MECHANIKA
TEORETYCZNA
I STOSOWANA
1/2, 22 (1984)

A TRANSVERSELY ISOTROPIC LAYER PRESSED ONTO A RIGID BASE WITH A PROTRUSION OR PIT

BOGDAN ROGOWSKI

Łódź, Technical University
Institute of Construction Engineering

The author solves problem pointed out in the title, in which the effects of transverse anisotropy and body forces are taken into account, by means of Hankels transforms and the displacement potentials. The three-part mixed boundary value contact problem is reduced to the solution of triple integral equations and some conditions. These equations are solved by expansion of the function describing the displacement and stress states into a Fourier cosine series, which leads to two infinite sets of linear simultaneous algebraic equations. The part of the lower surface of the plate which does not contact with the base, is an annulus, the inner or outer radii of which are not known a priori and are determined.

Numerical results are shown for the relation among the pressure and weight of the layer, its thickness, the annular region and the magnitudes of the protrusion or pit of the base in cadmium and magnesium single crystals and fiber-reinforced composite materials. They are compared with these of the isotropic layer to show the effect of anisotropy. The variation of the stress concentration factors at the adges of the contact are plotted versus the ratio of the radii of the contact regions for dissimilar materials and layer thickness.

1. Introduction

The indentation problems of an elastic, isotropic half-space by a rigid cone [1], sphere [2] or truncated cone [3], in the case of circular contact region, contact problems of a isotropic half-space or layer pressed onto a rigid base with a protrusion or a pit [4, 5], pressed by a concave rigid punch (a transversely isotropic case) [6], problem of two isotropic half-space pressed against each other with a rigid paraboloidal inclusion between them [7] problems involving annular contact or uncontact region, and a two-dimensional cases [8, 9], have been analyzed.

In the present paper, the axisymmetric contact problems of a transversely-isotropic layer pressed onto a rigid base with a cylindrical protrusion (Problem I) or a pit (Problem II) are considered.

2. Basic equations and displacement functions

We denote as usual the cylindrical polar coordinates of a point by (r, Θ, z) where the z-axis is chosen as the axis of geometric and elastic symmetry. In the case of the torsionless axisymmetric problems the stress-strain relations for a transversely isotropic solid are as follows:

$$\begin{aligned} \sigma_{rr} &= c_{11}e_{rr}+c_{12}e_{\Theta\Theta}+c_{13}e_{zz}, \\ \sigma_{\Theta\Theta} &= c_{12}e_{rr}+c_{11}e_{\Theta\Theta}+c_{13}e_{zz}, \\ \sigma_{zz} &= c_{13}(e_{rr}+e_{\Theta\Theta})+c_{33}e_{zz}, \\ \sigma_{rz} &= c_{44}e_{rz}, \\ \sigma_{r\Theta} &= \sigma_{\Theta z} = 0, \end{aligned} \tag{2.1}$$

where

$$e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\Theta\Theta} = \frac{u}{r}, \quad e_{zz} = \frac{\partial w}{\partial z}, \quad e_{rz} = \frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}, \tag{2.2}$$

u, 0, w are radial, circumferential and axial components of the displacement vector and c_{ij} are elastic constants.

Let $\gamma = \varrho_1 g$ be the body force density acting vertically, where ϱ_1 is the mass density and g the gravitational constant.

The displacement equations of equilibrium are:

$$\begin{aligned} & c_{11}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru)\right]+(c_{13}+c_{44})\frac{\partial^2 w}{\partial r\,\partial z}+c_{44}\frac{\partial^2 u}{\partial z^2} = 0, \\ & c_{44}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+(c_{13}+c_{44})\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru)\right]+c_{33}\frac{\partial^2 w}{\partial z^2} = \gamma. \end{aligned} \tag{2.3}$$

The particular part of the displacement components corresponding to γ and the clamping uniform pressure p_0 in the z direction may be obtained separately as:

$$\begin{aligned} u(r) &= \frac{c_{13}}{2c}r(2p_0+\gamma h), \\ w(z) &= \frac{1}{2c_{33}}z\gamma(z-h)-\frac{c_{11}+c_{12}}{2c}z(2p_0+\gamma h), \end{aligned} \tag{2.4}$$

where h is a geometric and c material parameter

$$c = (c_{11}+c_{12})c_{33}-2c_{13}^2. \tag{2.5}$$

This gives the stress state

$$\begin{aligned} \sigma_{rr}(z) &= \sigma_{\Theta\Theta}(z) = \frac{c_{13}}{2c_{33}}\gamma(2z-h), \\ \sigma_{zz}(z) &= -p_0-\gamma(h-z), \\ \sigma_{zr} &= 0, \end{aligned} \tag{2.6}$$

which satisfies the conditions

$$\sigma_{zz}(h) = -p_0, \quad \sigma_{zz}(0) = -p_0-\gamma h = -p_e,$$
$$\int_0^h \sigma_{rr}(z)dz = 0 \tag{2.7}$$

and in addition

$$w(r,0) = 0, \quad w(r,h) = -\frac{c_{11}+c_{12}}{2c} h(2p_0+\gamma h). \tag{2.8}$$

To solve the homogeneous equilibrium equations (2.3) we introduce the displacement potentials $\varphi_1(r,z)$ and $\varphi_2(r,z)$ as defined by [10]

$$u = \frac{\partial}{\partial r}(k\varphi_1+\varphi_2), \quad w = \frac{\partial}{\partial z}(\varphi_1+k\varphi_2). \tag{2.9}$$

Eqs. (2.3) are satisfied if [10]

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{s_i^2}\frac{\partial^2}{\partial z^2}\right)\varphi_i(r,z) = 0, \quad i = 1,2 \tag{2.10}$$

where

$$s_{1,2} = \frac{1}{2}(\alpha\pm\beta), \quad \alpha = \varepsilon\sqrt{2(\varkappa+1)}, \quad \beta = \varepsilon\sqrt{2(\varkappa-1)},$$
$$\varepsilon = \left(\frac{c_{11}}{c_{33}}\right)^{\frac{1}{4}}, \quad \varkappa = \sqrt{\frac{c_{33}}{c_{11}}}\left[\frac{1}{2}\frac{c_{11}}{c_{44}}-\frac{c_{13}}{c_{33}}\left(\frac{1}{2}\frac{c_{13}}{c_{44}}+1\right)\right], \tag{2.11}$$
$$k = q+\sqrt{q^2-1}, \quad q = \frac{c_{11}c_{33}-c_{13}^2}{2c_{44}(c_{13}+c_{44})}-1$$

are dimensionless parameters of the material.

The stress components corresponding to Eqs. (2.9) are:

$$\begin{pmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\end{pmatrix} = -G_1(k+1)\frac{\partial^2}{\partial z^2}(\varphi_1+\varphi_2)-2G\begin{pmatrix}\dfrac{1}{r}\dfrac{\partial}{\partial r}\\ \dfrac{\partial^2}{\partial r^2}\end{pmatrix}(k\varphi_1+\varphi_2),$$
$$\sigma_{zz} = G_1(k+1)\frac{\partial^2}{\partial z^2}(s_1^{-2}\varphi_1+s_2^{-2}\varphi_2), \tag{2.12}$$
$$\sigma_{zr} = G_1(k+1)\frac{\partial^2}{\partial r\,\partial z}(\varphi_1+\varphi_2),$$

where G and G_1 denote the shear moduli in the z-plane and along the z-axis, respectively.

