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APPLICATION OF WAVE METHOD IN INVESTIGATION OF DRIVE SYSTEMS COMPARISONS WITH OTHER METHODS

AMALIA PIELORZ

IPPT PAN, Warszawa

1. Introduction

The paper concerns dynamic investigations of drive systems with variable and constant shaft cross-sections using the wave solution of motion equations. The model of a drive system consists of shafts and rigid bodies with constant mass moments of inertia with respect to the axis of rotation. Rigid bodies are loaded by external moments which practically can be arbitrary. Considerations concern those systems where supporting bearings eliminate flexural deformations and shafts are mainly torsionally deformed. Damping appearing in these systems is taking into account by an equivalent damping, which is compared with a damping continuously distributed in the case of a unilaterally fixed rod torsionally deformed. Moreover, results obtained by means of the wave method are compared with suitable results obtained by means of the rigid finite element method and with the method of separation of variables.

It should be pointed out that dynamic investigations of drive systems are carried mostly out by means of discrete models, [1]. In literature also discrete-continuous models are used likewise in the present paper, [2, 3], which more precisely describe real systems but require slightly different methods for solution. The method of separation of variables the most often is applied in these studies. It allows, in principle, to consider undamped systems and to determine natural frequencies and eigenfunctions, [2 - 4]. Using the wave solution of motion equations one can determine displacements, strains and velocities in arbitrary shaft cross-sections at an arbitrary time instant.

2. Wave method in investigation of drive systems

In this section the wave method is presented in the case of the discrete-continuous model of a drive system with variable cros-section of shafts. The method is based on the utilization of wave solution of appropriate motion equations. It can be applied for shafts with a constant and variable cross-section, however in the last case the functions representing variable cross-sections should be such that the equations of motion have solutions of the d'Alembert type.

2.1. Drive systems with variable shaft cross-section. Consider a multi-mass drive system consisting of an arbitrary number of rigid bodies connected by means of shafts. The shafts consist of segments with variable polar moment of inertia. The method proposed may be easily applied to the discussion of models for drive systems with an arbitrary number of segments, however in order to get clearer and simpler analytical formulae the analysis is limited to the case when each shaft consists of two segments, Fig. 1.

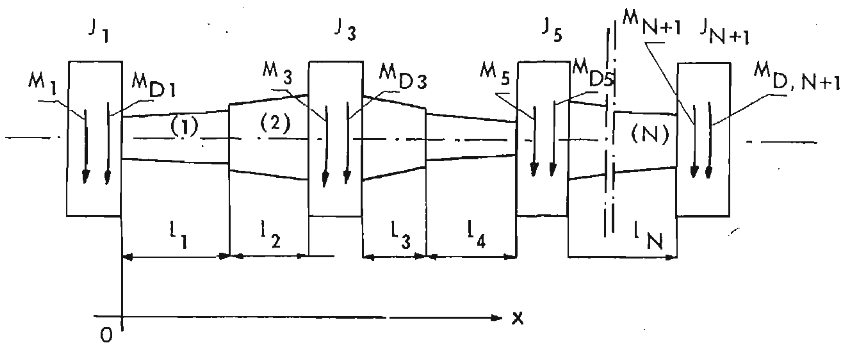

Fig. 1. Model of a drive system

The shafts are deformable only in torsion-like manner and their central axes, together with elements settled on them, coincide with the main axis of the drive system. It is assumed that the x axis is parallel to the main axis of the drive system, and that its origin coincides with the location of the left end of the first shaft in an undisturbed state at time instant $t = 0$. Moreover, damping is taken into account by means of an equivalent damping.

The i-th shaft segment, $i = 1, 2, \ldots, N$, where N is an even number, is characterized by the length l_i, density ϱ, shear modulus G and variable polar moment of inertia J_{0i} which is described by the function:

$$J_{0i}(x) = J_{pi}\left(\frac{x-b_{0i}}{L_{i-1}-b_{0i}}\right)^4, \tag{1}$$

where: $J_{0i}(b_{0i}) = 0$, $J_{pi} = J_{0i}(L_{i-1})$, $L_i = l_1+l_2+ \ldots +l_i$. If $b_{0i} \to -\infty$ then function (1) is constant, therefore shaft cross-sections can be constant, piece-wisely constant, variable and piece-wisely variable. Other forms of function J_{0i}, suitable motion equations for which have the solution of the d'Alembert type, one can find in [5 - 7]. The rigid bodies of the system, with mass moments of inertia J_i, are loaded by external moments $M_i(t)$. The moments of an equivalent damping, also loaded these bodies, are assumed in the form:

$$\begin{aligned} M_{Di}(t) &= -D_i\Theta_{i,t}(x,t) \quad \text{for} \quad x = L_{i-1}, \quad i = 1, 3, \ldots, N, \\ M_{D,N+1}(t) &= -D_{N+1}\Theta_{N,t}(x,t) \quad \text{for} \quad x = L_N, \end{aligned} \tag{2}$$

where D_i are the coefficients of the equivalent damping of viscous type, Θ_i angular displacements of shafts, and comma denotes partial differentiation. Moments M_{Di} act in selected shaft cross-sections, and thus this aassumption allows to apply motion equations without damping. Moreover, it is assumed that displacements and velocities of shaft cross-sections are equal to zero at time instant $t = 0$.

Under above assumptions, the determination of angular displacements and velocities of the system shown in Fig. 1 is reduced to the solution of wave equations, [7],

$$\Theta_{i,tt} - c^2\left(\Theta_{i,xx} + \frac{2}{x - b_{0i}}\Theta_{i,x}\right) = 0, \quad i = 1, 2, \ldots, N \tag{3}$$

with boundary conditions;

$$\begin{gathered} M_1(t) - J_1\Theta_{1,tt} + GJ_{01}\Theta_{1,x} - D_1\Theta_{1,t} = 0 \quad \text{for} \quad x = 0, \\ J_{0,i-1}\Theta_{i-1,x} = J_{0i}\Theta_{i,x} \quad \text{for} \quad x = L_{i-1}, \quad i = 2,4, \ldots, N, \\ M_i(t) - J_i\Theta_{i,tt} + GJ_{0i}\Theta_{i,x} - GJ_{0,i-1}\Theta_{i-1,x} - D_i\Theta_{i,t} = 0 \quad \text{for} \quad x = L_{i-1}, \\ i = 3,5, \ldots, N-1, \\ M_{N+1}(t) - J_{N+1}\Theta_{N,tt} - GJ_{0N}\Theta_{N,x} - D_{N+1}\Theta_{N,t} = 0 \quad \text{for} \quad x = L_N, \\ \Theta_{i-1} = \Theta_i \quad \text{for} \quad x = L_{i-1}, \quad i = 2, 3, \ldots, N \end{gathered} \tag{4}$$

and with initial conditions

$$\Theta_i(x\,0) = \Theta_{i,t}(x, 0) = 0, \quad i = 1, 2, \ldots, N, \tag{5}$$

where $c^2 = G/\varrho$. In the case of constant polar moment of inertia J_{0i} equations (3) become clasical wave equations.

