here the static pressure $p(X_\alpha)$, is given in the region of simple wave by (cf.[9])

\[ p(\lambda) = -4C_2 [v_1^2(\lambda) + v_2^2(\lambda)] + p_0. \] \hspace{1cm} (5.3)
The requirement that \( u \) is continuous on \( x_2 = 0 \) implies continuity of \( v_1 \) as well. Indeed, substituting (4.2), (4.3) into the compatibility condition (2.9) and integrating the first equation with boundary conditions (3.2) and conditions for region \( \hat{0} \) we obtain

\[
U_h v_1 + u = 0, \quad U_h \dot{v}_1 + \dot{u} = 0,
\]

substracting these equations for the final values of the simple wave parameters \( \hat{\lambda}, \hat{\mu} \) (regions 3,\( \hat{3} \)), \( U_h[v_1 - \dot{v}_1] + [u - \dot{u}] = 0 \) on \( x_2 = 0 \), where \( U_h \) is the apparent speed of the line of contact of the waves with the interface (fig.1). It is obvious that if \( u = \dot{u} \) then also

\[
v_1 = \dot{v}_1 \quad \text{on} \quad x_2 = 0. \quad (5.4)
\]

Substituting (5.2) and (5.3) into (4.4), and using (5.4) and the relation \( p_0 - \hat{p}_0 = \hat{c}^2 - c^2 \) connecting the constant pressures of region 0 and \( \hat{0} \), we obtain the conditions of continuity in the following form

\[
u = \dot{u}, \quad (v_1 = \dot{v}_1),
\]

\[
(C_2 v_2 - \hat{C}_2 \dot{v}_2) v_1 = 0,
\]

\[
c^2[1 + \eta(v_1^2 + v_2^2)] v_2 = \hat{c}^2[1 + \hat{\eta}(\dot{v}_1^2 + \dot{v}_2^2)] \dot{v}_2,
\]

\[
(C_2 - \hat{C}_2) v_1^2 = 0.
\]

It is evident that system (5.5) reduces to two independent nontrivial equations for \( v_1 = 0 \), i.e. for shocks travelling in the direction normal to the interface \( x_2 = 0 \) (cf.[10,11]). It also reduces to two equations if the medium is characterised by the neo-Hookean strain-energy function, i.e. when the constants \( C_2 \) and \( C_3 \) in (5.1) are zero (we take \( C_1 = C \)).

Let us assume that the strain-energy function is

\[
W = \rho \sigma(I_1) = C(I_1 - 3).
\]

In such materials the shock speed (3.3) is independent of the direction of propagation (3.1) and of the deformation gradient (3.4)

\[
V_0^2 = \frac{2}{\rho} C
\]

and the transverse shocks propagate with constant speed, without change in the form. Hence, the propagation of transverse shocks waves in a neo-Hookean material follows the same pattern as that of small amplitude waves in a homogeneous linear isotropic elastic solid.

We also note that the components \( \partial T_{i\alpha} / \partial x_{j\beta} \) in the acoustic tensor (2.10) are constant. Since the characteristic equation for the propagation condition (2.9a) for simple waves can be written in the form

\[
det(\partial^2 T_{i\alpha} / \partial x_{j\beta} - U_h^2 \delta_{ij}) = 0,
\]

(5.8)
\[ \mathbf{N} = \frac{N}{\sin \theta(\lambda)} = (1, -\tau, 0), \quad \tau = \cot \theta(\lambda), \quad U_h = \frac{U}{\sin \theta_0} \]

and the speed \( U_h \) of the point \( Q \) (ref.4.1) is constant, we conclude that the characteristic roots of (5.8) that define speed \( U(\lambda) \) of the reflected simple waves (and by a similar argument speed \( U(\mu) \) of the refracted simple waves) are independent of the wave parameter and that the wave speed is constant across the wave; hence according to the admissibility criterion (cf.[1]), such solutions do not represent simple waves.

We modify the solution pattern assuming now that region 2 is a plane shock wave centered on \( Q \), with strength \( m \) and direction of propagation \( \mathbf{N} = (\sin \theta, -\cos \theta, 0) \). An analogous assumption applies to region \( \hat{2} \). Eqs. of motion (2.4) are now replaced by the jump conditions (2.5) connecting the corresponding quantities in regions 1 and 3 across the wave. From (3.3) we calculate the reflected shock speed

\[ V^2 = \frac{2C}{\rho}, \quad (5.9) \]

which is exactly the speed of the incident shock.