The displacement functions $\varphi_1(r,z)$ and $\varphi_2(r,z)$ may be taken in the following forms, noting some symmetries of the stress state, the condition at infinity and the conditions

$\sigma_{zr}(r, 0) = \sigma_{zr}(r, h) = 0$:

$$\varphi_i(\varrho, \zeta) = (-1)^{i+1} \frac{s_{i\pm 1}}{G_1(k+1)\beta} \int_0^\infty \left\{ \frac{p(x)}{x} \left[e^{-s_i x\zeta} + \frac{1}{\operatorname{sh} s_i x} \left(e^{-s_i x} \operatorname{ch} s_i x\zeta - g_1(x) \operatorname{ch} s_i x(1-\zeta) \right) \right] + \frac{\omega(x)}{x \operatorname{sh} s_i x} [\operatorname{ch} s_i x\zeta - g_2(x) \operatorname{ch} s_i x(1-\zeta)] \right\} J_0(x\varrho)\,dx, \quad i = 1, 2 \tag{2.13}$$

where $\varrho = r/h$, $\zeta = z/h$ and $J_0(x\varrho)$ is Bessel function of the first kind of order zero, $p(x)$ and $\omega(x)$ are unknown functions which are to be determined by the boundary conditions and the functions $g_1(x)$ and $g_2(x)$ are defined as follows

$$g_i(x) = \frac{1}{\operatorname{sh} \alpha x + \alpha\beta^{-1} \operatorname{sh} \beta x} \cdot \begin{cases} \operatorname{ch} \beta x + \alpha\beta^{-1} \operatorname{sh} \beta x - e^{-\alpha x}, & i = 1 \\ 2\beta^{-1}(s_1 \operatorname{sh} s_1 x - s_2 \operatorname{sh} s_2 x), & i = 2. \end{cases} \tag{2.14}$$

The displacement w and stresses σ_{zz}, σ_{rz} of interest which correspond to Eqs. (2.9), (2.12) and (2.13) are:

$$w(\varrho, \zeta) = \frac{s_1 s_2}{G_1(k+1)\beta h} \int_0^\infty \left\{ p(x) \left[k e^{-s_2 x\zeta} - e^{-s_1 x\zeta} + e^{-s_1 x} \frac{\operatorname{sh} s_1 x\zeta}{\operatorname{sh} s_1 x} - k e^{-s_2 x} \frac{\operatorname{sh} s_2 x\zeta}{\operatorname{sh} s_2 x} + g_1(x) \left(\frac{\operatorname{sh} s_1 x(1-\zeta)}{\operatorname{sh} s_1 x} - k \frac{\operatorname{sh} s_2 x(1-\zeta)}{\operatorname{sh} s_2 x} \right) \right] + \omega(x) \left[\frac{\operatorname{sh} s_1 x\zeta}{\operatorname{sh} s_1 x} - k \frac{\operatorname{sh} s_2 x\zeta}{\operatorname{sh} s_2 x} + g_2(x) \left(\frac{\operatorname{sh} s_1 x(1-\zeta)}{\operatorname{sh} s_1 x} - k \frac{\operatorname{sh} s_2 x(1-\zeta)}{\operatorname{sh} s_2 x} \right) \right] \right\} J_0(x\varrho)\,dx + \frac{\gamma h^2}{2c_{33}} \zeta(\zeta-1) - \frac{c_{11}+c_{12}}{2c} h\zeta(2p_0+\gamma h), \tag{2.15}$$

$$\sigma_{zz}(\varrho, \zeta) = \frac{1}{\beta h^2} \int_0^\infty \left\{ p(x) \left[s_2 e^{-s_1 x\zeta} - s_1 e^{-s_2 x\zeta} + s_2 e^{-s_1 x} \frac{\operatorname{ch} s_1 x\zeta}{\operatorname{sh} s_1 x} - s_1 e^{-s_2 x} \frac{\operatorname{ch} s_2 x\zeta}{\operatorname{sh} s_2 x} - g_1(x) \left(s_2 \frac{\operatorname{ch} s_1 x(1-\zeta)}{\operatorname{sh} s_1 x} - s_1 \frac{\operatorname{ch} s_2 x(1-\zeta)}{\operatorname{sh} s_2 x} \right) \right] + \omega(x) \left[s_2 \frac{\operatorname{ch} s_1 x\zeta}{\operatorname{sh} s_1 x} - s_1 \frac{\operatorname{ch} s_2 x\zeta}{\operatorname{sh} s_2 x} - g_2(x) \left(s_2 \frac{\operatorname{ch} s_1 x(1-\zeta)}{\operatorname{sh} s_1 x} - s_1 \frac{\operatorname{ch} s_2 x(1-\zeta)}{\operatorname{sh} s_2 x} \right) \right] \right\} x J_0(x\varrho)\,dx - p_0 - \gamma h(1-\zeta), \tag{2.16}$$

$$\sigma_{zr}(\varrho, \zeta) = -\frac{s_1 s_2}{\beta h^2} \int_0^\infty \left\{ p(x) \left[e^{-s_2 x\zeta} - e^{-s_1 x\zeta} + e^{-s_1 x} \frac{\operatorname{sh} s_1 x\zeta}{\operatorname{sh} s_1 x} - e^{-s_2 x} \frac{\operatorname{sh} s_2 x\zeta}{\operatorname{sh} s_2 x} - g_1(x) \left(\frac{\operatorname{sh} s_2 x(1-\zeta)}{\operatorname{sh} s_2 x} - \frac{\operatorname{sh} s_1 x(1-\zeta)}{\operatorname{sh} s_1 x} \right) \right] + \omega(x) \left[\frac{\operatorname{sh} s_1 x\zeta}{\operatorname{sh} s_1 x} - \right.\right.$$

$$-\frac{\operatorname{sh}s_2x\zeta}{\operatorname{sh}s_2x}-g_2(x)\left(\frac{\operatorname{sh}s_2x(1-\zeta)}{\operatorname{sh}s_2x}-\frac{\operatorname{sh}s_1x(1-\zeta)}{\operatorname{sh}s_1x}\right)\Bigg]\Bigg\}xJ_1(x\varrho)dx;$$

$$0\leqslant\zeta\leqslant 1. \qquad (2.17)$$

The displacement u and stresses σ_{rr} and $\sigma_{\Theta\Theta}$ may be expressed similarly.

Superimposing Eqs. (2.4), (2.6) and (2.15) - (2.17) we obtain the representations for the stress and displacement. Especially, the displacement w and the stresses σ_{zz} and σ_{zr} on the layer surfaces $\zeta = 0$ and $\zeta = 1$ are given as:

$$\begin{aligned}
G_1hw(\varrho,0) &= C^{-1}\int_0^\infty \{p(x)[1-g_1(x)]-\omega(x)g_2(x)\}J_0(x\varrho)dx,\\
G_1hw(\varrho,1) &= -C^{-1}\int_0^\infty \omega(x)J_0(x\varrho)dx-G_1h^2\frac{c_{11}+c_{12}}{2c}(p_0+p_e),\\
\sigma_{zz}(\varrho,0) &= -h^{-2}\int_0^\infty xp(x)J_0(x\varrho)dx-p_e,\\
\sigma_{zz}(\varrho,1) &= -h^{-2}\int_0^\infty x\{p(x)g_2(x)+\omega(x)[1-g_3(x)]\}J_0(x\varrho)dx-p_0,\\
\sigma_{zr}(\varrho,0) &= \sigma_{zr}(\varrho,1) = 0,
\end{aligned} \qquad (2.18)$$

where

$$p_e = p_0+\gamma h, \qquad (2.19)$$

$$g_3(x) = 1-\frac{(1-g_2)(1+g_2)}{1-g_1} = 1-\frac{\operatorname{ch}\alpha x-1-\alpha^2\beta^{-2}(\operatorname{ch}\beta x-1)}{\operatorname{sh}\alpha x+\alpha\beta^{-1}\operatorname{sh}\beta x}, \qquad (2.20)$$

$$C = (k+1)(k-1)^{-1}(s_2^{-1}-s_1^{-1}), \qquad (2.21)$$

are pressure, known function and material parameter, respectively.