Upon the introduction of the following nondimensional quantities

$$\begin{gathered} \bar{x} = x/(l_1 + l_2), \quad \bar{t} = ct/(l_1 + l_2), \quad \bar{\Theta}_i = \Theta_i/\Theta_0, \quad \bar{D}_i = D_i(l_1 + l_2)/(J_1 c), \\ \bar{M}_i = M_i(l_1 + l_2)^2/(J_1\Theta_0 c^2), \quad K_i = J_{pi}\varrho(l_1 + l_2)/J_i, \quad E_i = J_1/J_i, \\ B_i = J_{pi}/J_{p,i-1}, \quad \bar{l}_i = l_i/(l_1 + l_2), \quad \bar{J}_{0i} = J_{0i}/J_{pi}, \quad \bar{b}_{0i} = b_{0i}/(l_1 + l_2) \end{gathered} \tag{6}$$

relations (3) - (4), omitting bars for convenience, are

$$\Theta_{i,tt} - \Theta_{i,xx} - \frac{2}{x - b_{0i}}\Theta_{i,x} = 0, \quad i = 1, 2, \ldots, N, \tag{7}$$

$$\begin{gathered} M_1 - \Theta_{1,tt} + K_1 J_{01}\Theta_{1,x} - D_1\Theta_{1,t} = 0 \quad \text{for} \quad x = 0, \\ J_{0,i-1}\Theta_{i-1,x} = B_i J_{0i}\Theta_{i,x} \quad \text{for} \quad x = L_{i-1}, \quad i = 2, 4, \ldots, N, \\ E_i M_i - \Theta_{i,tt} + K_i J_{0i}\Theta_{i,x} - K_{i-1}E_i J_{0,i-1}\Theta_{i-1,x}/E_{i-1} - E_i D_i\Theta_{i,t} = 0 \\ \text{for} \quad x = L_{i-1}, \quad i = 3, 5, \ldots, N-1, \\ E_{N+1}M_{N+1} - \Theta_{N,tt} - K_N J_{0N} E_{N+1}\Theta_{N,x}/E_N - E_{N+1}D_{N+1}\Theta_{N,t} = 0 \quad \text{for} \quad x = L_N, \\ \Theta_{i-1} = \Theta_i \quad \text{for} \quad x = L_{i-1}, \quad i = 2, 3, \ldots, N, \end{gathered} \tag{8}$$

where Θ_0 is the constant value of an angular displacement.

The solutions of the problem (7), (8) and (5) are sought in the form

$$\begin{gathered} \Theta_i(x, t) = \frac{1}{x - b_{0i}}\left[f_i(t - x + L_{i-1}) + g_i(t + x - L_{i-1})\right] \quad \text{for} \quad x = 1, 3, \ldots, N-1, \\ \Theta_i(x, t) = \frac{1}{x - b_{0i}}\left[f_i(t - x + L_i) + g_i(t + x - L_i)\right] \quad \text{for} \quad i = 2, 4, \ldots, N, \end{gathered} \tag{9}$$

where functions $(x - b_{0i})^{-1}f_i$ and $(x - b_{0i})^{-1}g_i$ represent waves, caused by an external loading, propagating in the i-th shaft segment in the direction consistent and opposite to the direction of the x axis, respectively. In the arguments of functions f_i and g_i it was taken into account that the first perturbation occurs in the i-th shaft segment at the time instant $t = 0$ in the cross-section $x = L_{i-1}$ or $x = L_i$, respectively, where $L_i = l_1+l_2+ +\ldots+l_i$. Furthermore it is assumed that the functions f_i and g_i are equal to zero for negative arguments. If the function J_{0i} is constant and equations (7) become classical wave equations then the sought solution (9) consists only of the sums of functions f_i and g_i with the same arguments, [8, 9].

Upon the substitution the assumed form of solution (9) into boundary conditions (8), upon denoting the largest argument in each equality by z and using the function (1) to the description of the variation of polar moment of inertia one gets the following system of equations for unknown functions f_i and g_i:

$$
\begin{aligned}
& r_{1i}f_i'(z)+r_{2i}f_i(z) = r_{3i}g_i'(z-2l_i)-r_{2i}g_i(z-2l_i) \\
& \quad +r_{4i}f_{i-1}'(z-l_{i-1}-l_i) \quad \text{for} \quad i = 2,4,\ldots,N, \\
& r_{1,i+1}g_i'(z)+r_{2,i+1}g_i(z) = -r_{3,i+1}f_i'(z-2l_i)-r_{2,i+1}f_i(z-2l_i) \\
& \quad +r_{4,i+1}'g_{i+1}'(z-l_i-l_{i+1}) \quad \text{for} \quad i = 1,3,\ldots,N-1, \\
& f_1''(z)+r_{11}f_1'(z)+r_{21}f_1(z) = C_1M_1(z)-g_1''(z)+r_{31}g_1'(z)-r_{21}g_1(z), \\
& f_i''(z)+r_{1i}f_i'(z)+r_{2i}f_i(z) = C_iM_i(z)-g_i''(z)+r_{3i}g_i'(z) \\
& \quad -r_{2i}g_i(z)+r_{4i}f_{i-1}'(z) \quad \text{for} \quad i = 3,5,\ldots,N-1, \\
& g_i''(z)+r_{1,i+1}g_i'(z)+r_{2,i+1}g_i(z) = C_iM_{i+1}(z)-f_i''(z) \\
& \quad +r_{3,i+1}'f_i'(z)-r_{2,i+1}f_i(z)+r_{4,i+1}'g_{i+1}'(z) \\
& \qquad \text{for} \quad i = 2,4,\ldots,N-2, \\
& g_N''(z)+r_{1,N+1}g_N'(z)+r_{2,N+1}g_N(z) = C_NM_{N+1}(z)-f_N''(z) \\
& \quad +r_{3,N+1}f_N'(z)-r_{2,N+1}f_N(z),
\end{aligned}
\tag{10}
$$

