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BASIC EQUATIONS OF SYMMETRIC ELASTIC-PLASTIC DEFORMATIONS OF AN INCOMPLETE TOROIDAL SHELL

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1. Formulation of the problem

In the case when kinematic nonlinearities is taken into account the elastic deformations may have essential influence on both the initiations of elastic-plastic process and its termination. Basic equations for symmetric, rigid-plastic deformations of a toroidal shell, loaded by in-plane bending and pressure, have been derived in [2]. The purpose of this paper is to generalize these results to the case when elastic-plastic deformations are allowed

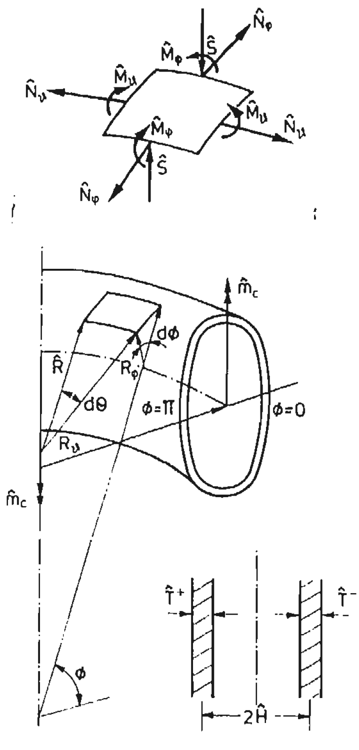

Fig. 1

for. Similarly as in [1] we consider deformation of an incomplete torus as the rotationally-symmetric problem in broader sense, since circumferential displacements are allowed for (Fig. 1). Thus, geometric assumptions investigated in [1], as well as the notation introduced there, have been retained. Let the parameter of change of principal curvature κ be defined as an increment of angle in the circumferential direction

$$\kappa = d\vartheta/d\theta - 1 \tag{1.1}$$

The Cauchy strain rates derived in [1] have the following form:

$$(\delta\varepsilon_j^c)^{\pm} = \delta\partial_j \pm \delta\varkappa_j, \quad j = \varphi, \vartheta \tag{1.2}$$

where

$$\delta\partial_\varphi = \frac{-\delta u_r' \sin\varphi + \delta u_z' \cos\varphi}{R_\varphi}, \qquad \delta\varkappa_\varphi = \frac{\delta\varphi'}{R_\varphi}, \tag{1.3}$$

$$\delta\partial_\vartheta = \frac{\delta u_r}{R}(1+\kappa) + \delta\kappa\left(1+\frac{u_r}{R}\right), \qquad \delta\varkappa_\vartheta = \frac{-\delta\varphi \sin\varphi(1+\kappa) + \delta\kappa\cos\varphi}{R},$$

$$\delta\varphi = -\frac{\delta u_r' \cos\varphi + \delta u_z' \sin\varphi}{R}.$$

δ and ( )' denote the derivation with respect to time-like parameter τ and material, angular, meridional coordinate, respectively. Other symbols used in (1.3) denote:

φ, ϕ — angular, meridional coordinate in deformed and undeformed state,
ϑ, θ — angular, circumferential coordinate in deformed and undeformed state,
∂_φ, ∂_ϑ — elongations at the middle surface,
$\varkappa_\varphi = \hat{\varkappa}_\varphi \hat{H}$, $\varkappa_\vartheta = \hat{\varkappa}_\upsilon \hat{H}$ — parameters of increments of the middle surface curvature change of unit angle,
$R_\varphi = \hat{R}_\varphi(\hat{H}, R = \hat{R}_\vartheta)\hat{H}$ — radii of undeformed element,
$u_r = \hat{u}_r(\hat{H}, u_z = \hat{u}_z)\hat{H}$ — radial and axial middle surface displacements.

All quantities are dimensionless, $\hat{H}$ is the half distance between sandwich sheets (^denotes dimension quantity).

