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BOUNDARY ELEMENTS FOR THERMO-ELASTO-PLASTICITY OF METALS

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1. Introduction

Singularities contained in the boundary element method represent one of its main advantages over finite elements, enabling a better approximation of stress distributions due to shape changes and/or variation of boundary conditions, like supports, contacts etc. Also, the resulting system of equations is smaller with BEM than with FEM, but, unfortunately, its matrix is fully populated. In this paper governing equations and numerical results (for a metal wall and plate bending) are given for the initial strain concept of BEM elasto-plastic formulation, bearing in mind body forces and thermal loads. For the case of large displacements, an updated Lagrange technique has been developed and is also reported in this paper.

2. Theoretical background

Mechanical properties of metals may be divided into elastic, plastic and viscous. In this paper, viscous influence has been neglected. In the elastic domain a complete reversibility can be observed, and at each point of time there is a unique relationship of loads and deformations, while for the plastic regime permanent deformations are due to occur, which are dependent upon the history of loading. If only small strain rates apply:

$$\dot{\varepsilon}_{ij} = \frac{1}{2}(\dot{u}_{i,j}+\dot{u}_{j,i}) = \dot{\varepsilon}^{e}_{ij}+\dot{\varepsilon}^{t}_{ij}+\dot{\varepsilon}^{p}_{ij} \tag{1.1}$$

an incremental formulation may be written by a formal multiplication with a time step $dt > 0$, which is also typical in the classical elasto-plastic analysis. For the description of material properties, elasticity and plasticity laws are required, where the last one is composed of a yield criterion and flow rule. Considering a bilinear material, one dimensional yield stress formulation reads as:

$$\sigma_y = \sigma_0 + E_t\,\varepsilon^p/(1-E_t/E), \tag{1.2}$$

and the yield criterion is:

$$F(\sigma, k, \Theta) = |\sigma| - \sigma_y(k, \Theta) = 0, \tag{1.3}$$

which is dependent on stress (σ), hardening (k) and temperature (θ). For the multidimensional modelling the Mises-Huber criterion is applicable:

$$F(\sigma_{ij}, k, \Theta) = \sqrt{3/2 S_{ij} S_{ij}} - \sigma_y(k, \Theta) = 0, \tag{1.4}$$

where S_{ij} is the deviatoric stress and k material hardening coefficient:

$$k = W^p = \int \sigma_{ij} d\varepsilon_{ij}^p. \tag{1.5}$$

For the plastic flow Prandtl-Reuss equation may be used in its incremental form:

$$d\varepsilon_{ij}^p = S_{ij} d\lambda. \tag{1.6}$$

In order to corelate the unidimensional state, a comparative stress shall be used:

$$\sigma_v = \sqrt{3/2 S_{ij} S_{ij}}, \tag{1.7}$$

and also a corresponding comparative plastic strain increment:

$$d\varepsilon_v^p = \sqrt{2/3 d\varepsilon_{ij}^p d\varepsilon_{ij}^p}. \tag{1.8}$$

Using the associated yield criterion, the proportionality factor is:

$$d\lambda = 3/2 d\varepsilon_v^p / \sigma_v. \tag{1.9}$$

Additionally, there are also incremental equilibrium conditions, valid both in the elastic and in the plastic regime:

$$\dot{\sigma}_{ij,i} + \dot{b}_j = 0, \quad \dot{\sigma}_{ij} = \dot{\sigma}_{ji}, \quad \dot{p}_i = \dot{\sigma}_{ij} n_j. \tag{1.10}$$

For the elastic strain increment:

$$\dot{\varepsilon}_{ij}^e = \dot{\varepsilon}_{ij} - \dot{\varepsilon}_{ij}^t - \dot{\varepsilon}_{ij}^p, \tag{1.11}$$

the Hookean constitutive law is applicable:

$$\dot{\sigma}_{ij} = 2G(\dot{\varepsilon}_{ij} - \dot{\varepsilon}_{ij}^t - \dot{\varepsilon}_{ij}^p) + 2G\nu\delta_{ij}(\dot{\varepsilon}_{kk} - \dot{\varepsilon}_{kk}^t - \dot{\varepsilon}_{kk}^p)/(1-2\nu), \tag{1.12a}$$

where $\dot{\varepsilon}_{ij}^p$ represents initial strain. Analogously by defining:

$$\dot{\sigma}_{ij}^p = 2G\dot{\varepsilon}_{ij}^p + 2G\nu\delta_{ij}\dot{\varepsilon}_{kk}^p/(1-2\nu),$$

the above formulation reads for the initial stress:

$$\dot{\sigma}_{ij} = 2G(\dot{\varepsilon}_{ij} - \dot{\varepsilon}_{ij}^t) + 2G\nu\delta_{ij}(\dot{\varepsilon}_{kk} - \dot{\varepsilon}_{kk}^t)/(1-2\nu) - \dot{\sigma}_{ij}^p. \tag{1.12b}$$

Incorporating kinematic relations (1.1) into (1.12a) and (1.12b), and then to the equilibrium equations (1.10a), the generalised Navier-Lamé equation is obtained:

$$\dot{u}_{j,kk} + u_{k,kj}/(1-2\nu) = \dot{\bar{b}}_j/G, \tag{1.13}$$

where generalised body forces $\dot{\bar{b}}_j$ include thermal loading.
The direct solution of this equation leads to the finite differences or to the finite element technique, while by converting this equation into an integral form, the boundary element method may be derived.