Since by (4.2b) \( V = U_h \sin \theta = V_0 \sin \theta / \sin \theta_0 \), the equality \( V = V_0 \) can be satisfied only if

\[ \theta = \pi - \theta_0. \quad (5.10) \]

The constant state of region 1 is defined by equations (3.2) (3.4) and (3.5). Applying (5.10) in the jump conditions (2.5) across region 2 we obtain the constant values of region 3

\[ v^B_1 = (m_0 + \varepsilon m) \sin \theta_0, \]
\[ v^B_2 = (-m_0 + \varepsilon m) \cos \theta_0, \]
\[ u^B = -(m_0 + \varepsilon m)V_0, \quad (5.11) \]

where \( \varepsilon \) is +1 or −1, depending on the orientation of the reflected shock polarization vector \( \mathbf{a} \) with respect to the \( x_3 \) - axis.

The speed of the refracted shock is \( \hat{V}^2 = 2\hat{C}/\rho \).

Since by (4.3b) \( \hat{V} = \hat{U}_h \sin \hat{\theta} = V_0 \sin \hat{\theta} / \sin \theta_0 \), we find that

\[ \sin \hat{\theta} = \frac{\hat{V}}{V_0} \sin \theta_0. \quad (5.12) \]

Applying (5.12) in the jump conditions (2.5) across region \( \hat{2} \) we obtain the constant values of region 3

\[ \hat{v}^B_1 = \varepsilon \hat{m} \frac{\hat{V}}{V_0} \sin \theta_0, \]
\[
\hat{v}_2^B = -\hat{\varepsilon} \hat{m} \sqrt{1 - \left(\frac{\hat{V}}{V_0}\right)^2 \sin^2 \Theta_0},
\]
\[
\hat{u}^B = -\hat{\varepsilon} \hat{m} \hat{V},
\]

where \( \hat{\varepsilon} = \pm 1 \).

The requirement that the solution of equation (5.12) for \( \hat{\varepsilon} \) must be real imposes on \( \Theta_0 \) the condition
\[
\sin \Theta_0 \leq \frac{V_0}{\hat{V}},
\]
which may restrict, depending on the material constants in both semi-spaces, the interval of variation of the angle of incidence.

![Fig. 2. Incident, reflected and refracted shocks](image)

a) for \( \hat{V} < V_0 \) and \( 0 < \Theta_0 < \frac{\pi}{2} \); b) for \( \hat{V} < V_0 \) and \( \Theta_0 = \frac{\pi}{2} \); c) for \( \hat{V} > V_0 \) and \( 0 < \Theta_0 < \Theta_\varepsilon \); d) for \( \hat{V} > V_0 \) and \( \Theta_0 = \Theta_\varepsilon \)

Relations (5.10), (5.12) and (5.14) have a simple geometrical interpretation. Though the speeds \( \hat{V} \) and \( V \) (= \( V_0 \)) depend on the material constants only, the corresponding directions of propagation depend on the angle of incidence as well. Let \( Q_0 \) and \( Q \) identify the position of point of contact of the incident, reflected and refracted wave on the \( x_1 \) - axis at the instant \( t = 0 \) and \( t = 1 \), respectively; the distance from \( Q \) to \( Q_0 \) is then equal \( U_h \) (ref.(4.1)). Let \( \hat{K} \) and \( K \) be two semicircles with centre at \( Q_0 \), and radius \( \hat{r} = \hat{V} \), \( r = \hat{V} \), and \( U_h \) be large enough for \( Q \) to be outside of \( \hat{K} \) and \( K \) (fig.2). The tangents to \( \hat{K} \) and \( K \) issuing from \( Q \) form with the \( x_1 \) - axis the angles \( \Theta \) and \( \Theta_0 \) (fig.2a) that satisfy eqs.(5.10) and
(5.12). The position of point \( \hat{Q} \) changes with angle of incidence \( \Theta_0 \), thus forming two one-parameter families of tangents which represent the families of reflected and refracted waves at unit time after passing through \( Q_0 \). The semicircles \( \hat{K} \) and \( K \) are envelopes of these two families; they are called wave curves.

If \( \hat{V} < V_0 \), the point \( \hat{Q} \) moves toward \( \hat{K} \) as \( \Theta_0 \) increases to \( \frac{\pi}{2} \). For \( \Theta_0 = \frac{\pi}{2} \) (grazing incidence) the incident and reflected wave coincide, and the refracted wave assumes its extreme position, its angle \( \Theta_c \) given by the equation \( \sin \Theta_c = \hat{V}/V_0 \) (fig.2b).