where:

$$
\begin{aligned}
r_{11} &= K_1+D_1, \quad r_{21} = -K_1/b_{01}, \quad r_{31} = K_1-D_1, \\
r_{1i} &= (L_{i-1}-b_{0,i-1})^2(L_{i-2}-b_{0,i-1})^{-2}+B_i, \quad i = 2,4,\ldots,N, \\
r_{2i} &= B_i(L_{i-1}-b_{0i})^{-1}-(L_{i-1}-b_{0,i-1})(L_{i-2}-b_{0,i-1})^{-2}, \quad i = 2,4,\ldots,N, \\
r_{3i} &= B_i-(L_{i-1}-b_{0,i-1})^2(L_{i-2}-b_{0,i-1})^{-2}, \quad i = 2,4,\ldots,N, \\
r_{4i} &= 2(L_{i-1}-b_{0,i-1})(L_{i-1}-b_{0i})(L_{i-2}-b_{0,i-1})^{-2}, \quad i = 2,4,\ldots,N, \\
r_{4i}' &= 2B_i(L_{i-1}-b_{0,i-1})(L_{i-1}-b_{0i})^{-1}, \quad i = 2,4,\ldots,N, \\
r_{1i} &= K_i+K_{i-1}E_i(L_{i-1}-b_{0,i-1})^2E_{i-1}^{-1}(L_{i-2}-b_{0,i-1})^{-2}+E_iD_i, \\
&\qquad i = 3,5,\ldots,N-1, \\
r_{2i} &= K_i(L_{i-1}-b_{0i})^{-1}-K_{i-1}E_i(L_{i-1}-b_{0,i-1})E_{i-1}^{-1}(L_{i-2}-b_{0,i-1})^{-2}, \\
&\qquad i = 3,5,\ldots,N-1, \\
r_{3i} &= -r_{1i}+2K_i, \quad r_{3i}' = -r_{3i}-2E_iD_i, \quad i = 3,5,\ldots,N-1, \\
r_{4i} &= 2K_{i-1}E_i(L_{i-1}-b_{0,i-1})(L_{i-1}-b_{0i})E_{i-1}^{-1}(L_{i-2}-b_{0,i-1})^{-2}, \\
&\qquad i = 3,5,\ldots,N-1,
\end{aligned}
\tag{11}
$$

$$r'_{4i} = 2K_i(L_{i-1}-b_{0,i-1})(L_{i-1}-b_{0i})^{-1}, \quad i = 3, 5, \ldots, N-1,$$
$$r_{1,N+1} = K_N E_{N+1}(L_N-b_{0N})^2 E_N^{-1}(L_{N-1}-b_{0N})^{-2}+E_{N+1}D_{N+1},$$
$$r_{2,N+1} = -K_N E_{N+1} E_N^{-1}(L_{N-1}-b_{0N})^{-2}(L_N-b_{0N}),$$
$$r_{3,N+1} = r_{1,N+1}-2E_{N+1}D_{N+1},$$
$$C_i = E_i(L_{i-1}-b_{0i}), \quad i = 1, 3, \ldots, N-1,$$
$$C_i = E_{i+1}(L_i-b_{0i}), \quad i = 2, 4, \ldots, N.$$

Equations (10) are differential equations with constant coefficients, however the arguments of several functions of the right-hand sides of these equations are shifted. These equations can be solved numerically by means of the finite difference method or analytically, as it was presented in [8, 9] for a drive system with a constant shaft cross-section.

2.2. Numerical results. Numerical calculations for nondimensional angular displacements $\Theta(x, t)$ are carried out in the case of a two-mass drive system. The method of finite differences with $\Delta z = 0.025$ is applied in order to solve equations (10) for $N = 2$, and next displacement functions are determined according to formulae (9).

The two-mass drive system is characterized by the following nondimensional parameters, (6): $l_1 = l_2 = 0.5$, $K_1 = 0.01$, $E_3 = 0.1$, $B_2 = 0.8$, 1.0, 1.25, $D_1 = D_3 = 1.0$, the parameters $b_{01} = -20$, -1000, $b_{02} = b_{01}-l_1$ occuring in formula (1), and $\Theta_0 = 1$ [rad], $c = 5000$ [m/s]. The effect of the quotient of polar moments of inertia of shaft segments $B_2 = J_{p2}/J_{p1}$ is investigated for the nondimensional external moments $M_1(t) = 7\cdot 10^{-5}\exp(-0.0044t)\cdot\sin(\pi t/70)$ and $M_3(t) = 0$. Displacements $\Theta(x, t)$ are plotted out in Fig. 2 for the three selected shaft cross-sections $x = 0$, 0.5, 1.0 and for $b_{01} = -20$, -1000. It follows from Fig. 2 that for any time instant differences between displacements for moderately changing shaft cross-sections with $b_{01} = -20$ and $b_{01} = -1000$ are small, and that the effect of the quotient $B_2 = J_{p2}/J_{p1}$ on displacements is most observable in the cross-section $x = 0.5$.

In the case of a two-mass drive system the effect of the lengths of shaft segments on displacements $\Theta(x, t)$ was also considered, namely for the lengths of shaft segments $l_1 = 0.2$, 0.4, 0.6, 0.8, $l_2 = 1.0-l_1$, and for $K_1 = 0.01$, 1.0 and $B_2 = 1.25$. All calculations indicated that this effect on angular displacements in shaft cross-sections under consideration was inconsiderable. For this reason, the appropriate diagrams for displacements are not presented in the paper.