The radii of curvature in current configuration r_φ, r_ϑ are related to R_φ, R_ϑ by:

$$r_\varphi = R_\varphi/\varphi', \qquad r_\vartheta = R_\vartheta \frac{\cos\phi}{(1+\kappa)\cos\varphi}. \tag{1.4}$$

According to [2] the dimensionless, generalized stresses have the following form:

$$N_j = \frac{1}{2}(\sigma_j^+ + \sigma_j^-)$$
$$M_j = \frac{1}{2}(-\sigma_j^+ + \sigma_j^-), \quad j = \varphi, \vartheta \tag{1.5}$$

where $N_j = \hat{N}_j/2\hat{\sigma}_0\hat{T}$, $M_j = \hat{M}_j/2\hat{\sigma}_0\hat{H}\hat{T}$, $\sigma_j = \hat{\sigma}_j/\hat{\sigma}_0$,
$\hat{\sigma}_0$ — tensile yield stress of sandwich sheet material,
$\hat{T}$ — initial thickness of sandwich sheets.

Superscripts + or − denote quantities evaluated in the exterior or interior sandwich sheets respectively.

Moreover, the system of equations of equilibrium derived in [2, 3] will be applied here in the following fashion:

$$
\begin{aligned}
&(RN_\varphi)'+R_\varphi\sin\varphi(1+\kappa)N_\vartheta+\varphi' RS = 0\\
&(RS)'-R_\varphi(1+\kappa)M_\vartheta\cos\varphi-\varphi' RN_\varphi+RR_\varphi p_n = 0\\
&(RM_\varphi)'+R_\varphi(1+\kappa)M_\vartheta\sin\varphi+RR_\varphi S = 0,
\end{aligned}
\tag{1.6}
$$

where $S = \hat{S}/2\hat{\sigma}_0\hat{T}$ — shearing force (reaction), $p_n = \hat{p}_n\hat{H}/2\hat{\sigma}_0\hat{T}$ — normal pressure. For the uniform presenting of the final system of equations, we transform the upper system (1.6) to the convenient, incremental fashion:

$$
\begin{aligned}
&R\delta N_\varphi'+R'\delta N_\varphi+R_\varphi[\sin\varphi(1+\kappa)\delta N_\vartheta+N_\vartheta(1+\kappa)\cos\varphi\delta\varphi+N_\vartheta\sin\varphi\delta\kappa]+\\
&\quad+RS\delta\varphi'+R\varphi'\delta S = 0,\\
&R\delta S'+R'\delta S+R_\varphi[(1+\kappa)M_\vartheta\sin\varphi\delta\varphi-(1+\kappa)\cos\varphi\delta M_\vartheta-M_\vartheta\cos\varphi\delta\kappa+R\delta p_n]-\\
&\quad+RN_\varphi\delta\varphi'-R\varphi'\delta N_\varphi = 0,\\
&R\delta M_\varphi'+R'\delta M_\varphi+R_\varphi[(1+\kappa)\delta M_\vartheta\sin\varphi+(1+\kappa)M_\vartheta\cos\varphi\delta\varphi+\\
&\quad+M_\vartheta\delta\kappa\sin\varphi+R\delta S] = 0.
\end{aligned}
\tag{1.7}
$$

2. Physical relations

The material of working sheets of perfect sandwich-wall is assumed to be elastic/perfectly-plastic, isotropic, perfectly incompressible and to obey the Huber-Mises-Hencky yield condition. Under the assumption of plane stress state, this condition takes the form:

$$
(\sigma_\varphi^\pm)^2-\sigma_\varphi^\pm\sigma_\vartheta^\pm+(\sigma_\vartheta^\pm)^2 = 1 \tag{2.1}
$$

The similarity of deviators will be generally applied in this paper as the physical law. From among six possibilities postulated in Table 1 which complete those discussed in [2] to the case when elastic strains are allowed for we shall use here the Prandtl-Reuss theory.