If \( \hat{V} > V_0 \), the angle of refraction \( \hat{\Theta} \) is greater than the angle of incidence \( \Theta_0 \) (fig.2c). The point \( \hat{Q} \) moves toward \( \hat{K} \) with increasing \( \Theta_0 \) and meets \( \hat{K} \) when \( \Theta_0 = \Theta_c \); we have then a "grazing refraction" (fig.2d). The critical value \( \Theta_c \) of the angle of incidence is given by the eq. \( \sin \Theta_c = V_0/\hat{V} \) (ref.5.14).

The remaining two unknown quantities, the strengths of the reflected and refracted shock \( m \) and \( \hat{m} \), will be found from the boundary conditions (5.5) connecting the constant states of region 3 and \( \hat{3} \) across the interface. Analysing the jump conditions (5.11),(5.13) and the relation (5.12) it is easy to see that if \( \hat{u} = u \) then also \( \hat{v}_1 = v_1 \), as for the pattern in the form of the simple wave. Conditions (5.5) are now reduced to two nontrivial equations involving the components of the particle velocity and deformation gradient in both half-spaces as function of \( m \) and \( \hat{m} \) (ref. also (5.4))

\[
\hat{u} = \hat{v} \Rightarrow (v_1 = \hat{v}_1), \quad Cv_2 = \hat{C}\hat{v}_2 \text{ at } x_2 = 0, \quad (5.15)
\]

with an additional equation \( \rho_0 - \hat{\rho}_0 = \hat{C} - C \) relating the constant pressures of region 0 and \( \hat{0} \). Solving eqs.(5.15), with the aid of (5.11) and (5.13), for \( m \) and \( \hat{m} \)

\[
\varepsilon m = \frac{\beta \cos \Theta_0 - \sqrt{\beta - \sin \Theta_0}}{\beta \cos \Theta_0 + \sqrt{\beta - \sin \Theta_0}} m_0,
\]

\[
\varepsilon \hat{m} = \frac{2\beta \sqrt{\beta} \cos \Theta_0}{\beta \cos \Theta_0 + \sqrt{\beta - \sin \Theta_0}} m_0,
\]

completes the reflection-refraction solution in the assumed form. Since speeds \( \hat{V} \) and \( \hat{V} \) are constant throughout the medium, the solution satisfies Lax's stability criterion (cf.[3]) for shocks; henced, the reflected and the refracted wave are shocks, and they are represented by the families of planes (ref.(4.2)) defined by

\[
\sin \Theta_0 x_1 - \cos \Theta_0 x_2 - V_0 t = 0
\]

and by

\[
\sin \Theta_0 x_1 - \sqrt{\left(\frac{V_0}{\hat{V}}\right)^2 - \sin^2 \Theta_0} x_2 - V_0 t = 0
\]
respectively.

We note here that eqs.(5.16) agree with the expressions obtained for the amplitude rations in the reflection-refraction problem for SH waves in linear theory (cf.[5], page 184).

Inspection of eqs. (5.16) leads to the following observations.

1 – The incident shock is completely refracted if \( m = 0 \). The corresponding equation

\[
\beta^2 \cos^2 \Theta_0 - \beta \left( \frac{\rho}{\rho'} \right) \sin^2 \Theta_0 = 0, \quad \beta = \frac{C}{\tilde{C}},
\]

(5.19)

shows that a combination of material properties and angle of incidence is possible for which there is no reflected wave. If \( \rho = \rho' \) and \( C \neq \tilde{C} \), such a combination is particularly simple

\[
\beta = \frac{C}{\tilde{C}} = \tan^2 \Theta_0,
\]

(5.20)

2 – The strength \( m \) of the reflected shock is a monotone function of \( \Theta_0 \), \( 0 < \Theta_0 < \Theta_c \), so \( \dot{e}m \) changes the sign in the neighbourhood of its zero (fig.3), thus changing the direction of polarisation. 3 – The expression for \( \ddot{e}m \) is positive for

![Fig. 3. Relation between the reflected shock strength and incident angle. For the interface characterised by \( \beta > 1 \) and \( \beta_1(1/\beta) \) the strength is the same for the incident angle \( \Theta'_2 \), \( \Theta''_2(\Theta_1) \)](image)

all \( \Theta_0 \in (0, \Theta_c) \), hence \( \dot{e} = 1 \); it also shows that reflection without refraction is not possible.
Fig. 4. Relation between the refracted shock strength and incident angle for some values of the parameter $\beta$ ($\rho = \hat{\rho}$)

4 – The strengths of the incident, reflected and refracted shocks are connected by the formula

$$(m_0 + \varepsilon m)V_0 = \hat{m}\hat{V}. \quad (5.21)$$

For "grazing incidence" (fig.2b) we have $m_0 = -\varepsilon m$ and the superposed incident and reflected shocks produce zero displacement the reflected wave also disappears. For "grazing refraction" (fig.2d), when $\Theta_0 = \Theta_c$, we have $m = m_0$, hence $\hat{m} = 2m_0(V_0/\hat{V})$.