3. Comparisons

The method, proposed in the paper for investigations of drive systems torsionally deformed, takes into account all reflectons of waves during the work of the drive system and leads to solving ordinary differential equations with a retarded argument. These equations are derived under the assumption that equivalent damping may be considered in boundary conditions. In real systems damping is distributed continuously. For the present, methods for solving appropriate motion equations with damping are not sufficiently effective to be used in investigations of discrete-continuous models of drive systems. A comparative

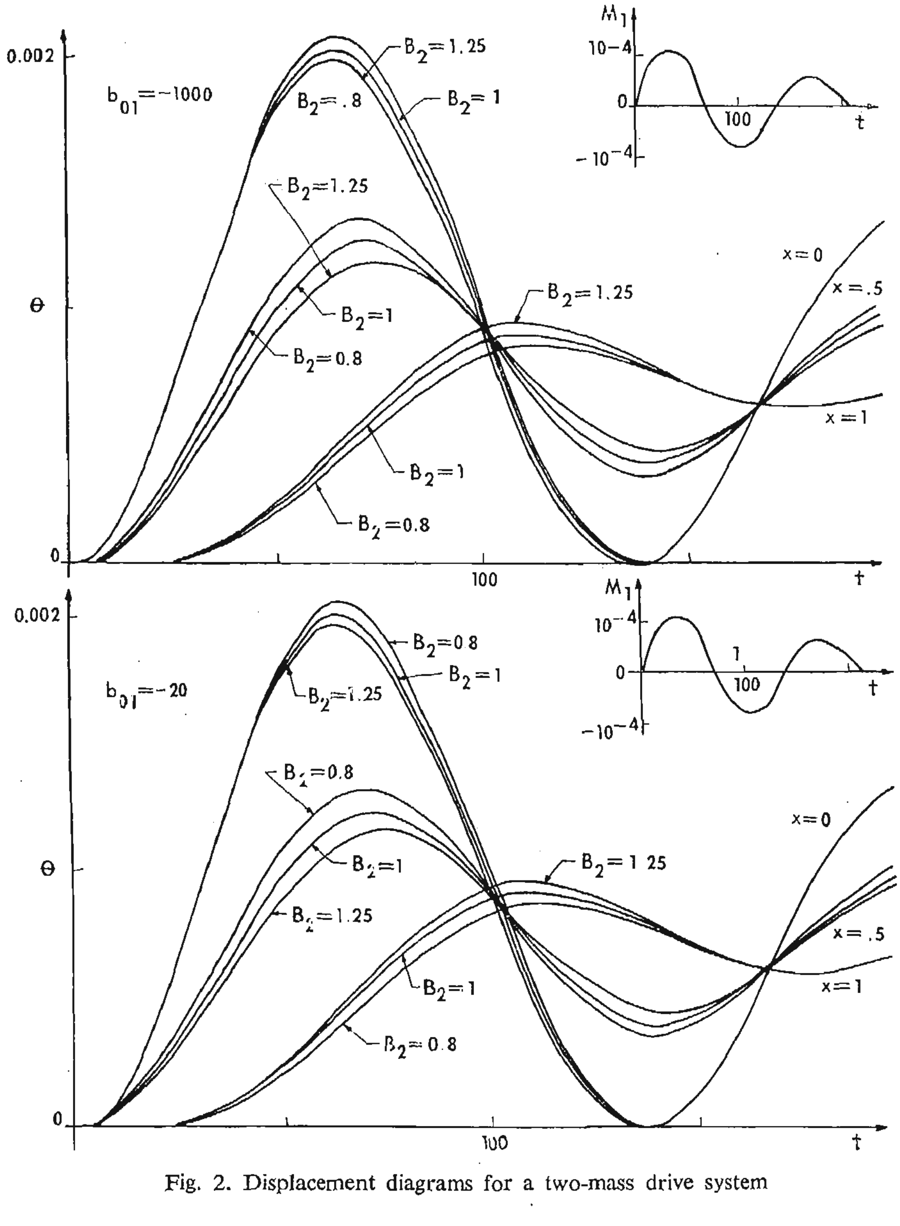

Fig. 2. Displacement diagrams for a two-mass drive system

analysis for the dissipative wave equation and the classical wave equation with the equivalent damping is performed in the case of a unilaterally fixed rod with a constant cross-section. Also for simple systems, results obtained by means of the wave method is compared with appropriate results obtained by means of the rigid finite element method and the method of separation of variables.

3.1. Damping in dissipative wave equation and equivalent damping. The solution of the dissipative wave equation is now compared with the solution of the classical wave equation with the equivalent damping. The comparison is accomplished for the rod right-handly fixed the free end of which is loaded at time instant $t = 0$ by a constant torque, Fig. 6 with $J = 0$.

3.1.1. Solution for dissipative wave equation. The discussion of the system under consideration, Fig. 6 with $J = 0$, taking into account the damping continuously distributed is reduced to solving the dissipative wave equation, which in appropriate nondimensional quantities analogous to (6) has the form:

$$\Theta_{,tt}+2h\Theta_{,t}-\Theta_{,xx} = 0, \tag{12}$$

with the following initial conditions:

$$\Theta = \Theta_{,t} = 0 \quad \text{for} \quad t = 0 \tag{13}$$

and boundary conditions:

$$\begin{aligned} \Theta_{,x} &= -M_0 \quad \text{for} \quad x = 0, \\ \Theta &= 0 \quad \text{for} \quad x = 1, \end{aligned} \tag{14}$$

where h is a nondimensional damping coefficient, M_0 is a nondimensional constant torque, and bars are omitted for convenience.

Upon the introduction of the transformation:

$$\Theta = e^{-ht}v \tag{15}$$

equation (12) takes the form:

$$v_{,tt}-v_{,xx}-h^2v = 0. \tag{16}$$

By executing the Laplace transformation relations (16), (14) are:

$$(s^2-h^2)\tilde{v}-\frac{d^2\tilde{v}}{dx^2} = 0, \tag{17}$$

$$\tilde{v} = 0 \quad \text{for} \quad x = 1 \quad \text{and} \quad \frac{d\tilde{v}}{dx}+\tilde{M}_0 = 0 \quad \text{for} \quad x = 0, \tag{18}$$

where by wavy lines the Laplace transformation of suitable functions are marked.

The solution of equation (17) for conditions (18) has the form:

$$\tilde{v}(x;s) = \sum_{n=0}^{\infty}\sum_{k=0}^{1}(-1)^{n+k}\frac{\tilde{M}_0}{(s^2-h^2)^{1/2}}\exp(-x_{kn}(s^2-h^2)^{1/2}), \tag{19}$$

where $x_{kn} = (-1)^k x+2(k+n)$.