3. Integral formulation

Starting from the weighted residual formulation of the equilibrium equation:

$$\int (\dot{\sigma}_{jk,j}+\dot{b}_k)u_k^*\,dV = 0,$$

where the weighting displacement function u_k^* depends on the Kelvin fundamental solution for an infinite domain. Equation (2.1) may be converted into a generalised Somigliana equation:

$$\dot{u}_i(\xi) = \int (u_{ik}^*\dot{p}_k - p_{ik}^*\dot{u}_k)\,dA + \int (u_{ik}^*\dot{b}_k + u_{ik,k}^*\dot{\Theta})\,dV + \int \sigma_{ijk}^*\dot{\varepsilon}_{jk}^p\,dV, \tag{2.2}$$

with initial deformations $\dot{\varepsilon}_{ij}^p$, while u_{ij}^* and p_{ij}^* are displacements or tractions at point $\vec{r}$ due to the unit body force in i direction at source point ξ in Kelvin's space, σ_{ijk}^* being stress kernel. The volume integral, which includes contributions of body forces and thermal loads, may be transformed into a contour integral form. Alternatively particular solutions $(\bar{u},\bar{p})$ [4] can also be applied, transforming equation (2.2) into:

$$\dot{u}_i(\xi)-\dot{\bar{u}}_i(\xi) = \int \big(u_{ik}^*(\dot{p}_k-\dot{\bar{p}}_k)-p_{ik}^*(\dot{u}_k-\dot{\bar{u}}_k)\big)\,dA + \int \sigma_{ijk}^*\dot{\varepsilon}_{jk}^p\,dV. \tag{2.3}$$

Bringing the source point ξ to the contour, the basic integral equation is obtained for nodes on the boundary:

$$C_{ik}(\dot{u}_k-\dot{\bar{u}}_k) = \int \big(u_{ik}^*(\dot{p}_k-\dot{\bar{p}}_k)-p_{ik}^*(\dot{u}_k-\dot{\bar{u}}_k)\big)\,dA + \int \sigma_{ijk}^*\dot{\varepsilon}_{ik}^p\,dV. \tag{2.4}$$

For the iterative process, stress values have to be determined. At the interior, these can be evaluated from the Somigliana equation (2.3), performing the derivation and bearing in mind the Hookean relationship (1.12a) of the elastic part of total displacements:

$$\sigma_{ij}-\bar{\sigma}_{ij} = \int \big(U_{kij}^*(p_k-\bar{p}_k)-P_{kij}^*(u_k-\bar{u}_k)\big)\,dA - \delta_{ij}\,\alpha E\Theta/(1-2\nu) + $$
$$+\int \Sigma_{ijkm}^*\varepsilon_{km}^p\,dV - D_{ij}\,\varepsilon_{km}^p(\xi), \tag{2.5}$$

where all tensors with an asterisk (*) are derived from the Kelvin fundamental solution, while vectors with a dash ($^-$) are particular solutions for body forces and temperature field with constant gradients.
For boundary nodes (2D) two stress tensor components appear to be known, and the third component may be determined by means of numerical derivatives.

4. Discretization and algebraization

In the case of elasto-plastic computation by the boundary element technique, nodalization is required not only on the contour, but also in one part of the interior where plastic zone is due to appear. Internal cells are to be used for the volume integration of plastic strain contributions, but they do not increase the number of algebraic equations. With N boundary nodes and $2N$ unknowns, only a system of $2N$ equations is obtained:

$$\underline{\underline{H}}\underline{u} = \underline{\underline{G}}\underline{p}+\underline{b}+\underline{\underline{S}}\underline{\varepsilon}^p. \tag{4.1}$$

Taking into account the prescribed boundary values, the system is written as:

$$\underline{A}\underline{x} = \underline{f} + \underline{S}\varepsilon^p, \tag{4.2}$$

and its solution is:

$$\underline{x} = \underline{m} + \underline{K}_1 \varepsilon^p. \tag{4.3}$$

Stresses have to be evaluated at N boundary nodes and M internal points (i.e. $3*(M+N)$ equations):

$$\underline{\sigma} = \hat{\underline{G}}\underline{p} + \hat{\underline{H}}\underline{u} + \hat{\underline{b}} + (\hat{\underline{S}} + \underline{D})\underline{\varepsilon}^p. \tag{4.4}$$