Table 1

	Deformation theory of plasticity	Incremental theory of plasticity — elastic strains — neglected	Incremental theory of plasticity — elastic strains — allowed for
Small strains	$e_i = \psi s_i$ Hencky-Ilyushin H-I	$\delta e_i = \delta\psi s_i$ Levy-Mises L-M	$\delta e_i = \delta\psi s_i+\frac{1}{2G}\delta s_i$ Prandtl-Reuss P-R I
Large strains	$e_i^H = \Lambda s_i^C$ Nadai-Davis N-D I	$\delta e_i^H = \delta\Lambda s_i^C$ Nadai-Davis N-DII	$\delta e_i^H = \delta\Lambda s_i^C+\frac{1}{2G}\delta s_i^C$ Prandtl-Reuss P-R II

C — Cauchy stress tensor, H — Hencky strain measure, $i = \varphi, \vartheta, z$. Variants of constitutive equations

Since in the problem considered we deal with irrotational deformation, we need here only the principal components of Hencky strain tensor instead of its tensorial expansions in broader case of deformation [4]. Introducing dimensionless scalar factors $\delta\psi^{\pm} = \delta\hat{\psi}^{\pm}/3\hat{\sigma}_0$ and dimensionless Kirchhoff's modulus $G = \hat{G}/\hat{\sigma}_0$, we obtain:

$$
\begin{aligned}
(\delta\varepsilon_{\varphi}^{C})^{\pm} &= \frac{1}{6G}(2\delta\sigma_{\varphi}^{\pm} - \delta\sigma_{\vartheta}^{\pm}) + \delta\psi^{\pm}(2\sigma_{\varphi}^{\pm} - \sigma_{\vartheta}^{\pm}) \\
(\delta\varepsilon_{\vartheta}^{C})^{\pm} &= \frac{1}{6G}(2\delta\sigma_{\vartheta}^{\pm} - \delta\sigma_{\varphi}^{\pm}) + \delta\psi^{\pm}(2\sigma_{\vartheta}^{\pm} - \sigma_{\varphi}^{\pm})
\end{aligned}
\tag{2.2}
$$

Equations of internal equilibrium (1.7) and kinematic relations (1.2) - (1.4), together with the physical equations (2.1-2.2), constitute system of fundamental governing equations. They determine the vector of unknown variables

$$
\Gamma = \begin{bmatrix} \varphi \\ u_r \\ u_z \\ N_{\varphi} \\ M_{\varphi} \\ S \\ p_n \\ \kappa \end{bmatrix}
\tag{2.3}
$$

where in (2.3) the two last are loadings parameters. We assume the vector as the basic in our approach, other uknowns N_{ϑ}, M_{ϑ} and scalar factors $\delta\psi^{\pm}$ can be simply eliminated by applying (2.1) and (2.2). In this way the problem may be reduced to the solution of system of six nonlinear, coupled, partial differential equations, linear with respect to the time-like parameter τ.

3. Basic differential equations

In the P-R I formulation we consider time derivatives of first six components of vector Γ (2.3) three kinematic — δu_r, δu_z, $\delta\varphi$ and three static — δN_{φ}, δM_{φ}, δS as basic uknowns for the governing system of differential equations:

$$
\begin{aligned}
\delta\varphi' &= \frac{1}{2}R_{\varphi}(\delta F^{+} - \delta F^{-}), \\
\delta u_r' &= -R_{\varphi}\delta\varphi\cos\varphi - \frac{1}{2}R_{\varphi}(\delta F^{+} + \delta F^{-})\sin\varphi, \\
\delta u_z' &= -R_{\varphi}\delta\varphi\sin\varphi + \frac{1}{2}R_{\varphi}(\delta F^{+} + \delta F^{-})\cos\varphi, \\
\delta N_{\varphi}' &= R_{\varphi}\{\delta N_{\varphi}\sin\phi - \delta N_{\vartheta}(1+\kappa)\sin\varphi - N_{\vartheta}[(1+\kappa)\cos\varphi\delta\varphi + \sin\varphi\delta\kappa]\}/R - \\
&\quad + \varphi'\delta S - \frac{1}{2}R_{\varphi}(\delta F^{+} - \delta F^{-})S,
\end{aligned}
\tag{3.1}
$$