Upon the retransformation and the use of relation (15) the solution of equation (12) is the following function:

$$\Theta(x, t) = M_0 \sum_{n=0}^{\infty} \sum_{k=0}^{1} (-1)^{k+n} H(t-x_{kn}) \int_0^t e^{-hz} I_0\big(h(z^2-x_{kn}^2)^{1/2}\big)\,dz, \tag{20}$$

where $I_0(x)$ is the Bessel function. From formula (20) it is seen that it is more comfortable to consider the derivative of function $\Theta(x, t)$ with respect to time. This derivative has the form:

$$\Theta_{,t}(x, t) = M_0 \sum_{n=0}^{\infty} \sum_{k=0}^{1} (-1)^{k+n} H(t-x_{kn}) e^{-ht} I_0\big(h(t^2-x_{kn}^2)^{1/2}\big) = \tag{21}$$

$$= M_0 \sum_{n=0}^{\infty} \sum_{k=0}^{1} (-1)^{n+k} H(t-x_{kn}) \sum_{r=0}^{\infty} \frac{\big(h^2(t^2-x_{kn}^2)/4\big)^r}{(r!)^2}\, e^{-ht},$$

where $H(t)$ is the Heaviside function.

3.1.2. Equivalent damping. The discussion of the fixed rod to the free end of which a constant torque and an equivalent damping moment are applied is reduced, in nondimensional quantities, to solving the equation

$$\Theta_{,tt} - \Theta_{,xx} = 0 \tag{22}$$

with initial conditions (13) and with the following boundary conditions:

$$\begin{aligned} &\Theta_{,x} = -M_0 + D\Theta_{,t} \quad \text{for} \quad x = 0, \\ &\Theta = 0 \quad \text{for} \quad x = 1, \end{aligned} \tag{23}$$

where D is a nondimensional coefficient of equivalent damping.

For the solution of the form:

$$\Theta(x, t) = f(t-x) + g(t+x), \tag{24}$$

we have

$$\begin{aligned} &g(z) = -f(z-2), \\ &f'(z)(1+D) = M_0 + (1-D)g'(z), \end{aligned} \tag{25}$$

from where:

$$f'(z) = \frac{M_0}{1+D} \sum_{k=0}^{n} (-1)^k \left(\frac{1-D}{1+D}\right)^k \quad \text{for} \quad 2n \leqslant z < 2(n+1). \tag{26}$$

For example for cross-section $x = 0$ and for $2n \leqslant t < 2(n+1)$

$$\Theta_{,t}(0, t) = \frac{M_0}{1+D} (-1)^n \left(\frac{1-D}{1+D}\right)^n. \tag{27}$$

Function (27) is a piece-wisely constant function.

3.1.3. Numerical results. In Fig. 3 are shown diagrams of velocities in cross-sections $x = 0$ and $x = 0.5$ of the considered rod for damping coefficients $h = D = 0.1$, 0.2 and 0.5 and for $M_0 = 1.0$ obtained according to formulae (21) (continuous lines) and formulae (26) (dashed lines). From Fig. 3 it follows that velocities obtained for the equiva-

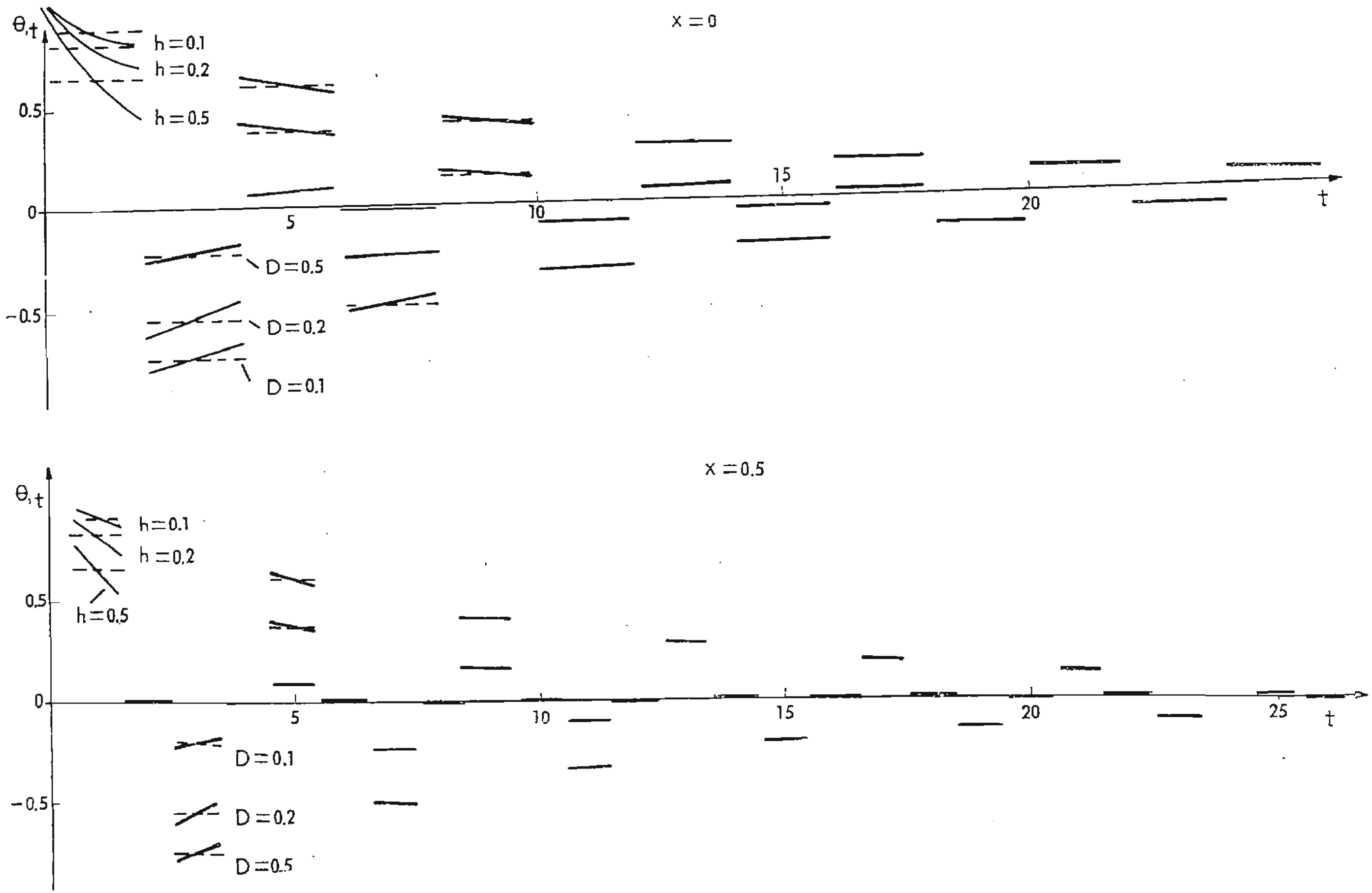

Fig. 3. Velocity diagrams for damping continuously distributed (———) and for equivalent damping (- - -)

lent damping in successive intervals of time beginning with even numbers are approximately average velocities obtained for the damping continuously distributed. Moreover, results for the both types of damping coincide for $t \geqslant 8$. Analytical formulae are considerably simpler in the case of the equivalent damping.