3. Boundary conditions

Consider axisymmetric contact problem of a thick plate of height h pressed onto a rigid base with a cylindrical protrusion (Problem I, Fig. 1a) or a pit (Problem II, Fig. 1b).

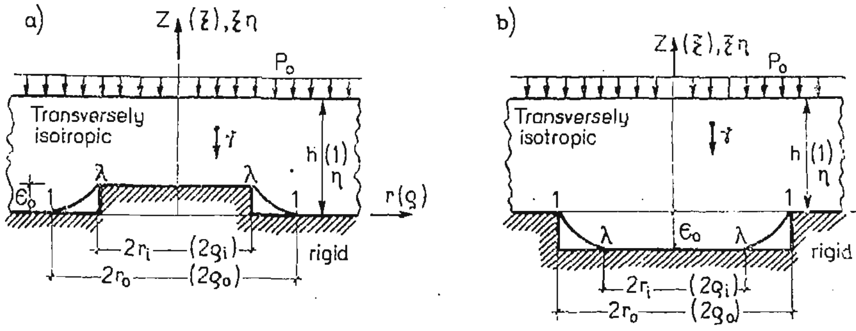

Fig. 1. Geometry of the problems (a) Problem I (b) Problem II

The layer is pressed by uniform pressure and the effect of gravity is also taken into account. We assume, that the magnitude of depth ϵ_0 is small.

In the second problem, if the pressure is small (without the body force) and the lower surface of the plate does not make contact with the bottom of the pit, the stress state of the plate is equivalent to that of the plate with a penny-shaped crack, which is analyzed by COLLINS [11] and author [12] for isotropic and transversely isotropic cases, respectively.

In the present paper we analyze, in the second problem, a three-part mixed boundary value problem where the applied pressure and the weight of the layer are so large, that a part of the lower surface of the layer $0 \leqslant \varrho \leqslant \varrho_i$, makes contact with the bottom of the pit.

The uncontact region is annular and it inner ϱ_i or outer ϱ_0 radius is not known a priori in second or first problem, respectively.
The radii ϱ_i and ϱ_0 depend on p_0, γ, ε_0, h and ϱ_0 or ϱ_i, respectively, and on the material properties of the layer.

The boundary conditions are:

$$w(\varrho, 0) = \begin{cases} \pm\epsilon_0, & 0 \leqslant \varrho \leqslant \varrho_i, \\ 0, & \varrho_0 \leqslant \varrho, \end{cases} \tag{3.1}$$

$$\sigma_{zz}(\varrho, 0) = 0, \quad \varrho_i < \varrho \leqslant \varrho_0 \quad \text{or} \quad \varrho_i \leqslant \varrho < \varrho_0, \tag{3.2}$$

$$\frac{dw(\varrho, 0)}{d\varrho} = 0, \quad \varrho = \varrho_0 \quad \text{or} \quad \varrho = \varrho_i, \tag{3.3}$$

$$\sigma_{zr}(\varrho, 0) = \sigma_{zr}(\varrho, 1) = 0, \quad \varrho \geqslant 0, \tag{3.4}$$

$$\sigma_{zz}(\varrho, 1) = -p_0, \quad \varrho \geqslant 0, \tag{3.5}$$

where the upper and the lower of the double sings denote the cases of the first and second problems, respectively. With the help of Eqs. (2.18) for the shearing stresses the boundary conditions (3.4) are satisfied automatically.

4. The triple integral equations

Get a new unknown function $t(x)$ and set as follows:

$$\begin{aligned} \omega(x) &= -g_2(x)t(x), \\ p(x) &= [1-g_3(x)]t(x), \end{aligned} \tag{4.1}$$

the boundary values of the displacement and stresses which correspond to Eqs. (2.18) are then given by

$$\begin{aligned} G_1 hw(\varrho, 0) &= C^{-1} \int_0^\infty t(x) J_0(x\varrho)\, dx, \\ \sigma_{zz}(\varrho, 0) &= -h^{-2} \int_0^\infty xt(x)[1-g_3(x)] J_0(x\varrho)\, dx - p_e, \\ G_1 hw(\varrho, 1) &= C^{-1} \int_0^\infty t(x) g_2(x) J_0(x\varrho)\, dx - G_1 h^2 \frac{c_{11}+c_{12}}{2c}(p_0+p_e), \\ \sigma_{zz}(\varrho, 1) &= -p_0 \end{aligned} \tag{4.2}$$

with the aid of the relation (2.20) in the first Eq. (2.18). The condition (3.5) is satified automatically.
The remaining boundary conditions (3.1) - (3.3) lead to the triple integral equations with the unknown function $t(x)$:

$$w(\varrho, 0) = (G_1 Ch)^{-1} \int\limits_0^\infty t(x) J_0(x\varrho)\, dx = \begin{cases} \pm \epsilon_0, & 0 \leqslant \varrho \leqslant \varrho_i, \\ 0 & \varrho_0 \leqslant \varrho, \end{cases} \tag{4.3}$$

$$\sigma_{zz}(\varrho, 0) = -h^{-2} \int\limits_0^\infty x t(x) [1 - g_3(x)] J_0(x\varrho)\, dx - p_e = 0, \quad \varrho_i < \varrho \leqslant \varrho_0 \quad \text{or } \varrho_i \leqslant \varrho < \varrho_0, \tag{4.4}$$

under condition

$$\frac{dw(\varrho, 0)}{d\varrho} = -(G_1 Ch)^{-1} \int\limits_0^\infty x t(x) J_1(x\varrho)\, dx = 0, \quad \varrho = \varrho_0 \quad \text{or} \quad \varrho = \varrho_i. \tag{4.5}$$

We use here the series expansion method to solve the above integral equations. This technique reduces the mixed boundary value problem to the solution of an infinite set of simultaneous linear algebraic equations [4], which are easier to solve than complicated integral equations.