6. Concluding remarks

The geometry of the reflected shock wave is given by (5.17); the shock travels through the half-space $x_2 > 0$ with a constant speed $V (= V_0)$ in the direction
\( \Theta = \pi - \Theta_0 \). The direction of polarisation and strength of this shock depend on the material constants \( C \) and \( \dot{C} \) and the incident shock parameters \( \Theta_0 \) and \( m_0 \). Equation (5.17) (for \( \rho = \beta \)) determines the incident angle \( \bar{\Theta}_0 \) for which there is no reflection. Since \( (V_0/V)^2 = C/\dot{C} = \tan^2 \bar{\Theta}_0 \), such an angle is within the admissibility interval \((0, \Theta_c)\) defined by (5.12). If \( C > \dot{C} \), the reflection ratio \( R = \varepsilon m/m_0 \) is a decreasing function of \( \Theta_0 \), and it is positive for \( \Theta_0 \in (\bar{\Theta}_0, \Theta_c) \). Hence, the reflected wave is a shock propagating into a deformed body while:
(a) – increasing the strain level if \( \Theta_0 \in (0, \bar{\Theta}_0) \),
(b) – decreasing the strain level if \( \Theta_0 \in (\bar{\Theta}_0, \Theta_c) \). The situation is reversed for \( C < \dot{C} \).

![Diagram](image)

**Fig. 5.** Relations between the reflected shock strength, critical angle and the parameter \( \beta \).

The family \( \mathcal{F} \) of curves representing the reflection ratio \( R = \varepsilon m/m_0 \) as a function of \( \Theta_0 \) for various values of the parameter \( \beta = (C/\dot{C}) \) is shown in figure 3. It can be seen that the family curves intersect at some points, thus indicating that a given incident shock combined with two different composite materials may produce the same reflected shock. Indeed, if a family curve \( \beta \) passes through a point \((\Theta_0, R)\) then the curve \( \beta_1 \) defined by

\[
\beta_1 = \frac{\beta \sin^2 \Theta_0}{\beta - \sin^2 \Theta_0},
\]

also passes through this point; since \( \beta_1 - \sin^2 \Theta_0 = \sin^4 \Theta_0/(\beta - \sin^2 \Theta_0) > 0 \), condition (5.14) is satisfied, thus confirming that the curve \( \beta_1 \) belongs to family \( \mathcal{F} \). In the trivial case of \( \beta = 1 \) the ratio \( R = \varepsilon m/m_0 \) is zero for arbitrary \( \Theta_0 \); from
(6.1) we find the parameter $\beta_1$ defining the associated curve to be $\beta_1 = \tan^2 \Theta_0$, and the point of intersection is $(\Theta_0, 0)$ (ref.(5.17)).

Suppose now that $\beta_1 = 1/\beta$. From (6.1) it follows that the curves defined by $\beta$ and $\beta_1 = 1/\beta$ intersect at $(\Theta_0, R)$ if and only if

$$\sin^2 \Theta_0 = \frac{\beta}{1 + \beta^2}, \quad (6.2)$$

eq.(6.2) has a unique solution $\Theta_0$ for arbitrary $\beta > 0$. Since in this case $\sin \Theta_0 \leq 1/\sqrt{2}$, $\Theta_0$ is restricted to the interval $(0, \pi/4]$. Hence for an arbitrary incident angle $\Theta_0 \in (0, \pi/4]$ a medium characterised by the ratio $\beta = C/\hat{C}$ can be found such that the reflected shock wave in this medium and in the medium composed of the same materials but in the reverse order are the same (figure 5).