3.2. Wave method and rigid finite element method. The comparison of the wave method with the rigid finite element method is performed for angular displacements of an undamped two-mass drive system with constant cross-section, Fig. 1 for $N = 2$, $J_{01} = J_{02} =$ const and $M_{D1}(t) = M_{D3}(t) = 0$. The system is loaded by the external moment applied to rigid body (1), which is described in nondimensional quantities by function $M_1(t) = 0.00001\sin(\pi t/4)$. In calculations $K_1 = 0.1$ and $E_3 = 0.1$, (6), are assumed.

The motion of the drive system under consideration using the method proposed in the paper is described by equations (10) with $b_{01} \to -\infty$. Displacements $\Theta(x, t)$ of shaft cross-sections of the drive system for $x = 0, 0.5, 1.0$ are obtained by means of the finite difference method with $\Delta z = 0.025$. Diagrams of these displacements are shown in Fig. 4.

Motion equations for the undamped two-mass drive system using the rigid finite element method have the form, [10],

$$\left(J_1 + \frac{1}{2} R_0\right)\ddot{\Theta}_1 + \frac{GJ_{01}}{\Delta l}(\Theta_1 - \Theta_2) = \frac{J_1 \Theta_0 c^2}{l^2} M_1(t),$$

$$R_0 \ddot{\Theta}_i + \frac{GJ_{01}}{\Delta l}(-\Theta_{i-1} + 2\Theta_i - \Theta_{i+1}) = 0 \quad \text{for} \quad i = 2, 3, \ldots, N, \tag{28}$$

$$\left(J_3 + \frac{1}{2} R_0\right)\ddot{\Theta}_N + \frac{GJ_{01}}{\Delta l}(-\Theta_{N-1} + \Theta_N) = 0,$$

where N is the number of finite elements, lengths of extreme and remaining elements are $\Delta l/2$ and Δl respectively, $R_0 = J_{01}\varrho\Delta l$ and Θ_i is the displacement of the i-th element.

Introducing the appropriate nondimensional quantities (6) equations (28) take the form:

$$r_1\ddot{\Theta}_1 + r_4(\Theta_1 - \Theta_2) = M_1(t),$$

$$r_2\ddot{\Theta}_i + r_4(-\Theta_{i-1} + 2\Theta_i - \Theta_{i+1}) = 0 \quad \text{for} \quad i = 2, 3, \ldots, N-1, \tag{29}$$

$$r_3\ddot{\Theta}_N + r_4(-\Theta_{N-1} + \Theta_N) = 0,$$

where: $r_1 = 1 + \Delta l K_1/2l$, $r_2 = \Delta l K_1/l$, $r_3 = \Delta l K_1/2l + 1/E_3$, $r_4 = K_1 l/\Delta l$, and bars are omitted for convenience.

Displacements Θ_i for finite elements are obtained from equations (29) by means of the Runge-Kutta method with $\Delta t = 0.01$ and initial conditions:

$$\Theta_i(0) = \dot{\Theta}_i(0) = 0. \tag{30}$$

Diagrams for these displanements, in nondimensional time, are shown in Fig. 5 taking into account 5 finite elements.

From the comparison of the displacement diagram for the cross-section $x = 0$ in Fig. 4 with the diagram of function Θ_1 in Fig. 5 it follows that the character of the both curves is similar, and that the suitable maximum displacements obtained by means of the both methods differ from each other by about 8 per cent. However, the displacement

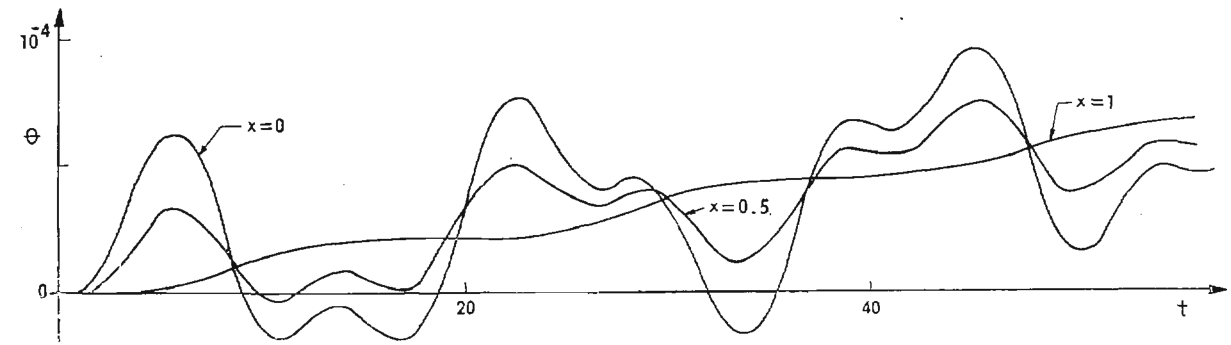

Fig. 4. Displacements of cross-sections $x = 0$, 0.5, 1.0 obtained from equations (10)

