PLASTIC ZONE SIZE OF DUGDALE TYPE CRACKS IN A SELF-STRESSED TWO-PHASE MEDIUM WITH PARTIALLY PLASTIFIED MATRIX MATERIAL

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Abstract

Different boundary-value problems are considered concerning the elastic-plastic and fracture behaviour of brittle fibres-ductile matrix composites under thermal loading conditions in both the cases of absence and presence of cracks within the matrix phase. A model of the plastic deformation process is proposed with regard to a single unit cell of the fibre-reinforced composite. Numerous details of the deformation process within this unit cell are investigated by use of the above mentioned model including the possible failure mechanisms of the fibre-matrix bond. If applied together with the known crack model of the Dugdale type the proposed model for the plastic deformation process of a composite unit cell is shown to imply useful conclusions concerning the thermal crack growth of radial Dugdale type cracks within the matrix phase.

Introduction

The investigation of the interaction between the stress fields caused by the presence of different inhomogeneities is a problem of great practical importance. This is actually the basic problem of the mechanics of the composite materials. Of special interest from the point of view of fracture mechanics of the composite structures are the questions concerning the interaction between the structural components and existing cracks within these structures. Both the cases of mechanical and thermal loading of cracked composites have been since long studied and different models of interaction have been already considered by means of both micromechanical analysis and macromechanical theories. The essential features of these two different approaches were characterized in a paper by SMITH [1]. The fibre-reinforced composites consisting of ductile matrices strengthened by continuous brittle fibres form a large class of the commonly used composite materials.

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Thereby numerous investigations concerning the plastic behaviour of fibre-reinforced composites have been performed for example in the papers of HILL [2], SPENCER [3], MULHERN et al. [4], COOPER and PIGGOTT [5]. Comprehensive surveys about the state of the art are given in the Conference Proceedings of the 1975 ASME Winter Annual Meeting [6] as well as in the books of SPENCER [7], KOPIOV and OVCINSKIJ [8] and PIGGOTT [9]. Further, a problem of basic interest represents the micromechanical aspect in thermal cracking of unidirectionally reinforced composites. Thereby, definite progress has been already made in a series of papers by HERRMANN [10 - 12] and HERRMANN and associates [13 - 15] concerning the elastic and viscoelastic behaviour of a cracked unit cell of a low fibre concentration composite under the conditions of different thermal loading. In a recent work by HERRMANN and MIHOVSKY [16] the plastic behaviour of an uncracked unit cell and the mechanisms of failure of the fibre-matrix interface have been analyzed for the case of isothermal longitudinal extension of the composite. The model of the plastic deformation process proposed in [16] is especially attractive for the study of the behaviour of cracks situated within the matrix phase. It is shown in the present paper that this model is applicable to the problem of thermal loading of the composite. Moreover, if combined with the Dugdale model solution of HERRMANN [11] for a crack situated within the matrix phase this above mentioned model of the plastic deformation process implies useful conclusions concerning the fracture behaviour of the considered unit cell of a unidirectionally reinforced composite.

Statement of the problem

A unidirectionally reinforced fibrous composite with continuous fibres and relatively small fibre volume fraction is considered. The fibre material is linear elastic with Young’s modulus $E_f$, Poisson’s ratio $\nu_f$ and the thermal expansion coefficient $\alpha_f$. The material of the matrix is elastic-perfectly plastic with corresponding elastic constants $E_m$ and $\nu_m$, thermal expansion coefficient $\alpha_m$ and tensile yield stress $\sigma_y$. The thermoelastic properties of the fibre and matrix materials as well as the yield stress of the latter are assumed to be temperature independent.

A unit cell of this fibre-reinforced composite in the sense of the well-known model of two coaxial fibre-matrix cylinders is studied in the following where if referred to a cylindrical coordinate system $(r, \theta, z)$ the fibre and the matrix occupy the regions $(0 \leq r \leq r_f, 0 \leq \theta \leq 2\pi, -\infty < z < +\infty)$ and $(r_f \leq r \leq r_m, 0 \leq \theta \leq 2\pi, -\infty < z < +\infty)$, respectively. Thus, equation $r = r_f$ is the equation of the fibre-matrix interface.