The locus of points at which two "neighbouring" family curves $\beta$ and $\beta_1 = \beta + d\beta$, $d\beta \to 0$, intersects forms an envelope of $F$

$$R = -\tan^2(\Theta_0 - \frac{\pi}{4}), \quad (6.3)$$

with the family parameter $\beta$ constrained by

$$\beta = 2\sin^2 \Theta_0. \quad (6.4)$$

This curve has a specific property that to each of its points there corresponds one and only one combination of incident shock and material properties. Moreover, since on this curve the derivative $\partial R/\partial \beta = 0$, eq.(6.3) gives the minimum values of the reflection ratio $R = \epsilon m/m_0$ for fixed $\Theta_0$ and varying $\beta$.

The refracted shock (5.19) travels through the half-space $x_2 < 0$ with a constant speed $\hat{V}$, in the direction $\hat{\Theta}$ defined by (5.12); its strength is given by eq.(5.16)$_2$, and it propagates into an unstrained medium, loading the material.

The family $F$ of curves representing the refraction ratio $R = \hat{m}/m_0$ as a function of $\Theta_0$ for various values of $\beta$ is shown in fig.4. The curves intersect at some points, thus indicating that different combinations of material properties and angle of incidence are possible for which the refracted shock is the same. The analytic expression of such combinations, however, is more complex than in the case of reflection (ref.(6.1)), as here the refraction angle $\hat{\Theta}$ depends on both the incident angle $\Theta_0$ and the material ratio $\beta$.

On the other hand, it is easy to show that for the combinations that produce equal reflected shocks, the corresponding refracted shocks are not equal. Indeed, by (5.12), for a fixed $\Theta_0$ the refraction angle $\hat{\Theta}$ is a monotone function of $\beta$, thus assuming different values for different values of $\beta$. This means that different material combinations lead to a given shock being refracted in different directions. A similar conclusion concerning the shock strength $\hat{m}$ can be derived from expression (5.21). Denoting by $R$ the reflection ratio corresponding, for a fixed $\Theta_0$, to $\beta$ and
$\beta_1$ related by (6.1) and by $\hat{R}$ and $\hat{R}_1$ the associated refraction ratios, we obtain from (5.21)

\[
(1 + R)\sqrt{\beta} = \hat{R}, \\
(1 + R)\sqrt{\beta_1} = \hat{R}_1,
\]

(6.5) eliminating $R$ we get

\[
R_1 = \hat{R}_1 \sqrt{\frac{\beta_1}{\beta}}.
\]

(6.6) Since by assumption $\beta_1 \neq \beta$, it follows that $\hat{m}_1 \neq \hat{m}$.

In the special case of total refraction the corresponding refraction angle and refracted shock strength are $\hat{\Theta} = \pi/2 - \Theta_0$ and $\hat{m} = \sqrt{\beta} m_0$. Of course, in the trivial case of identical materials ($\beta = 1$) there is also a total (apparent) refraction: $m = 0$ for arbitrary value of $\Theta_0$, the "refracted" planes (4.2) become extensions of the "incident" planes (3.1) and the shock travels throughout the medium unobstructed.

The other special case refers to the reflected shocks represented by the envelope (6.3). Substitution from (6.4) into (5.12) gives the refraction angle $\hat{\Theta} = \pi/4$. Hence, the refracted shocks associated with the reflected shocks given by (6.3) and (6.4) have a fixed direction of propagation $\Theta_0 = \pi/4$; strength $\hat{m}$ of these shocks is given by

\[
\hat{m} = \frac{\sqrt{2} m_0 \sin \Theta_0 \sin 2\Theta_0}{\cos^2(\Theta_0 - \pi/4)}.
\]

(6.7) The method of solution used in this paper is based on an assumption that solutions must be real. For this reason it is required that the incident angle $\Theta_0$ satisfies the condition: $0 < \Theta_0 < \Theta_c$ where $\Theta_c$ is a certain critical angle determined by the material properties of the composite medium. We have $\Theta_c < \pi/2$ for $\beta < 1$ and $\Theta_c = \pi/2$ for $\beta > 1$. The case of total reflection when $\Theta_0$ exceeds its critical angle leads to a complex solution and is beyond applicability of the present method. In linear theory the solution for this case is composed of infinitesimal waves and Stoneley waves. For $\beta > 1$, the case of "grazing incidence" is illustrated in fig.2b; the corresponding reflection and refraction shock strengths are $m = m_0$ and $\hat{m} = 0$, and the case represents a zero motion. The technique used in [12] to derive a special solution for this case through a limit procedure is not applicable here, since, the solution waves have finite amplitudes. To include these particular cases in the problem considered here it would be necessary to modify the assumed reflection-refraction pattern.

References


**Streszczenie**


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