5. Analysis

We employ a nondimensional geometric parameters

$$\lambda = \frac{r_i}{r_0}, \quad \eta = \frac{h}{r_0}. \tag{5.1}$$

These parameters describe the contact regions and are generally unknown quantities, because r_i or r_0 are unknown in the second and first problems, respectively.
Then, Eqs. (4.3) and (4.4), and condition (4.5) can be rewritten to the form

$$w(\varrho, 0) = (G_1 Cr_0)^{-1} \int\limits_0^\infty t(x) J_0(x\varrho)\, dx = \begin{cases} \pm \epsilon_0, & 0 \leqslant \varrho \leqslant \lambda, \\ 0, & 1 \leqslant \varrho, \end{cases} \tag{5.2}$$

$$\sigma_{zz}(\varrho, 0) = -r_0^{-2} \int\limits_0^\infty x t(x) [1 - g_3(x\eta)] J_0(x\varrho)\, dx - p_e = 0, \quad \lambda < \varrho \leqslant 1 \quad \text{or} \quad \lambda \leqslant \varrho < 1, \tag{5.3}$$

$$\frac{dw(\varrho, 0)}{d\varrho} = -(G_1 Cr_0)^{-1} \int\limits_0^\infty x t(x) J_1(x\varrho)\, dx = 0, \quad \varrho = 1 \quad \text{or} \quad \varrho = \lambda. \tag{5.4}$$

The function $g_3(x\eta)$ tends exponentially to zero as $x\eta$ tends to infinity, is continuous for any $x \in (0, \infty)$, because $\alpha \in R_+$ and its limit is equal unity for $x\eta \to 0$.

When η tends to infinity i.e. in the case of the half-space, then the function $g_3(x\eta)$ identically equals zero. For the isotropic solid, i.e. when $s_2 \to s_1 \to 1$ and $k \to 1$, the function $g_3(x)$ assumes the form:

$$g_3(x) = 2\,\frac{x+x^2+e^{-x}\operatorname{sh}x}{\operatorname{sh}2x+2x} \tag{5.5}$$

and does not depend on the material properties of the solid. For our solid the boundary functions $g_3(x)$, $g_1(x)$ and $g_2(x)$ depend on the material properties and the solution of the integral equations depends on the anisotropic properties of the material.

Now, interchanging the variable ϱ in $\lambda \leqslant \varrho \leqslant 1$ to Φ in $0 \leqslant \Phi \leqslant \pi$

$$\varrho = \frac{1}{\sqrt{2}}[1+\lambda^2-(1-\lambda^2)\cos\Phi]^{\frac{1}{2}}, \tag{5.6}$$

the variables ϱ and Φ correspond each other and $\varrho = \lambda$ is $\Phi = 0$ and $\varrho = 1$ to $\Phi = \pi$.

We put the function $t(x)$ in the form of integral as follows:

$$t(x) = \int_0^{\pi} F(\psi)J_1(xR)\,d\psi, \qquad R = \frac{1}{\sqrt{2}}[1+\lambda^2-(1-\lambda^2)\cos\psi]^{\frac{1}{2}}, \tag{5.7}$$

where $F(\psi)$ is an arbitrary continuous function in the interval $0 \leqslant \psi \leqslant \pi$.
Substituting Eq. (5.7) into $w(\varrho, 0)$ of Eq. (5.2) and $dw(\varrho, 0)/d\varrho$ of Eq. (5.4), we obtain:

$$w(\varrho, 0) = (G_1 C r_0)^{-1}\int_0^{\pi}\frac{1}{R}F(\psi)H(R-\varrho)\,d\psi, \tag{5.8}$$

$$\frac{dw(\varrho, 0)}{d\varrho} = -(G_1 C r_0)^{-1}\int_0^{\pi}\frac{1}{R}F(\psi)\delta(R-\varrho)\,d\psi. \tag{5.9}$$

Since the argument of the delta function is $R-\varrho \neq 0$ in the intervals $0 \leqslant \varrho < \lambda$ and $1 < \varrho$ because of $\lambda \leqslant R \leqslant 1$, the radial gradient of the displacement $w(\varrho, 0)$ is always equal to zero on the contact surface $< 0, \lambda)\cup(1, \infty)$ independent of the function $F(\psi)$. On the other hand, using

$$H(R-\varrho) = \begin{cases} 1, & 0 \leqslant \varrho \leqslant \lambda, \quad (0 \leqslant \lambda \leqslant R \leqslant 1), \\ \begin{cases} 0, & 0 \leqslant \psi < \phi, \quad (R < \varrho), \\ 1, & \phi \leqslant \psi \leqslant \pi, \quad (R \geqslant \varrho), \end{cases} \\ 0, & 1 \leqslant \varrho, \quad (\lambda \leqslant R \leqslant 1 \leqslant \varrho), \end{cases} \tag{5.10}$$

we see that the displacement $w(\varrho, 0)$ is equal constant in the interval $0 \leqslant \varrho < \lambda$, is a function of ϱ in the interval $\lambda \leqslant \varrho \leqslant 1$ and equals zero in the remaining one $1 \leqslant \varrho$, independent of the function $F(\psi)$.

Integrating in Eq. (5.9), we obtain

$$\frac{dw(\varrho, 0)}{d\varrho} = -\frac{4}{G_1 C r_0(1-\lambda^2)}\cdot\frac{F(\Phi)}{\sin\Phi}, \qquad 0 < \Phi < \pi \tag{5.11}$$

and zero in the contact region.

Because the layer contacts smoothly at the edges $\varrho = 1$ or $\varrho = \lambda$ with the rigid base in the first and second problems, respectively, the gradient of $w(\varrho, 0)$ must be finite at $\varrho \to 1$ or $\varrho \to \lambda$, respectively. That is equivalent to the conditions (5.4), which lead to

$$F(\pi) = 0 \quad \text{or} \quad F(0) = 0 \tag{5.12}$$

in the first and second problems, respectively.
The integral representation of the function $t(x)$ by Eq. (5.7) satisfies the displacement conditions, if the function $F(\psi)$ satisfies first or second Eq. (5.12), respectively. It should be noted that, rigorously speaking, we have two unknown functions $F_+(\psi)$ and $F_-(\psi)$ for first and second problems, respectively. We assume also, that these unknown functions take a finite and non-zero possitive or negative values in the intervals $0 < \psi \leqslant \pi$ or $0 \leqslant \leqslant \psi < \pi$ in the first and second problems, respectively. Then the displacement $w(\varrho, 0)$ is continuous in the interval $\lambda \leqslant \varrho \leqslant 1$, it gradient takes definite and non-zero values negative or positive in $\lambda < \varrho < 1$, tends to minus infinity at contact edge $\varrho \to \lambda^+$ or infinity at $\varrho \to 1^-$, and equals to zero for $\varrho = 1$ or $\varrho = \lambda$ in the first and second problem respectively, and consequently, the slope of $w(\varrho, 0)$ at contact edges $\varrho = \lambda$ and $\varrho = 1$ coincides with that of the contact face.

The function $F(\psi)$ can be expressed by a Fourier cosine series

$$F(\psi) = \frac{1}{\pi} p_e r_0^2 R \sum_{n=0}^{\infty} a_n \cos n\psi. \quad 0 \leqslant \psi \leqslant \pi, \tag{5.13}$$

where a_n are unknown coefficients, which are to be determined by the boundary conditions (5.3), and for which from the conditions (5.12), we obtain

$$\sum_{n=0}^{\infty} (-1)^n a_n = 0 \quad \text{or} \quad \sum_{n=0}^{\infty} a_n = 0. \tag{5.14}$$