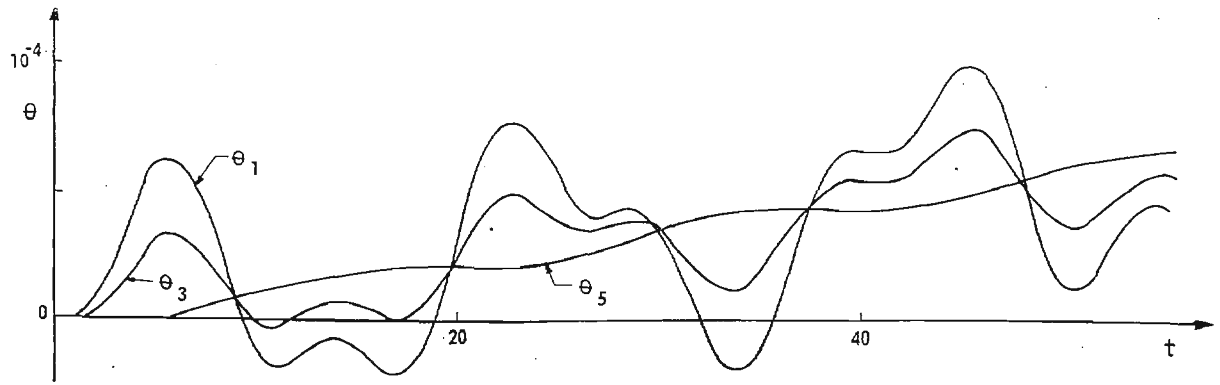

Fig. 5. Displacements Θ_1, Θ_3, Θ_5 obtained from equations (29)

curves of cross-sections $x = 0.5$ and $x = 1.0$ do not differ practically from corresponding curves Θ_3, Θ_5 presented in Fig. 5. Additionally one may note that the execution time of numerical calculations is much longer when the rigid finite element method is applied.

3.3. Wave method and method of separiation of variables. In this section the forced vibrations of the undamped system shown in Fig. 6 is considered using wave solutions of motion equations and the method of separation of variables. In the both cases displacements of cross-section $x = 0$ are determined (i.e. for the cross-section where the rigid body is attached to the rod), and the amplitude-frequency curve is plotted out for the nondimensional external moment $M(t) = a_0 \sin pt$. The rod is characterized by polar moment of inertia J_0, shear modulus G, density ϱ and length l.

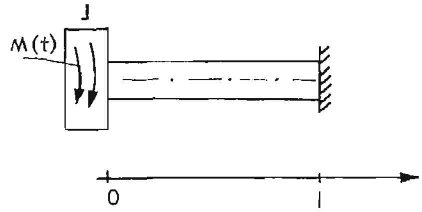

Fig. 6. Simple mechanical system

3.3.1. Wave solution. The determination of nondimensional displacements in the elastic element of the system shown in Fig. 6 is reduced to solving motion equations (22) with initial conditions (13) and the following boundary conditions:

$$\begin{aligned} &M(t) - \Theta_{,tt} + K\Theta_{,x} = 0 \quad \text{for} \quad x = 0, \\ &\Theta = 0 \quad \text{for} \quad x = 1, \end{aligned} \tag{31}$$

where bars, denoting nondimensional quantities, are omitted for convenience and $K = J_0 \varrho l$.

Substituting (24) into boundary conditions (31) we have:

$$\begin{aligned} f''(z) + Kf'(z) &= a_0 \sin pz + f''(z-2) - Kf'(z-2), \\ g(z) &= -f(z-2), \end{aligned} \tag{32}$$

where function $f(z)$ is assumed to be zero for negative arguments. Equations (32) are solved numerically by means of the finite difference method.

3.3.2. Method of separation of variables. The solution for the forced vibrations of the undamped system presented in Fig. 6 is now sought in the form:

$$\Theta(x, t) = \sum_{n=1}^{\infty} T_n(t)\Theta_n(x), \tag{33}$$

where T_n are unknown functions depending on time, and Θ_n are eigenfunctions which are determined from equation (22) with boundary conditions (31) for $M(t) = 0$. We have then, analogously as in [4],

$$\begin{aligned} &\Theta_n(x) = \sin \omega_n x - \frac{K}{\omega_n} \cos \omega_n x, \\ &\gamma_n^2 = \int_0^1 \Theta_n^2(x)\,dx + \frac{1}{K}\Theta_n^2(0) = \frac{1}{2}\left(1 + \frac{K^2}{\omega_n^2}\right) + \frac{1}{4\omega_n}\left(\frac{K^2}{\omega_n^2} - 1\right)\sin 2\omega_n + \frac{K}{\omega_n^2}\cos^2\omega_n, \end{aligned} \tag{34}$$

where ω_n are natural frequencies, and γ_n^2 can be obtained from an orthogonality principle for the discussed example when eigenfunctions are assumed to be identical. As it is seen from $(34)_2$ a nonintegral term appears in the orthogonality principle. Such a term occurs in the case of discrete-continuous systems. For such systems it is more convenient to use Lagrange equations in coordinates T_n:

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{T}_n}\right)+\frac{\partial E_p}{\partial T_n} = H_n(t), \qquad n = 1,2,..., \tag{35}$$

where H_n are generalized external forces corresponding to coordinates T_n and they are determined from the expression for the work of an external loading on an infinitesimal displacement $\delta\Theta(x, t)$. In the case under discussion the loading acts in cross-section $x = 0$.

Energies E_k and E_p in nondimensional quantities:

$$\bar{E}_{k,p} = \frac{l^2}{Jc^2\Theta_0^2} E_{k,p}, \quad \bar{H}_n(\bar{t}) = \frac{l^2}{Jc^2\Theta_0^2} H_n(t), \quad \bar{\gamma}_n^2 = \frac{1}{l\Theta_0^2}\gamma_n^2, \tag{36}$$

take the form, [4],

$$E_p = \frac{K}{2}\int\limits_0^1 \left(\frac{\partial\Theta}{\partial x}\right)^2 dx, \quad E_k = \frac{K}{2}\int\limits_0^1 \left(\frac{\partial\Theta}{\partial t}\right)^2 dx+\frac{1}{2}\left(\frac{\partial\Theta(0, t)}{\partial t}\right)^2, \tag{37}$$

where bars are omitted for convenience, and the Lagrange equations remain in the form (35).