The following is assumed with regard to the cracked composite unit cell. The crack is situated within the matrix phase and is presented in the cross section of the unit cell (cf. Fig. 1) by a straight line cut along the polar axis $\theta = 0$. The crack occupies the segment $r_1 \leq r \leq r$, so that $r_1$ and $r$ denote the radial coordinates of the left and the right crack tip, respectively. The crack length of the actual crack is $2L = r_r - r$.

The well-known elastic-plastic model of a crack proposed by DUGDALE [17] will be applied in the following analysis. The crack of length $2L = R_r - R_l$ shown in Fig. 1 presents the imaginary crack in the sense of this model. The segments $R_l \leq r \leq r_1$ and $r_r \leq r \leq R$,
Fig. 1. Dugdale type crack configuration in the cross section of a composite unit cell

present the thin plastic zones at the tips of the actual crack. The plastic zone lengths at
the left and the right crack tips are thus \( s_l = r_l - R_l \) and \( s_r = R_r - r_r \), respectively.

A quasi-static thermal loading of the unit cell will be considered which implies a tem-
perature distribution over the cross section of the form (cf. Fig. 1 for notation)

\[
T(r, \theta) = \begin{cases} T_f, & 0 \leq r \leq r_f \\ T_m, & r_f \leq r \leq r_m \end{cases}
\]

(1)

The temperatures \( T_f \) and \( T_m \) do not depend on the axial coordinate \( z \) and are constants
at each given instant of the deformation process so that the latter is viewed as a sequence
of stress-strain states of the cell corresponding to a sequence of stationary temperature
distributions of the form (1).

From the viewpoint of a quasi-static crack propagation behaviour such loading con-
ditions are of interest only which result in the appearance of tensile circumferential stresses
within the uncracked matrix phase. According to Herrmann [18] these conditions are
satisfied for the elastic state of the unit cell provided the inequality

\[
(1 + \nu_f) \alpha_f (T_f - T_o) - (1 + \nu_m) \alpha_m (T_m - T_o) > 0,
\]

(2)

holds true where \( T_o \) is the temperature of the unstressed initial state. Under the simplifying
assumption \( T_o = 0 \) relation (2) is obviously satisfied if, for example, \( T_f = 0 \) and \( T_m < 0 \).
This case will be actually considered in the following calculations. The thermal loading
process will be thus viewed as a process of monotonous quasi-static decrease of the itself
negative temperature of the matrix phase. The accepted loading conditions provide ob-
viously an axisymmetric state of stress within the uncracked composite cell. The axial
symmetry together with the standard assumptions of perfect fibre-matrix contact and
generalized plane strain imply the evident result that the normal stresses within the uncracked unit cell are at the same time principal ones and depend on the radial coordinate only.

The elastic state of the considered unit cell for both cases of absence and presence of a crack in the matrix phase has been described in detail by Herrmann [11, 18]. These elastic solutions concern a cell with a traction-free external surface \( r = r_m \) and traction-free or partially loaded crack surfaces in the sense of the applied Dugdale model. These same conditions are supposed to apply in the here considered elastic-plastic problem as well.

The condition of axial symmetry together with the assumed scheme of loading implies certain obvious features of the elastic-plastic state of the uncracked unit cell in accordance with the above mentioned elastic solution [11]. These are that the plastic zone presents itself an infinitely long cylinder \( (r_f \leq r \leq r_c, 0 \leq \theta \leq 2\pi, -\infty < z < +\infty; r_c < r_m) \) and spreads with developing thermal loading, i.e. with decreasing matrix temperature, into the matrix coating. The equation of the current elastic-plastic boundary could be then written in the form \( r_c = r_c(T_m) \).

Finally, it will be assumed that the crack length \( 2l \) is small compared with the radius \( r_m \) and that the crack itself is situated relatively far-away from the fibre. This implies the possibility of neglecting the effect of the crack on the stress-strain state within the matrix region just surrounding the fibre where the plastic deformation process actually develops. Then, the latter could be viewed as an axisymmetric one as in the case of an uncracked unit cell.