Substituting Eq. (5.13) into Eq. (5.8) and integrating, we obtain the displacement:

$$w(\varrho, 0) = \begin{cases} \dfrac{p_e r_0 a_0}{G_1 C} = \pm \epsilon_0, & 0 \leqslant \varrho \leqslant \lambda, \\ \dfrac{p_e r_0 a_0}{G_1 C} \left\{1 - \dfrac{1}{\pi}\left[\Phi + \dfrac{1}{a_0} \displaystyle\sum_{n=1}^{\infty} \dfrac{a_n}{n} \sin n\phi\right]\right\}, & \lambda \leqslant \varrho \leqslant 1, \ 0 \leqslant \phi \leqslant \pi, \\ 0, \quad 1 \leqslant \varrho & \end{cases} \tag{5.15}$$

and the relation

$$\epsilon_0 = \pm \frac{p_e r_0 a_0}{G_1 C}. \tag{5.16}$$

Eq. (5.16) gives the relation between p_e, ϵ_0, r_0 or r_i and a_0 where a_0 depends on the parameters of the contact regions λ and η and on the properties of the material.
Substituting Eq. (5.13) into Eq. (5.7) and the result into Eqs. (5.3), and using some relations for Bessel functions, we obtain:

$$t(x) = -p_e r_0^2 \sum_{=0}^{\infty} a_n \frac{\partial Z_n(x)}{\partial x}, \tag{5.17}$$

$$\sigma_{zz}(\varrho, 0) = p_e \sum_{n=0}^{\infty} a_n \int_0^{\infty} [1-g_3(x\eta)] \frac{\partial Z_n(x)}{\partial x} x J_0(x\varrho) dx - p_e, \tag{5.18}$$

$$\sum_{n=0}^{\infty} a_n \int_0^{\infty} [1-g_3(x\eta)] \frac{\partial Z_n(x)}{\partial x} x J_0(x\varrho) dx = 1, \quad \lambda < \varrho \leqslant 1 \quad \text{or} \quad \lambda \leqslant \varrho < 1, \tag{5.19}$$

where

$$Z_n(x) = J_n\left[\frac{x}{2}(1+\lambda)\right] J_n\left[\frac{x}{2}(1-\lambda)\right]. \tag{5.20}$$

Multiplying the both sides of Eq. (5.19) by ϱ, using the formula $x\varrho J_0(x\varrho) = \partial[\varrho J_1(x\varrho)]/\partial\varrho$, integrating with respect to ϱ and using the formula $\partial[J_0(x\varrho)]/\partial x = -\varrho J_1(x\varrho)$, we obtain:

$$\sum_{n=0}^{\infty} a_n \int_0^{\infty} [1-g_3(x\eta)] \frac{\partial Z_n(x)}{\partial x} \cdot \frac{\partial J_0(x\varrho)}{\partial x} dx = -\frac{1}{2}\varrho^2 - c_i, \quad \lambda < \varrho \leqslant 1 \quad \text{or} \quad \lambda \leqslant \varrho < 1, \tag{5.21}$$

where c_i $(i = 1, 2)$ are unknown integral constants.
Using the Neumann's formula [13]

$$J_0(x\varrho) = Z_0(x) + 2\sum_{m=1}^{\infty} Z_m(x)\cos m\varphi, \quad \lambda \leqslant \varrho \leqslant 1 \tag{5.22}$$

in Eq. (5.21), we get

$$\sum_{n=0}^{\infty} a_n \int_0^{\infty} [1-g_3(x\eta)] \frac{\partial Z_n(x)}{\partial x} \left\{\frac{\partial Z_0(x)}{\partial x} + 2\sum_{m=1}^{\infty} \frac{\partial Z_m(x)}{\partial x} \cos m\varphi\right\} dx =$$

$$= -\frac{1}{4}[1+\lambda^2-(1-\lambda^2)\cos\varphi] - c_i, \quad 0 \leqslant \Phi \leqslant \pi, \quad i = 1, 2. \tag{5.23}$$

Assuming the coefficients a_n as

$$a_n = a_n' - c_i a_n'', \quad i = 1, 2 \tag{5.24}$$

and equating the coefficients of $\cos m\varphi$ in both sides of Eq. (5.23), we obtain two infinite systems of simultaneous equations for the determination of the coefficients a_n' and a_n'')

$$\sum_{n=0}^{\infty} a_n' A_{mn} = -\frac{1}{4}\left[(1+\lambda^2)\delta_{0m} - \frac{1}{2}(1-\lambda^2)\delta_{1m}\right],$$
$$\sum_{n=0}^{\infty} a_n'' A_{mn} = \delta_{0m}, \quad m = 0, 1, 2, \ldots \tag{5.25}$$

where A_{mn} denote the symmetrical matrix given by

$$A_{mn} = \int_0^{\infty} [1-g_3(x\eta)] \frac{\partial Z_n(x)}{\partial x} \frac{\partial Z_m(x)}{\partial x} dx, \quad m, n = 0, 1, 2, \ldots \tag{5.26}$$

and δ_{0m}, δ_{1m} are Kronecker's delta.

The constants c_i are determined from Eqs. (5.14) and (5.24) as:

$$\sum_{n=0}^{\infty}(-1)^n a_n' - c_1\sum_{n=0}^{\infty}(-1)^n a_n'' = 0,$$
$$\sum_{n=0}^{\infty} a_n' - c_2\sum_{n=0}^{\infty} a_n'' = 0 \tag{5.27}$$

in the first and second problems, respectively.

The presented three-part mixed boundary value problem given by Eqs. (5.2) and (5.3) under condition (5.4) is, therefore, reduced to one of solving two infinite systems of simultaneous equations (5.25). Solving these equations for dissimilar parameters λ and η of the contact regions and given material, we obtain the coefficients a_n from Eqs. (5.27) and (5.24) in both problems.

The other quantities of physical interest will be determined in the next sections.

6. Displacement and stress components on the surfaces of the layer

The displacement of the lower surface of the layer is given by Eqs. (5.15) and of the upper surface by Eqs. (4.2) and (5.17)
We have

$$w(\varrho, 1) = -\frac{p_e r_0}{G_1 C}\sum_{n=0}^{\infty} a_n G_2^n - \frac{c_{11}+c_{12}}{2c}(p_0+p_e)h, \tag{6.1}$$

where

$$G_2^n = \int_0^{\infty} g_2(x\eta)J_0(x\varrho)\frac{\partial Z_n(x)}{\partial x}dx, \quad n = 0, 1, 2, \ldots \tag{6.2}$$

are the convergent integrals.

The stress $\sigma_{zz}(\varrho, 1) = -p_0$ and $\sigma_{zz}(\varrho, 0)$ corresponding to Eq. (5.18) is:

$$\sigma_{zz}(\varrho, 0) = -p_e\left[1+\sum_{n=0}^{\infty} a_n\left(G_3^n + I_0^n + \varrho\frac{\partial}{\partial\varrho}I_0^n\right)\right], \tag{6.3}$$

where

$$G_3^n = \int_0^{\infty} g_3(x\eta)xJ_0(x\varrho)\frac{\partial Z_n(x)}{\partial x}dx,$$
$$I_0^n = \int_0^{\infty} J_0(x\varrho)Z_n(x)dx, \quad n = 0, 1, 2, 3, \ldots \tag{6.4}$$

are the convergent integrals.

The integrals I_0^n can be evaluated by the results in the paper of the author [14] where they are presented in analytical form by Gaussian hypergeometric functions. Since the continuous functions $g_2(x\eta)$ and $g_3(x\eta)$ converge exponentially to zero as the value of

x becomes large, the integrals G_2^n and G_3^n can be easily evaluated numerically, for example by the second rule of Simpson's numerical integral formula.