$$\delta M'_{\varphi} = R_{\varphi}\{\delta M_{\varphi}\sin\phi - \delta M_{\vartheta}(1+\kappa)\sin\varphi - M_{\vartheta}[(1+\kappa)\delta\varphi\cos\varphi + \sin\varphi\delta\kappa]\}/R - \tag{3.1 [cont.]}$$
$$+R_{\varphi}\delta S,$$
$$\delta S' = R_{\varphi}\{\delta S\sin\phi + \delta N_{\vartheta}(1+\kappa)\cos\varphi + N_{\vartheta}[\delta\kappa\cos\varphi - \delta\varphi(1+\kappa)\sin\varphi]\}/R +$$
$$+R_{\varphi}\left[\frac{1}{2}(\delta F^{+} - \delta F^{-})N_{\varphi} + \varphi'\delta N_{\varphi} - \delta p_{n}\right],$$

where, new uknown is defined

$$\delta F^{\pm} = 2\left[\frac{1}{6G}(\delta N_{\varphi} \mp \delta M_{\varphi}) + \delta\psi^{\pm}(N_{\varphi} \mp M_{\varphi})\right] - \left[\frac{1}{6G}(\delta N_{\vartheta} \mp \delta M_{\vartheta}) + \delta\psi^{\pm}(N_{\vartheta} \mp M_{\vartheta})\right] \tag{3.2}$$

and scalar factors $\delta\psi^{\pm}$ takes the form:

$$\delta\psi^{\pm} = \begin{cases} 0 & \text{for elastic domain.} \\ \Big\{[\delta u_r(1+\kappa)/R + \delta\kappa(1+u_r/R)] - \frac{1}{6G}[2(\delta N_{\vartheta} \mp \delta M_{\vartheta}) - & \\ \qquad + (\delta N_{\varphi} \mp \delta M_{\varphi})]\Big\}/[2(N_{\vartheta} \mp M_{\vartheta}) - (N_{\varphi} \mp M_{\varphi})] & \text{for elastic domain.} \end{cases} \tag{3.3}$$

To determine all eigth components of vector Γ (2.3), we have to complete the system of differential Eqs. (3.1) by two additional equations — equation of trajectory

$$h(\Gamma) = 0 \tag{3.4}$$

and definition of a monotonically increasing quantity as time-like parameter

$$\tau = d(\Gamma). \tag{3.5}$$

4. Numerical solution of the problem

The basic system of equations (3.1 - 3.5) as the system of partial, differential equations describes the initial/boundary problem. It can be reduced to two independent problems, initial and boundary problems, by discretization of time-like parameter τ along the trajectory (3.4) into intervals $\Delta\tau_i$ with $\tau_0 = 0$ corresponding to the beginning of the process. For each step of time τ_i, we assume, the simplest, linear extrapolation for vector Γ according to Euler's formula:

$$\Gamma|_{\tau=\tau_i} = \Gamma|_{\tau=\tau_{i-1}} + \delta\Gamma|_{\tau=\tau_{i-1}}, \qquad i \geqslant 1 \tag{4.1}$$

which is also valid for the derivatives with respect to ϕ variable of Γ. The above relation (4.1) must be completed by the condition at $\tau = \tau_0$ (the initiation of the process) which takes, generally, the form:

$$\Gamma|_{\tau=\tau_0} = \Gamma_0. \tag{4.2}$$

For the problem considered the vector of initial condition has the form:

$$\Gamma_0 = \begin{bmatrix} \phi \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4.3}$$

Other forms of the initial vector Γ_0 depend on the assumed material of the shell (for a rigid/perfectly-plastic material the seven or the eight component of Γ_0 must be different from zero) or on the considered initial, geometrical imperfections of torus.