Upon the substitution (33) into (37) and upon proper transformations we get:

$$\begin{aligned} E_p &= \frac{K}{2}\sum_{n=1}^{\infty}\omega_n^2\gamma_n^2 T_n^2(t), \\ E_k &= \frac{K}{2}\sum_{n=1}^{\infty}\gamma_n^2 T_n^2(t). \end{aligned} \tag{38}$$

Substituting (38) into (35) we have:

$$\ddot{T}_n(t)+\omega_n^2 T_n(t) = \frac{1}{K\gamma_n^2} H_n(t). \tag{39}$$

In the case of an external loading applied in cross-section $x = 0$, [4],

$$H_n(t) = M(t)\Theta_n(0) = -\frac{K}{\omega_n} M(t), \tag{40}$$

and equation (39) for $M(t) = a_0 \sin pt$ has the following solution

$$T_n(t) = -\frac{a_0}{\omega_n^2(\omega_n^2-p^2)\gamma_n^2}(\omega_n \sin pt - p\sin\omega_n t). \tag{41}$$

Displacement $\Theta(x,t)$ is then calculated according to the formula

$$\Theta(x,t) = a_0 \sum_{n=1}^{\infty} \frac{1}{\omega_n^2(\omega_n^2-p^2)\gamma_n^2} (\omega_n \sin pt - p \sin \omega_n t) \times \times \left(\frac{K}{\omega_n}\cos\omega_n x - \sin\omega_n x\right). \tag{42}$$

3.3.3. Numerical results. Numerical calculations are concentrated on the amplitude-frequency curve for the cross-section $x = 0$ with $K = 0.5$ and $a_0 = 1.0$. This curve on the base of the Lagrange equations for the undamped system can be easily determined from the formula (42). However, when the wave method is used the points of this curve are obtained from numerical solutions of equations (32) in the region of steady motion.

In Fig. 7 results obtained on the base of formula (42) are marked by a continuous line. According to this formula, the vibration amplitude is infinite for the frequencies

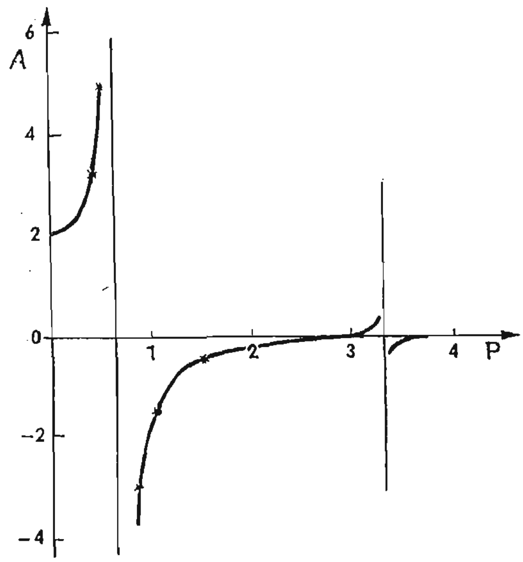

Fig. 7. Amplitude-frequency curve

of the external moment being equal to the successive natural frequencies of the system, because in the denominator of formula (42) differences $\omega_n - p$ occur. In the considered example $\omega_1 = 0.654$, $\omega_2 = 3.293$, $\omega_3 = 6.362$.

In Fig. 7 results obtained using wave solutions of motion equations are marked by stars for p smaller than the second natural frequency. It follows from Fig. 7 that stars lie practically on the continuous curve. However, from numerical calculations it follows that for p being equal to the first natural frequency and in the neighbourhood of this value the vibration amplitude is not infinite, because expressions $(\omega_n - p)^{-1}$ do not occur when the finite difference method is applied for the solution of equations (32). It appears moreover that the value of the amplitude for the resonance frequency is sensitive to a numerical integration step and there are some difficulties in its exact determination.

4. Final remarks

The method applied in the paper, based on the use of wave solutions of suitable motion equations, allows to determine displacements, strains and velocities in arbitrary shaft cross-sections of drive systems modelled by means of rigid bodies and elements torsionally deformed. These systems can be loaded by periodic and nonperiodic external forces. Using this method variable cross-sections, finite lengths and equivalent damping can be taken into account.

From comparisons for simple systems it follows that 1) the substitution of damping continuously distributed by an equivalent damping leads, beyond a short initial time interval, to practically the same results, 2) maximum values of displacements for the cross-section in which the external loading is applied, obtained by means of the wave method and the method of rigid finite elements differ by 8 per cent, while suitable curves coincide practically for remaining considered cross-sections, and the execution time of numerical calculations is considerably shorter when the wave method is used, 3) the application of the wave method in the investigation of forced vibrations for undamped systems does not lead to infinite amplitudes.

It should be pointed out that the wave method in the presented form leads to solving simple mathematical relations and it is more effective than other methods for considerations of discrete-continuous models of drive systems undergoing torsional deformations.

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Резюме

ИСПОЛЬЗОВАНИЕ ВОЛНОВОГО МЕТОДА В ИССЛЕДОВАНИЯХ ПРИВОДНЫХ СИСТЕМ И ЕГО СРАВНЕНИЕ С ДРУГИМИ МЕТОДАМИ

В работе предложено волновой метод для динамических исследований дискретно-непрерывной модели приводной системы с постоянными и переменными сечениями валов переносящих скручивающий момент. Затухание в исследованной системе учитываются при помощи фиктивного затухания действующего в избранных сечениях привода, что допускает применять уравнения движения без затухания.

Предлагаемый метод сравнено с другими методами на примере избранных простых систем торсионно деформированных. Именно, фиктивное затухание сравнено с затуханием непреривно разложенном и полученные при помощи волнового метода результаты сравнено с соответствующими результатами полученными при помощи метода жёстких конечных элементов и метода разделения переменных.

В данной форме волновой метод ведёт до простых математических формул. Кроме того, из сравнений простых систем следует, что он более эфективен чем другие методы анализа систем переносящих скручивающую нагрузку.

Streszczenie

WYKORZYSTANIE METODY FALOWEJ W BADANIACH UKŁADÓW NAPĘDOWYCH, PORÓWNANIE Z INNYMI METODAMI

W pracy zaproponowano metodę falową do badań dynamicznych dyskretno-ciągłego modelu układu napędowego poddanego odkształceniom skrętnym, o stałych i zmiennych przekrojach wałów. Tłumienie w badanym układzie uwzględnione jest poprzez tłumienie zastępcze działające w wybranych przekrojach wału, co umożliwiło przyjęcie równań ruchu bez tłumienia.

Proponowaną metodę porównano z innymi metodami na przykładzie wybranych prostych układów odkształcanych skrętnie. Mianowicie, tłumienie zastępcze porównano z tłumieniem rozłożonym w sposób ciągły, oraz wyniki otrzymane za pomocą metody falowej porównano z odpowiednimi wynikami uzyskanymi za pomocą metody sztywnych elementów skończonych i metody rozdzielenia zmiennych.

W podanej postaci metoda falowa prowadzi do prostych związków matematycznych. Ponadto z dokonanych porównań dla prostych układów wynika, że jest efektywniejsza od innych metod przy dyskusji układów poddanych odkształceniom skrętnym.

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