For the half-space problem ($\eta \to \infty$) these integrals are equal to zero. In this case the values of the solutions of two infinite sets of linear equations and the stress $\sigma_{zz}(\varrho, 0)$ do not depend on the material properties and the effect of transverse isotropy includes in the displacement w and relation (5.16) by the material constant $G_1 C$.

To determine a singularity of the contact stress, we consider its behaviour near the contact edges into material regions. Using the asymptotic expansion of $J_n(x\varrho)$ with large value of x [13], we obtain

$$x \frac{\partial Z_n(x)}{\partial x} \doteq -\frac{2}{\pi\sqrt{1-\lambda^2}}[\lambda \sin x\lambda - (-1)^n \cos x] + O(x^{-1}). \tag{6.5}$$

Thus Eq. (5.18) can be rewritten as follows:

$$\begin{aligned}\sigma_{zz}(\varrho, 0) = -p_e\Bigg\{1 + \sum_{n=0}^{\infty} a_n G_3^n - \sum_{n=0}^{\infty} a_n \int_0^{\infty}\Bigg[x\frac{\partial Z_n(x)}{\partial x} + \\ + \frac{2}{\pi\sqrt{1-\lambda^2}}(\lambda\sin x\lambda - (-1)^n\cos x)\Bigg] J_0(x\varrho)\,dx + \\ + \frac{2}{\pi\sqrt{1-\lambda^2}}\Bigg[\frac{\lambda}{\sqrt{\lambda^2-\varrho^2}} H(\lambda-\varrho)\sum_{n=0}^{\infty} a_n - \frac{1}{\sqrt{\varrho^2-1}} H(\varrho-1)\sum_{n=0}^{\infty}(-1)^n a_n\Bigg]\Bigg\}.\end{aligned} \tag{6.6}$$

It is apparent from Eq. (6.6) that the stress has the singular parts such as $(\lambda-\varrho)^{-1/2}$ in the first or $(\varrho-1)^{-1/2}$ in the second problem because of the conditions (5.14). The infinite integrals with respect to x in Eq. (6.6) are convergent because their integrands are $O(x^{-3/2}$ as $x \to \infty$. In addition, in the first problem $\sigma_{zz}(r_0, 0)$ equals zero, whereas in the second $\sigma_{zz}(r_i, 0)$ equals zero. Using Eq. (6.6) we can easily evaluate the stress concentration factors.

7. The stress concentration factors

In analogy with the stress intensity factors in the annular crack problem, we define the stress concentration factors at the edges of the contact regions by the expressions:

$$\begin{aligned} N_i &= \lim_{\varrho \to \lambda^-} \sqrt{2r_0(\lambda-\varrho)}\,\{\sigma_{zz}(\varrho, 0)\}_{\varrho<\lambda}, \\ N_0 &= \lim_{\varrho \to 1^+} \sqrt{2r_0(\varrho-1)}\,\{\sigma_{zz}(\varrho, 0)\}_{\varrho>1}, \end{aligned} \tag{7.1}$$

or in terms of the coefficients a_n:

$$\begin{aligned} N_i &= \frac{2p_e\sqrt{r_i}}{\pi\sqrt{1-\lambda^2}} \sum_{n=0}^{\infty} a_n, \\ N_0 &= \frac{2p_e\sqrt{r_0}}{\pi\sqrt{1-\lambda^2}} \sum_{n=0}^{\infty} (-1)^{n+1} a_n. \end{aligned} \tag{7.2}$$

for the inner contact edge and the outer one in the first and second problems, respectively. In the first problem $\sigma_{zz}(r_0, 0)$ equals zero, whereas in the second $\sigma_{zz}(r_i, 0)$ is zero.

8. The critical loads

Let $p_{cr} = (\gamma h \pm p_0)_{cr}$ be the indentated load giving the state in which the layer contacts the bottom of the pit only in the point $r = 0$.
Then, Eqs. (5.16) and (2.19), give:

$$(\gamma h \pm p_0)_{cr} = -\frac{\epsilon_0 G_1 C}{r_0 a_0(\lambda = 0)}, \tag{8.1}$$

where the upper and the lower of the double singns denote the cases of pressure and tension, respectively. When the load $p_e = \gamma h \pm p_0$ is above the critical value, the uncontacting area in the second problem will be an annulus $\lambda < \varrho < 1$, the inner circumference of which will shrink with a decreasing load, and when the load is less than a critical value, the uncontacting region will be a circle. After determining a_0 by Eqs. (5.25), (5.24) and $(5.27)_2$ for $\lambda = 0$ and known η, we obtain the critical load.

For the special case $r_0 = \text{const}$, $r_i \to 0$ and $\eta \to \infty$ (the half-space problem), Eq. (5.19) can be rewritten as follows

$$\sum_{n=0}^{\infty} a_n \int_0^{\infty} \frac{\partial}{\partial x}\left[\mathrm{J}_n^2\left(\frac{x}{2}\right)\right] x\mathrm{J}_0(x\varrho)\,dx = 1, \quad 0 \leqslant \varrho < 1. \tag{8.2}$$

Using the formula

$$\frac{2}{\pi}\int_0^{\infty} x\mathrm{J}_0(x\varrho)\frac{d}{dx}\left(\frac{\sin x}{x}\right) = -1, \quad 0 \leqslant \varrho < 1, \tag{8.3}$$

we have

$$\sum_{n=0}^{\infty} a_n \mathrm{J}_n^2\left(\frac{x}{2}\right) = -\frac{2}{\pi}\frac{\sin x}{x}. \tag{8.4}$$

Making use of the formulae [13]

$$\begin{gathered} \int_0^{\pi/2} \mathrm{J}_0(x\sin\Theta)\sin\Theta\, d\Theta = \frac{\sin x}{x}, \\ \mathrm{J}_0(x\sin\Theta) = \mathrm{J}_0^2\left(\frac{x}{2}\right) + 2\sum_{n=1}^{\infty} \mathrm{J}_n^2\left(\frac{x}{2}\right)\cos 2n\Theta, \end{gathered} \tag{8.5}$$

we see that

$$\begin{aligned} \sum_{n=0}^{\infty} a_n \mathrm{J}_n^2\left(\frac{x}{2}\right) &= -\frac{2}{\pi}\int_0^{\pi/2}\left[\mathrm{J}_0^2\left(\frac{x}{2}\right) + 2\sum_{n=1}^{\infty}\mathrm{J}_n^2\left(\frac{x}{2}\right)\cos 2n\Theta\right]\sin\Theta\, d\Theta = \\ &= -\frac{2}{\pi}\mathrm{J}_0^2\left(\frac{x}{2}\right) + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\mathrm{J}_n^2\left(\frac{x}{2}\right). \end{aligned} \tag{8.6}$$

It is apparent from Eq. (8.6) that

$$a_0 = -\frac{2}{\pi}, \quad a_n = \frac{4}{\pi} \cdot \frac{1}{4n^2-1}, \quad n = 1, 2, 3, \ldots \tag{8.7}$$

are the solution of Eq. (8.2).

