TRANSIENT AND STEADY VIBRATION OF HELICAL GEARS

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1. Introduction

In view of modern practical design, the article emphasizes the transient state of motion and considers the load distribution between cooperating teeth. Attention is given to engineering aspects of vibration problem of the gear for solving transient and steady state relative displacement \((u, u')\) of the wheels. Dynamic cooperation of teeth are dealt with and dynamic load distribution along the path and lines of contact changes stiffness and damping force of a system. A nonlinear differential equation of motion is integrated using the analytical-numerical methods taking into account random pitch errors (relation with kinematic errors is also shown). This paper describes methods to find the density function of the maximum dynamic load along the path of contact during start-stop operations under variable external loads and speeds. If we stop the clock, then we will find a load distribution on tooth like in fig. 1.

Load \(p(x, y)\) will produce the stresses \(\sigma_{H}, \sigma_{1}, \sigma_{2}, \sigma_{12}\), where \(x, y\) are variables along the path of contact and width of tooth. There are three important zones A, B, C shown in Fig. 1 (tip, pith and root)—when random densities of max. unit load have been found in these zones next we can apply appropriate theories to find factor of safety or the probability of damage (for example rain flow count method), Serensen et al. (1975) or Kocanda and Szala (1985).

The paper deals with a certain useful method solving dynamically indeterminate contact problems which are applied to find the density of loading of a helical gear, however many other important technical problems
(bearings, clutches, breakes, etc.) can be solved in similar way.

Fig 1. Load distribution on a tooth of helical gear.

2. Static unit load

A static load distribution is important because it makes possibility to compare dynamic solution with the static ones and then to find relation between density of probability of pith errors and kinematic errors of cooperating wheels. External moments acting on wheels (driver and follower) produces external force P. Rectangle of theoretical contact ABCD

Fig 2. Rectangle of theoretical contact ABCD.
in a plane of pressure is shown fig. 2, where: \(x, y\)-variables along lines of contact and path of contact, \(pb\)-base pith, \(\beta_b\)-base angle of h. gear, \(\epsilon\)-total contact ratio.

As a consequence, it appears to be possible to replace a rectangle of contact by a parallelogram ABCD (fig. 3).

\[\text{Fig 3. Computation rectangle of contact AECF.}\]

The new rectangle of contact AFCE is more convenient when we assume that in triangles ADE and BCF stiffness value is equal 0. In such a way we don't have to control theoretical lines of contact when Osy\(\geq pb\) and \(N=1, 2, \ldots, Ne\). Every number of pair of teeth \(N\) has got his own random pitch error \(\delta_p\) which can be covered nearly by the Gaussian parameter \(\sigma\) in \((-0.5 \delta_0, 0.5 \delta_0\) internal.

Consequently, a restoring force has a random character. An example of restoring-force-displacement diagram is plotted in fig. 4, either for 3 lines observed inside rectangle of contact (fig. 4a) or for 2 lines respectively (fig. 4b). When \(2<\epsilon<3\) the kinematic error (orthogonal to line contact direction) of wheels is described by \(\chi_{12}, \chi_{12}, \sigma_{13}, \sigma_{12}\), notation as in fig. 4. In particular cases, it will be found that
Fig. 4. Gaussian density describing restoring force of teeth when $2 < \varepsilon < 3$.

\begin{align}
1' \quad & \varepsilon < 2 \\
& x_1 = A_1 \sigma \Delta \varepsilon ; \quad \sigma_1 = \sigma \sqrt{1 - \Delta \varepsilon (1 - B_1^2)} ,
\end{align}

\begin{align}
2' \quad & 2 < \varepsilon < 3 \\
& x_1 = \sigma [A_1 + \Delta \varepsilon (C_1 - A_1)]; \quad \sigma_1 = \sigma \sqrt{B_1^2 - \Delta \varepsilon (B_1^2 - D_1^2)} ,
\end{align}

\begin{align}
3' \quad & \varepsilon < 4 \\
& x_1 = \sigma [C_1 + \Delta \varepsilon (G_1 - C_1)]; \quad \sigma_1 = \sigma \sqrt{D_1^2 - \Delta \varepsilon (D_1^2 - K_1^2)} ,
\end{align}

where

\[ \Delta \varepsilon = \varepsilon - \text{INT} \varepsilon, \]

\[ \sigma_\varepsilon = \sigma_1 / \cos \beta_b - \text{Gaussian parameter of kinematic errors}, \quad A_1 = 0.567, \quad B_1 = 0.814, \quad C_1 = -0.862, \quad D_1 = 0.679, \quad G_1 = -1.04, \quad K_1 = 0.56 \quad (\text{as an example} \quad \sigma = 0.58; \quad \varepsilon = 2.7; \quad \sigma_1 = 0.41). \]

According to equation (2.1) fig. 5 shows comparison of Gaussian parameter of kinematic errors $\sigma_\varepsilon$ with Gaussian parameter of pitch errors $\sigma$. In order to determine constant parameters $A_1, \ldots, K_1$, some simple computer programs were set up, the Author does not know analytical solution of this mathematical problem.