Replacing all components of vector Γ and its derivatives with respect to ϕ variable in the system of equations (3.1 - 3.5) by the relation (4.1) we obtain at each step of time $\tau_i (i \geqslant 1)$ the system of ordinary differential equations with respect to ϕ variable for the uknown increment $\delta\Gamma|_{\tau=\tau_{i-1}}$. Finally, for each step τ_i we need eight boundary conditions. Generally, they takes the form:

$$L_C\Gamma|_{\phi=\phi_C} = V_C, \quad L_D\Gamma|_{\phi=\phi_D} = V_D \tag{4.4}$$

where L_C, L_D are arbitrary operators and V_C, V_D — arbitrary vectors. In the case considered in this paper, the meridional section remains closed and symmetric about $\phi = 0$ so $\phi_C = 0$ and $\phi_D = \pi$. Thus, the operators I_C, L_D has following representation

$$L_C = L_D = [\delta, \delta, 0, 0, 0, \delta, \delta, \delta] \tag{4.5}$$

and first six components of V_C and V_D are equal to zero. Using the semi-inverse method (shooting method) the two-point boundary problem (4.4) ca be converted into Cauchy's problem. By employing Newton-Raphson's method one can determine, missing, initial components of vector Γ at $\phi = \phi_C$ by assuring the boundary conditions at $\phi = \phi_D$. To improve the effectiveness of the iterative (Newton-Raphson's) procedure, the Lagrange quadratic extrapolation formula is used in searching of uknown, missing, initial components of vector $\delta\Gamma$ at $\phi = \phi_C$. We start our calculation at $\tau = \tau_0$ where the initial vector Γ_0 is known (4.3) and the whole shell is elastic. Applying the numerical integration along the coordinate ϕ (for example Runge-Kutta IV) after extrapolation (4.1) and the usage of shooting method, we find the distribution of $\delta\Gamma|_{\tau=\tau_0}$ along ϕ variable. Then, we calculate $\Gamma|_{\tau=\tau_1}$ according to the scheme (4.1) and get the distribution $\delta\Gamma|_{\tau=\tau_1}$ for the new, initial values $\Gamma|_{\tau=\tau_1}$. The described procedure can be repeated along the trajectory (3.4) with increasing time-like parameter τ — (3.5). We assume the distribution of elastic and plastic zones at the beginning of each time step τ_i (it is identical to the distribution at $\tau = \tau_{i-1}$). The assumed zones are corrected by checking the condition (2.1) and scalar factor $\delta\psi^{\pm}$ at ϕ_j such as to fulfill the mentioned conditions as well as the boundary conditions (4.4). The procedure is continued to the appearance of singularity of equations (3.1 - 3.3) which corresponds to the zero of the denominator of $\delta\psi^{\pm}$, when

$$2\sigma_\varphi^{\pm} - \sigma_\vartheta^{\pm} = 0 \tag{4.6}$$

5. Numerical examples

The initial geometry of a torus is completely defined by two dimensionless radii which were assumed in numerical calculations as follows

$$R_\varphi = 50, \quad R_\vartheta = 1000. \tag{5.1}$$

To illustrate different processes of plastic deformations of the shell we restrict our considerations to two strictly determined cases of the loading trajectory (3.4) and the time-like parameter (3.5). In the first case we assume the loading stations and the independent parameter τ in the following way:

$$\begin{aligned} \kappa &= 0, \\ p_n &= \tau \end{aligned} \tag{5.2}$$

and in the second case

$$\begin{aligned} p_n &= c\kappa, \\ \kappa &= \tau \end{aligned} \tag{5.3}$$

where c is an arbitrary, real constant. On each step τ_i, the boundary conditions (4.4) takes the following, explicit form:

$$[\delta\varphi, \delta u_z, \delta S, \delta\kappa, \delta p_n] = [0, 0, 0, 0, \delta\tau] \quad \text{for } \phi = 0 \wedge \pi \tag{5.4}$$

in the case of the relations (5.2) and

$$[\delta\varphi, \delta u_z, \delta S, \delta\kappa, \delta p_n] = [0, 0, 0, \delta\tau, c\delta\tau] \quad \text{for } \phi = 0 \wedge \pi \tag{5.5}$$

in the second case (5.3).