The solution (8.7) satisfies the second condition (5.14). Substituting the value $a_0 = -2/\pi$ into Eq. (8.1), we get for larger values of h/r_0

$$(\gamma h \pm p_0)_{cr} = \frac{\pi}{2} G_1 C \frac{\epsilon_0}{r_0}. \tag{8.8}$$

Especially, if

$$p_0 \geqslant p_{0cr} = \frac{\pi}{2} G_1 C \frac{\epsilon_0}{r_0}, \tag{8.9}$$

or

$$p_0 \leqslant p_{0cr} = \gamma h - \frac{\pi}{2} G_1 C \frac{\epsilon_0}{r_0}, \tag{8.10}$$

for the pressure p_0 without the body force and for the tension p_0 and the weight γh, respectively a part of the lower surface of the thick plate is in contact with the bottom of the pit. When $\eta = h/r_0$ decreases, then the critical load also decreases. In the limiting case of $\eta \to 0$ the function $g_3(x\eta)$ tends to unity and the solution a_0 tends to infinity, and consequently the critical value of the load tends to zero.

If $p_e < \pi C G_1 \epsilon_0/2r_0$ the elastic body does not make contact with the bottom of the pit and the solutions are as follows:

$$\begin{aligned} t(x) &= \frac{2}{\pi} p_e r_0^2 \frac{d}{dx}\left(\frac{\sin x}{x}\right), \\ w(\varrho, 0) &= -\frac{2}{\pi} \cdot \frac{p_e r_0}{G_1 C} \sqrt{1-\varrho^2}\, H(1-\varrho), \\ \sigma_{zz}(\varrho, 0) &= -p_e\left[1 - \frac{2}{\pi} \arcsin\left(\frac{1}{\varrho}\right) + \frac{2}{\pi} \cdot \frac{1}{\sqrt{\varrho^2-1}}\right] H(\varrho-1), \\ N_0 &= \frac{2p_e\sqrt{r_0}}{\pi}. \end{aligned} \tag{8.11}$$

The formulas (8.11) agree with the results for the solid with a penny-shaped crack if we replace $-p_e$ by p_0, which are given by Collins [11] and author [12] for isotropic, i.e. $C = 1/(1-\nu)$, and transversely isotropic case, respectively. In the special case for a half-space problem and $r_i = 0$, the contact stress and the stress concentration factor N_0 do not depend on anisotropy of the material, whereas the displacement depends. In the layer contact problem, the stress and displacement fields, and the stress concentration factor depend on the material properties of the solid. By means of results present in the paper, the effect of transverse isotropy may be examined.

9. Numerical calculations

At present, we must determine the values $\lambda = r_i/r_0$ for given p_e, ϵ_0, h, r_i or r_0 and material constants. However, it is considerably difficult to determine the unknown ratio λ by the above procedure. Therefore, we determine the relationship among p_e, ϵ_0 and h from Eq. (5.16) under the condition that the ratios of the inner to outer radius $\lambda = r_i/r_0$ and $\eta = h/r_0 = h\lambda/r_i$ are given and solving the simultaneous equations (5.25).

To solve these equations we evaluate the infinite integrals A_{mn} involving the product of four Bessel functions by the following method. The element A_{mn} of m-th row and n-th column can be rewritten as

$$A_{mn} = \int_0^{x_1} \frac{\partial}{\partial x}[Z_m(x)]\frac{\partial}{\partial x}[Z_n(x)]\,dx + A'_{mn} -$$

$$- \int_0^{x_0} g_3(x\eta)\frac{\partial}{\partial x}[Z_m(x)]\frac{\partial}{\partial x}[Z_n(x)]\,dx, \tag{9.1}$$

where

$$A'_{mn} = \int_{x_1}^{0} \frac{\partial}{\partial x}[Z_m(x)]\frac{\partial}{\partial x}[Z_n(x)]\,dx \approx$$

$$\approx \frac{4}{\pi^2(1-\lambda^2)}\left\{\frac{\lambda^2\sin^2\lambda x_1}{x_1} - \lambda^3\mathrm{si}(2\lambda x_1) + (-1)^{m+n}\left[\frac{\cos^2 x_1}{x_1} + \mathrm{si}(2x_1)\right] - \right. \tag{9.2}$$

$$\left. - \lambda[(-1)^m + (-1)^n]\left[\frac{\sin\lambda x_1\cos x_1}{x_1} - \frac{1}{2}(1+\lambda)\mathrm{ci}[x_1(1+\lambda)] + \frac{1}{2}(1-\lambda)\mathrm{ci}[x_1(1-\lambda)]\right]\right\},$$

and x_1, is taken to be a very large value, and x_0 a large value.

The second term A'_{mn} is obtained by using the asymptotic approximation of Bessel function, integrating Eq. (9.2) by parts and using sine and cosine integral functions $\mathrm{si}(x)$ and $\mathrm{ci}(x)$. The first and third terms on the right hand side of Eq. (9.1) are integrated numerically with sufficient convergence by means of Simpson's rule taking $x_1 = 500$ and $x_0 = 20/\alpha\eta$. The algebraic equations are solved by truncation, i.e. we calculate only the first n roots of them. We can get numerically good results, taking $n = 15$ or $n = 10$ for $\lambda \leqslant 0{,}2$ or $\lambda > 0{,}2$ and $\eta \geqslant 1$, respectively, and $n = 20$ or $n = 15$ in the case $\lambda \leqslant 0{,}2$ or $\lambda > 0{,}2$ and $\eta < 1$. With a decreasing degree of anisotropy (E/E_1, G/G_1) the convergence in the numerical calculation becomes slower. For $E/E_1 \ll 1$ and $G/G_1 \ll 1$we must take more equations, respectively for E/E_1, $G/G_1 \gg 1$ we can take less.

10. Numerical results

Numerical results show the relations between p_e, ϵ_0, h, r_i, and r_0 (in Problems I and II) in cadmium and magnesium single crystals and fiber-reinforced composite materials such as E glass-epoxy and graphite-epoxy with fiber direction along z-axis, and they are

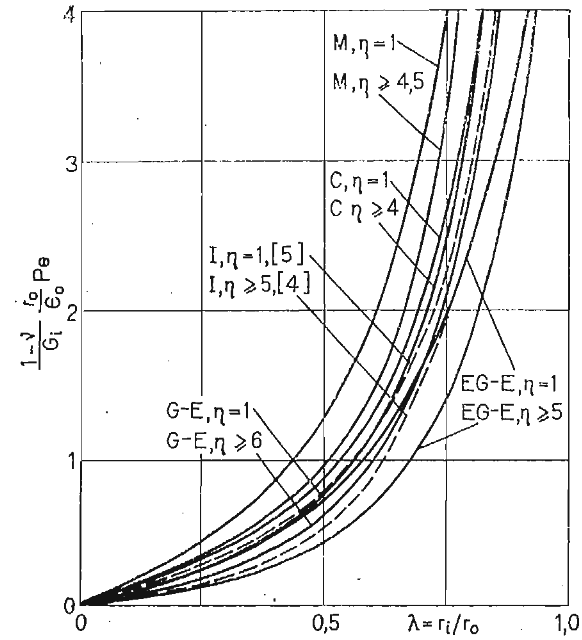

Fig. 2. Relations of p_e, ϵ_0, h, r_i and r_0 for dissimilar materials in Problem I