The maximum kinematic error $\delta_k$ is determined by the following equation

\[ \delta_k = Y \delta_0 / \cos \beta_b . \]
It therefore seems to be possible to account for maximum pitch error δ₀ using kinematic tests (with external static load P=0).

Let us consider the static load and deformation u if \( N=1, \ldots, N_e \) \((N_e \approx 10^3)\) and for every \( N, O_s \approx P_o \).

When

\[
0 \leq y < \Delta \xi \quad p_b \quad k=\text{INT} \xi + 1 ,
\]

\[
\Delta \xi \quad p_b \leq y \leq p_b \quad k=\text{INT} \xi ,
\]

where \( k \)-number of theoretical lines of contact (fig.3). The problem of evaluation of internal loadings is statically indeterminate.

External load for \( i=1, \ldots, n \) equals

\[
p_i = \sum_{l=1}^{n} \sum_{j=1}^{n} (u - \delta l) c_{j, i+(i-1)l} ,
\]

\[
c_{j, i+(i-1)l} = 0 \quad \text{when} \quad u<\delta l ,
\]

where

\[
\delta l = \delta M , \quad \delta 2 = \delta M - 1 , \quad \delta k = \delta M - k ,
\]

\[
\Delta x = \frac{E}{\mu} , \quad \Delta y = \frac{P_o}{n} , \quad \Delta \xi = q \Delta y ,
\]

\( c_{i,j} \) -matrices of stiffness in a new rectangle of contact like in fig.3, which is divided into three zones:
If \(0 \leq i \leq q\) and \(1 \leq j \leq m\), then unit loading equals

\[
\begin{align*}
\text{PA} &= \text{MAX}(P_{i,j}) \\
\text{PC} &= \text{MAX}(P_{k,j}) \\
\text{PB} &= \text{MAX}(P_{2,j}, \ldots, P_{k-1,j})
\end{align*}
\]

and if \(q < 1 \leq n\) then, \(\text{PB} = \text{MAX}(P_{1,j}, \ldots, P_{k-1,j}, P_{n})\)

\[
\begin{align*}
P_{i,j} &= (u - \delta_1) C_{j,1} \times (1 - \epsilon) n \quad \text{when } u - \delta_1 > 0 \\
P_{i,j} &= 0 \quad \text{when } u - \delta_1 \leq 0
\end{align*}
\]

\[\text{Fig. 6. Maximum static unit load as a function of number of repetitions (P-external load).}\]

Random errors change the value of maximum unit load \(PA\), \(PB\), \(PC\) for any number \(N\).

\[\text{Fig. 6 summarizes the results of numerical computation of the gear:} \]

\[
\begin{align*}
\epsilon_a &= 1.4 \; ; \; \epsilon_b = 1.2 \; ; \; \epsilon = 2.6 \; ; \; n = 5 \; ; \; m = 7, \; \delta_0 = 10 \, \mu m \; ; \\
C_{i,j} &= (0.7 + 2.0) \, kN/\mu m.
\end{align*}
\]
3. Dynamic unit load

Under the assumption stated before the equation of motion is based on Rys and Stachon theory (1987):

\[ u'' + u' \frac{v' + 10^4 T \sqrt{20C/M}}{v} + u \frac{10^8 C}{Mv^2} = \frac{10^8 (P+D)}{M^2} \]

if \( u > \text{MIN}\{\delta_i\} \), \( i = 1...k \), \hspace{1cm} (3.1)

or

\[ u'' = \frac{10^6 P}{Mv^2} \]