The process of acting of internal pressure is the elastic one. The first point of plastification of the shell (at $\phi = \pi/2$) corresponds to termination of the process (the limit

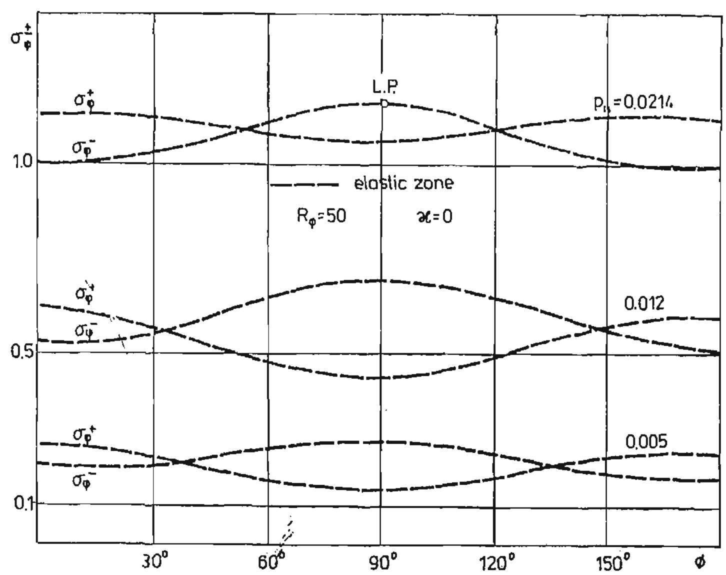

Fig. 2

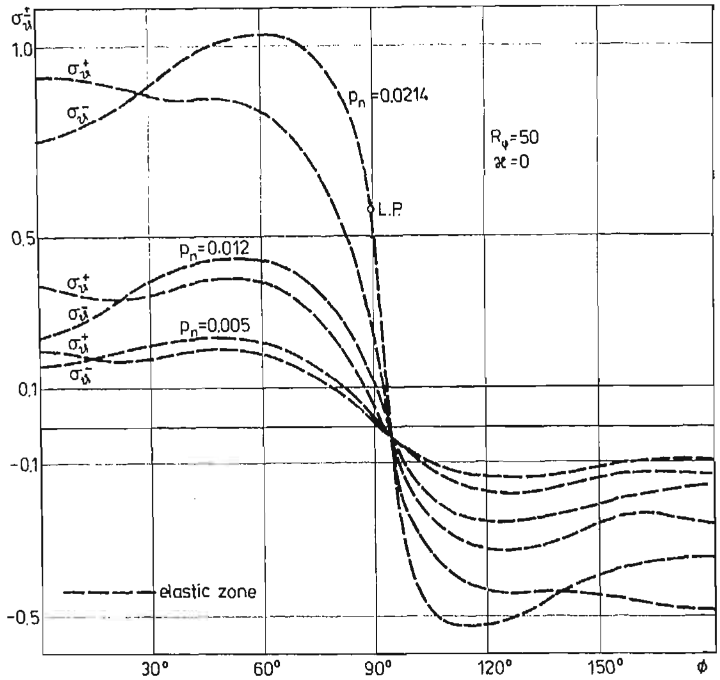
$\sigma_\vartheta^\pm$
1.0
σ_ϑ^+
σ_ϑ^-
p_n =0.0214
R_φ=50
$\varkappa$=0
L.P
0.5
p_n=0.012
σ_ϑ^+
σ_ϑ^-
p_n=0.005
σ_ϑ^+
σ_ϑ^-
0.1
-0.1
elastic zone
-0.5
30°
60°
90°
120°
150°
ϕ