Taking into account the known boundary values, it gives:

$$\underline{\sigma} = \hat{\underline{A}}\underline{x} + \hat{\underline{f}} + (\hat{\underline{S}} + \underline{D})\underline{\varepsilon}^p, \tag{4.5}$$

for $\underline{x}$ the solution of (4.3) has to be considered, rendering:

$$\underline{\sigma} = \hat{\underline{A}}\underline{m} + \hat{\underline{f}} + (\hat{\underline{A}}\underline{K}_1 + \hat{\underline{S}} + \underline{D})\underline{\varepsilon}^p = \underline{n} + \underline{K}_2\underline{\varepsilon}^p. \tag{4.6}$$

5. Solution procedure

In the preceeding formulation increments of plastic deformation have been taken as formally known. In the reality these values have yet to be determined by an incremental procedure:

A) At the first step a complete elastic computation is performed, using the full load (prescribed tractions, displacements, body forces and thermal loading):

$$\underline{x}^e = \underline{m}, \quad \underline{\sigma}^e = \underline{n}. \tag{5.1}$$

Consecutively, the load is to be adjusted to meet the yield criterion at the mostly loaded node:

$$L_0 = \sigma_0/\max(\sigma_v) \rightarrow \underline{x}_0 = L_0\underline{m}, \quad \underline{\sigma}_0 = L_0\underline{n}. \tag{5.2}$$

B) Next, an incremental part of the load is used. After the l-th step the load factor is determined selecting an increment ω:

$$L_l = L_{l-1} + L_0\omega, \tag{5.3}$$

and the unknown boundary values are:

$$\underline{x}_l = L_l\underline{m} + \underline{K}_1, \underline{\varepsilon}^p{}_l \quad \underline{\sigma}_l = L_l\underline{n} + \underline{K}_2\underline{\varepsilon}^p{}_l. \tag{5.4}$$

For the evaluation of plastic deformations, plastic strains are separated into accumulated strains (from the previous increments) and actual strain increments:

$$\varepsilon^p_{ij}(l) = \bar{\varepsilon}^p_{ij}(l-1) + \Delta\varepsilon^p_{ij}(l). \tag{5.5}$$

C) The plastic strain increment is determined iteratively [1]. The procedure starts from the old value at each of $M+N$ points, producing stress values:

$$\underline{\sigma}(l) = L_l\underline{n} + \underline{K}_2(\bar{\underline{\varepsilon}}^p(l-1) + \Delta\underline{\varepsilon}^p(l)) \tag{5.6}$$

Next, modified strains may be determined:

$$\varepsilon'_{ij} = \varepsilon_{ij} - \bar{\varepsilon}^p_{ij} = \varepsilon^e_{ij} + \varepsilon^t_{ij} + \Delta\varepsilon^p_{ij}, \tag{5.7}$$

and also deviatoric strains:

$$e'_{ij} = \varepsilon'_{ij} - \delta_{ij}\varepsilon'_{mm}/3. \tag{5.8}$$

Using the Hookean law of elasticity, deviatoric strains are evaluated from stresses:

$$e'_{ij} = S_{ij}/2G + \Delta\varepsilon^p_{ij}. \tag{5.9}$$

Plastic strain increments can now be evaluated using Prandtl-Reuss rule:

$$\Delta\varepsilon^p_{ij} = \Delta\lambda S_{ij}. \tag{5.10}$$

For the determination of the yield point of a bilinear material, the following relation is to be used:

$$\sigma_v(l) = \sigma_v(l-1) + E_t\Delta\varepsilon^p_v/(1-E_t/E). \tag{5.11}$$

From the last interval plastic strain increments are obtained:

$$^{old}(\Delta\varepsilon^p_v) = \left(3G\sqrt{2/3e'_{ij}e'_{ij}} - \sigma_v(l-1)\right)/\left(3G + E_t/(1-E_t/E)\right), \tag{5.12}$$

and now for the new interval the strain components become:

$$^{new}(\Delta\varepsilon^p_v) = \Delta\varepsilon^p_v e'_{ij}/\sqrt{2/3e'_{ij}e'_{ij}}. \tag{5.13}$$

By recursion, starting from equation (5.6) and repeating the procedure until the required convergence criterion:

$$^{old}(\Delta\varepsilon^p_v) \simeq {}^{new}(\Delta\varepsilon^p_v) \tag{5.14}$$

is met, the last load increment gives:

$$\underline{x} = \underline{x}^e + \underline{K}_1(\underline{\varepsilon}^p(l-1) + \Delta\underline{\varepsilon}^p(l)), \quad \underline{\sigma} = \underline{\sigma}^e + \underline{K}_2(\underline{\varepsilon}^p(l-1) + \Delta\underline{\varepsilon}^p(l)). \tag{5.15}$$