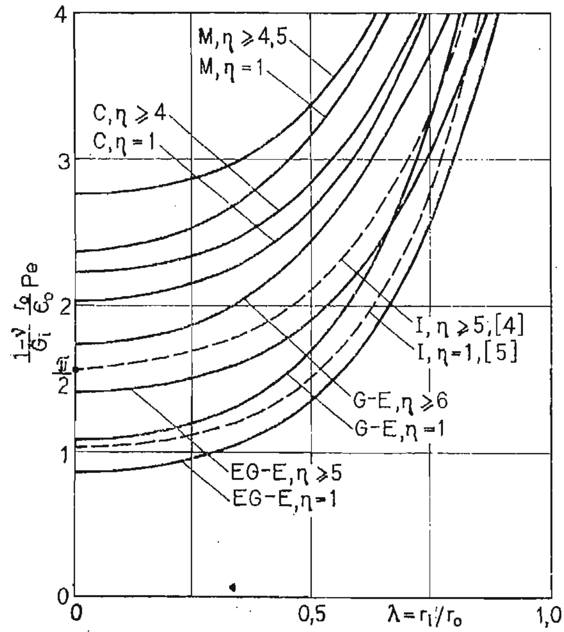

Fig. 3. Relations of p_e, ϵ_0, h, r_i and r_0 for dissimilar materials in Problem II

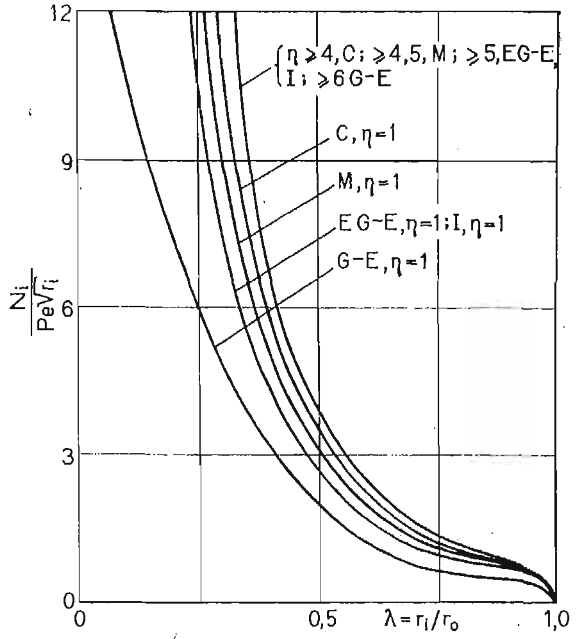

Fig. 4. The variation of N_i with λ for dissimilar materials and $\eta = h/r_0$ in Problem I

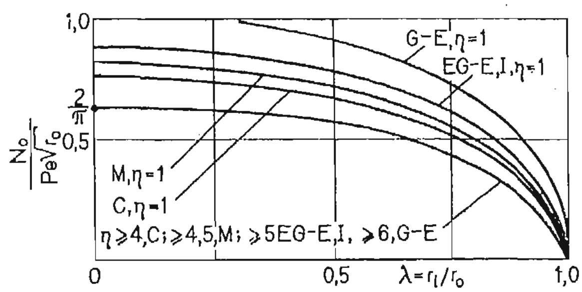

Fig. 5. The variation of N_0 with λ for dissimilar materials and $\eta = h/r_0$ in Problem II

compared with those of the isotropic material [4] to show the effect of anisotropy. The stress concentration factors are also shown graphically.

The elastic constant c_{ij} given by HUNTINGTON [15] and CHEN [16] are used. The values of s_1, s_2, k are

1,58; 0,98; 1,85   1,41; 0,70; 2,78

1,67; 0,34; 12,7   1,36; 0,23; 21,4

for cadmum, magnesium , E glass-epoxy, graphite-epoxy, respectively and for isotropic material 1; 1; 1; $G_i = 10^{10}$ N/m², $\nu = 0{,}30$.

As shown in Figs. 2 and 3 in each case, $(1-\nu)p_e r_0/G_i\, \epsilon_0$ increases with an increasing $\lambda = r_i/r_0$ and tends to the case of an elastic half-space, with corresponding material, with an increaing $\eta = h/r_0$. In each figure, the results indicated by the chain line, show those for isotropic material. These results agree with the ones of Refs. [4.5], where the weight is omitted ($p_e \to p_0$). Figs 4 and 5 show the variation of the stress concentration factors with $\lambda = r_i/r_0$ and $\eta = h/r_0$ for dissimilar materials. N_i (for the protrusion) is always greater than N_0 (for the pit) and becomes very large when $\lambda \to 0$. With the increasing of λ, N_i decreases and N_0 decreases slowly. The stress concentration factors are different for presented material and become small (N_0) or larger (N_i) as the layer becomes thick under the same protrusion or pit dimension, converging to the same values for an infinite body.

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Резюме

ТРАНСВЕРТАЛЬНО-ИЗОТРОПНЫЙ СЛОЙ ПРИЖИМАЕМЫЙ К ЖЕСТКОМУ ОСНОВАНИЮ С ВЫСТУПОМ ИЛИ ВПАДИНОЙ

Автор решил сформулированные в заглавие задачи, в которых принял во внимание эффекты трансфертальной анизотропии и собственного веса, при помощи интегральных преобразований Ханкеля и потенциалов перемещения. Смешанную краевую задачу сведено к тройным интеграль-

ным уравнениям и некоторым условиям. В результате представления неизвестной функции в виде ряда Фурье с неопределенными пока коэффициентами эти уравнения приводятся к решению двух бесконечных систем линейных алгебраических уравнений относительно коэффициентов ряда Фурье.

В каждой из указанных задач контактные давления имеют сингулярность на одной из границ области контакта. Область контакта распадается на два участка, разделяемых кольцевой областью с неизвестными внешним либо внутренным радиусами.

Численные результаты представляют зависимости между сжатием и собственным весом пластинки, ее толщиной, кольцевой областью и величинами выступа или впадины основания в кристалах кадмия и магния и в композитных материалах армированных волокнами. Сравнено их с результатами для изотропной среды, чтобы выяснить эффект анизотропий. Графически иллюстрируется изменение коэффициентов концентрации напряжения на краях выступа или впадины в зависимости от соотношения радиусов контактной области для разных материалов и толщин пластинки.

Streszczenie

WARSTWA POPRZECZNIE IZOTROPOWA DOCISKANA DO SZTYWNEGO PODŁOŻA Z WZNIESIENIEM ALBO ZAGŁĘBIENIEM

Autor rozwiązał sformułowane w tytule zagadnienia, w których uwzględnił efekty poprzecznej anizotropii i siły masowej, za pomocą transformacji Hankela i potencjałów przemieszczenia. Mieszane zagadnienie brzegowe sprowadzono do rozwiązania potrójnych równań całkowych i pewnych warunków. Te z kolei sprowadzono do dwóch układów nieskończonych równań algebraicznych liniowych za pomocą rozkładu funkcji określającej stany naprężenia i przemieszczenia w kosinusowy szereg Fouriera. Część dolnej powierzchni płyty, która nie kontaktuje się z podłożem, jest pierścieniem, którego promienie wewnętrzny albo zewnętrzny nie są znane a priori i zostały wyznaczone.

Wyniki liczbowe przedstawiają zależności między ciśnieniem i ciężarem płyty, jej grubością, pierścieniowym obszarem i wielkościami wzniesienia albo zagłębienia podłoża w materiałach z kadmu, magnezu i kompozytów zbrojonych włóknami. Porównywano je z wynikami dla ciała izotropowego w celu wyjaśnienia efektu anizotropii. Graficznie pokazano także jak zmieniają się współczynniki koncentracji naprężenia na brzegach wzniesienia lub zagłębienia w zależności od stosunku promieni określających obszary kontaktu dla różnych materiałów i grubości warstwy.

Praca została złożona w Redakcji dnia 1 lipca 1982 roku