Fig. 3

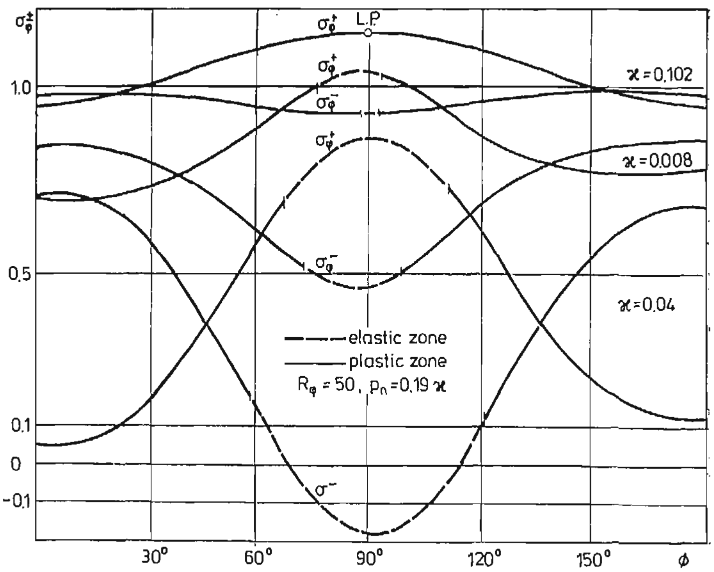
$\sigma_\varphi^\pm$
σ_φ^+ L.P
σ_φ^+
$\varkappa$=0.102
1.0
σ_φ^-
σ_φ^+
$\varkappa$=0.008
σ_φ^-
0.5
$\varkappa$=0.04
elastic zone
plastic zone
R_φ = 50 , p_n=0.19 $\varkappa$
0.1
0
-0.1
σ^-
30°
60°
90°
120°
150°
ϕ

Fig. 4

carrying capacity — L.P.) because the conditions (2.1) and (4.6) are satisfied simultaneously. Redistributions of meridional and circumferential stresses are shown in Fig. 2 and 3.

The influence of bending (the second analyzed case — (5.3) and (5.5)) causes antisymmetry of redistribution of meridional stresses as well as the great development of plastic

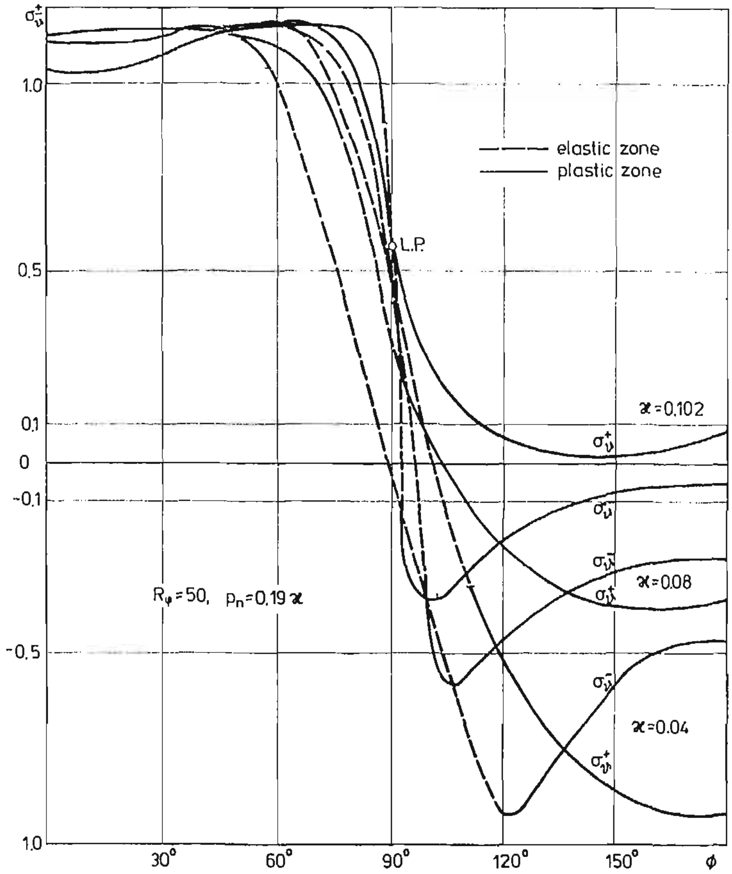

Fig. 5

zone — elastic zone exists only in the neighbourhood of $\phi = \pi/2$ (see Fig. 4,5). The limit point (4.6) — L.P. is also achieved at the edge of elastic zone at $\phi = \pi/2$, so the evolution of meridional strains is inconsiderable — Fig. 6. Similar results concerning to in-plane bending with internal pressure was obtained by C. R. Calladine [6]. The presented examples show also essential influence of type of material of working sheets compare the results with [5].