As an example of the described procedure, a thermally loaded metal wall analysis has been performed, using temperature dependent material properties $\sigma_y(\theta)$ and $E_t(\theta)$. The temperature field has been kept steady, 0°C at the upper and 480°C at the lower side. The yield point has been 310 MPa (20°C) and 175 MPa (500°C), and the tangent modulus of 36 GPa (20°C) and 30 GPa (500°C) respectively have been considered with linear variation between specified limits. Results of the analysis are given in Fig. 1.

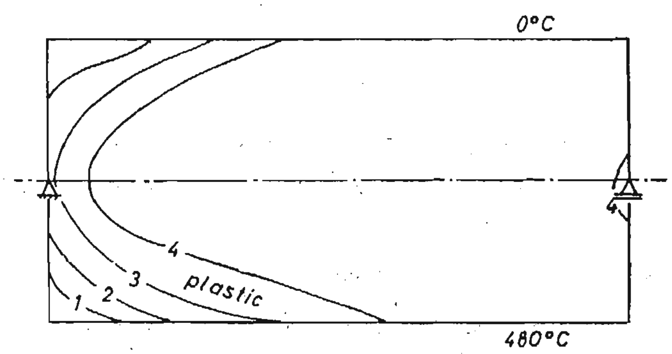

Fig. 1. Plastic zone development in a thermally loaded wall

6. Updated Lagrange routine

In many engineering problems there are small strains but large geometrical changes. For such cases an "updated" Lagrange procedure has been developed, consisting of the following steps:

A) Evaluation stops if a maximum displacement reaches a specified value.
B) New geometry is determined using computed displacements.
C) For each computational point, a rotation with respect to the previous position is determined.
D) Stresses and strains from the preceeding steps are added and rotated for the new geometry.
E) Correction of the yield point.
F) Correction of boundary conditions (supports and contacts).
G) Restart of the elasto-plastic computation (i.e. back to A).

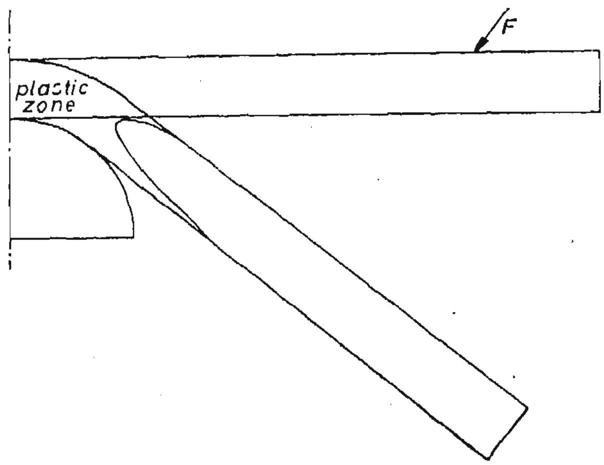

Fig. 2. Plastic zone development in a bending plate sheet

As an example of this kind of structural behaviour, plastic bending problem of a plate has been evaluated. Results of the computation are shown in Fig. 2 (plane strain case of a metal plate, bent over a rigid support cylinder).

7. Conclusion

In the above paper, theory of the boundary element method for plasticity problems has been demonstrated. Results of two typical problems have been presented.

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Р е з ю м е

КРАЕВЫЕ ЭЛЕМЕНТЫ В ЗАДАЧАХ ТЕРМО-УПРУГО-ПЛАСТИЧНОСТИ МЕТАЛОВ

В работе выведены управляющие уравнения и представлены численные результаты для двух тыпичных, упруго-пластических задач касающихся начальной деформации в формулировке метода краевых элементов. Учтено массовые силы и теоретическую нагрузку.

Для случая больших перемещеней, в работе развито современную формулировку метода Лагранжа, силлюстрированную на примере изгиба металлической пластинки на жесткой опоре.

S t r e s z c z e n i e

ELEMENTY BRZEGOWE W ZAGADNIENIACH TERMO-SPRĘŻYSTO-PLASTYCZNOŚCI METALI

W pracy zostały wyprowadzone równania rządzące i przedstawione wyniki liczbowe dwu typowych zagadnień sprężystoplastycznych dotyczących początkowego odkształcenia w sformułowaniu MEB. Uwzględniono siły masowe i obciążenia termiczne. W przypadku dużych przemieszczeń w pracy rozwinięto uwspółcześnioną metodę Lagrange'a, zilustrowaną przykładem zagadnienia zginania płyty metalowej na sztywnej podporze.

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