if \( u \leq \text{MIN}\{\delta_i\} \),

where

- \( u \) – relative dynamic displacement (\( \mu \)m),
- \( u' \) – relative velocity (\( \mu \)m/s),
- \( P = P(t) \) – external load (kN),
- \( v = v(t) \) – velocity of teeth (tooth/s),
- \( v' = v'(t) \) – acceleration of teeth (tooth/s^2),
- \( M \) – reduced mass (kg),
- \( T \) – coefficient of damping (-), \( T = (0.2 + 0.4) \) by Airapetow (1981),

\[
C = \sum_{i=1}^{k} \sum_{j=1}^{n} C_{j,i+(i-1)n} \left[ \begin{array}{c}
\end{array} \right]
\]

\[
D = \sum_{i=1}^{k} \sum_{j=1}^{n} \delta_i \cdot C_{j,i+(i-1)n} \left[ \begin{array}{c}
\end{array} \right]
\]

and \( C_{j,i+(i-1)n} = 0 \) if \( u \leq \delta_i \),

for \( i = 1...n \), \hspace{1cm} (3.2)

Time is connected with \( i \) and \( N \):

- if \( 1 = n \) then \( N = N + 1 \) and \( n = 1; i \leq N \leq N_0 \),
- \( v_0 = v(N = 1, i = 1) \) – input velocity,
- \( a_0 = v'(N = 1, i = 1) \) – input acceleration,
\[ v(N,1) = \sqrt{[v(N)]^2 + 2\omega(i - 1)/n} \quad (3.3) \]

\[ v(N) = \sqrt{[v(N-1)]^2 + 2\omega}, \quad 1 \leq i \leq n \quad (3.3) \]

If \( N = \text{const} \) and \( i = \text{const} \), interval of time is \( \Delta t = 1/n \), \( v(N,1) \) and equations (3.1) are linear on step \( \Delta t < \Delta t \).

\[ u'' + 2\gamma u' + u^2 u = S_1 \quad \text{and} \quad u > \min \{ \delta_i \} \quad (3.4) \]

\[ u'' = S_2 \quad \text{and} \quad u \leq \min \{ \delta_i \} \]

Thus the whole calculation procedure is based on testing of analytical solution and restrictions on time \( 0 < \Delta t \). Solution \( u(t) \) is plotted for \( N = 1, 2, \ldots, N_0 \) and maximum unit load \( P_a; P_b; P_c \) could be found with (2.6) on every step \( N, i \).

The numerical examples obtained thus are presented below

**DYNAMIC LOAD OF H. GEARS**

- max. ext. load (kN) = 95.0;
- min ext. load (kN) = 35.0
- max. pitch error (mm) = 0.010;
- reduced mass (kg) = 100.0
- min. velocity (tooth/s) = 1;
- max. velocity (tooth/s) = 50
- coeff. of damping (-) = 0.4;
- input accel. (tooth/s²) = 20

**max. unit load**

<table>
<thead>
<tr>
<th>transient vibr. in zones</th>
<th>steady vibr. in zones</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>3.743</td>
<td>8.597</td>
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<tr>
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<td>1.897</td>
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Fig. 7. shows the results, namely dynamic unit loads for the data: \( \varepsilon = 2.6; \quad m = 5; \quad m = 7; \quad \delta_0 = 10 \, [\mu\text{m}]; \quad N = 100 \, [\text{kg}]; \quad T = 0.4; \quad C_1 = (0.7 + 2.0) \, [\text{kN/\mu m}]; \quad v = (1 + 50) \, [\text{tooth/s}] \) and three variants of start conditions are considered.
Fig. 7. Dynamic unit load as a function of number repetitions

The observed tendency is a direct effect of dynamic shocks and tooth separation if \( a_0 = 20 \) and smooth cooperation of teeth if \( a_0 = 10 \) or 30 tooth/s².

References

(in Polish), Warsaw: PWN

Summary

NIEUSTALONE I USTALONE DRGANIA W PRZEKLADNI ZEBATEJ
O ZEBACH SKOSNYCH

W pracy analizowano statyczne i dynamiczne obciążenie zębów przekładni z uwzględnieniem losowych odchylek kinematycznych, zajęto się głównie problemem rozruchu z racji występowania wówczas zewnętrznych przeciążeń dynamicznych. Zапрzedstawiona metoda pozwala na określenie maksymalnych obciążen na jednostkę szerokości kola dla określonych stref współpracy w ujęciu losowym. Praca stwarza możliwość obliczania współczynników bezpieczeństwa lub prawdopodobieństwa uszkodzeń (np. złomu zmnęczeniowego, uszkodzenia powierzchni itp.) dla znanych warunków obciążenia przekładni.