The form of the limit carrying capacity curve in the $p_n - \kappa$ system will be discussed separately.

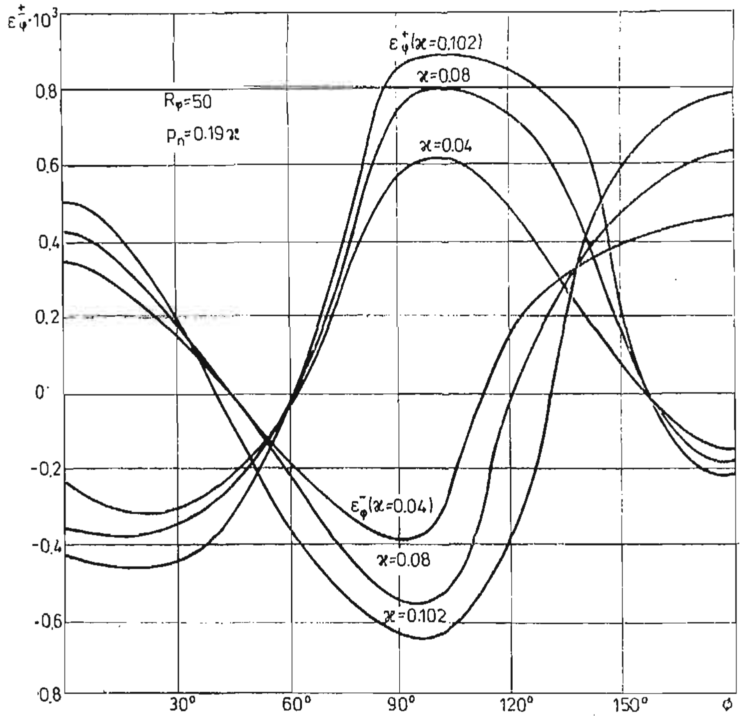

Fig. 6

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Резюме

ОСНОВЫЕ УПРАВНЕНИЯ СИММЕТРИЧНЫХ УПРУГОПЛАСТИЧЕСКИХ ДЕФОРМАЦИЙ ТОРОИДАЛЬНЫХ ОБОЛОЧЕК

В работе представлена основная система дифференциальных уравнений описывающих упругопластические деформации вафельной, торойдальной оболочки подвергнутой изгибу в плоскости главизны и действию нормального давления. Проблема сформулирована на основании теории малых деформации Прандтля-Рейсса (но конечных перемещений). Предполагается что материал несущих слоев упруго-идеальнопластический, несжимаемый и подчиняется условию текучести Хубера-Мизеса-Генки. Рассуждены разные варианты управления процессом и разные определения параметра времени. Предложен метод нумерического решения этой проблемы.

Streszczenie

PODSTAWOWE RÓWNANIA SYMETRYCZNYCH ODKSZTAŁCEŃ SPRĘŻYSTOPLASTYCZNYCH POWŁOKI TOROIDALNEJ

W pracy przedstawiono podstawowy układ równań różniczkowych opisujących sptężysto-plastyczne odkształcenia sandwiczowej powłoki toroidalnej zginanej w płaszczyźnie krzywizny głównej i obciążonej ciśnieniem normalnym. Problem został sformułowany w oparciu o teorię malych odkształceń Prandtla-Reussa (uwzględniono jednak skończone przemieszczenia). Założono, że materiał warstw nośnych toroidu jest sprężysto/idealnie-plastyczny, nieściśliwy i podlega warunkowi plastyczności Hubera-Misesa- Hencky'go. Rozważono możliwości różnorodnego sterowania procesem jak również różnych definicji umownego parametru czasowego. Zaproponowano także metodę numerycznego rozwiązania tego problemu.

Praca została złożona w Redakcji dnia 31 stycznia 1983 roku