With this in mind the thermal stress field within the cracked matrix phase could be considered to be a superposition of the following two fields. The first one is the elastic-plastic stress field for the uncracked unit cell while the second one is the field resulting from the presence of a Dugdale type crack.

### The plastic deformation process

The model of the plastic deformation process proposed in [16] will be generalized in the present paper with regard to the considered thermal loading problem. The possibility for such a generalization follows from the fact that this model is based in general upon certain effects of the fibre-reinforcement which are common for both the isothermal [16] and the here considered thermal problem. Firstly, to these effects belongs the so-called „shrinkage effect”, i.e. the appearance of compressive radial stresses over the fibre-matrix interface. One comes up with this effect provided relation (2) is satisfied which is actually the here considered case. Secondly, in accordance with the elastic solution [18] the fibre acts as a stress concentrator. Because of the local nature of this stress concentration effect one could expect that especially for the considered composites with low fibre volume fractions intensive plastic deformation and even fracture processes may develop within the immediate surrounding of a fibre whereas at a certain distance from the fibre-matrix interface the matrix material may deform still elastically. Thirdly, it is well-known from experimental observations that because of the strengthening effect of the fibre the be-
haviour of the composite „in the fibre direction” is rather elastic-like than perfectly-plastic. This implies the reasonable assumption that the fibre, consisting itself of linearly elastic material with a high stiffness, contributes due to the assumed perfect fibre-matrix contact to the development of a relatively large elastic part \( \varepsilon_f^e \) of the total axial strain \( \varepsilon_z \) within the plastificated region and prevents thus the occurrence of a corresponding large plastic part \( \varepsilon_f^p \). In other words in the course of the deformation process one should permanently account for the current elastic part of the axial strain. It is obvious that for the considered regime of thermal loading both the \( \varepsilon_f^e \)- and \( \varepsilon_f^p \)-strains should be monotonously increasing in absolute value functions in dependence of the absolute value of the matrix temperature. A reasonable restriction concerning the behaviour of the \( \varepsilon_f^e \)-strain is associated with the assumption that the matrix material is a perfectly-plastic one and its elastic response is thus limited. One should expect correspondingly that for the considered unit cell and type of loading there exists a certain critical value \( \varepsilon_f^c \) of \( \varepsilon_f^e \) such that upon reaching this value the current increments of the \( \varepsilon_f^e \)-strain become negligible with respect to the corresponding increments of \( \varepsilon_f^p \). Due to the concentration effect of the fibre this critical value \( \varepsilon_f^c \) should be first achieved over the fibre-matrix interface.

The account for the just introduced limiting characteristic \( \varepsilon_f^c \) implies the following natural description of the plastic deformation process. The plastic deformations appear first over the fibre-matrix interface and the plastic zone \( r_f \leq r \leq r_c \) spreads consequently into the matrix phase. Within this zone both the \( \varepsilon_f^e \) and \( \varepsilon_f^p \)-strains increase simultaneously up to the instant when \( \varepsilon_f^e|_{r=r_f} = \varepsilon_f^c \). At this instant a second plastic zone \( r_f \leq r \leq R_c \) where \( R_c < r_c \) appears within which the relation \( \varepsilon_f^e = \varepsilon_f^c \) holds true while the \( \varepsilon_f^p \)-strain further increases. The second plastic zone also spreads into the matrix phase having the first one, which occupies now the region \( R_c \leq r \leq r_c \), at its front \( r = R_c \).

The model of the plastic deformation process just considered implies a simple possible scheme of an approximate analysis of the elastic-plastic behaviour of the uncracked composite cell.

Analysis of the uncracked unit cell

In accordance with the standard assumption of the plasticity theory the total axial strain at each instant of the plastic deformation process is a sum of an elastic and a plastic part. As usually it will be assumed that the matrix material is plastically incompressible which implies the validity of the following relation within the plastic zone

\[
\varepsilon_f^\text{total} = \varepsilon_f^e + \frac{1}{3} \varepsilon^{(\text{str})} + \frac{1}{3} \varepsilon^{(\text{temp})},
\]

where \( \varepsilon_f^e \) is the deviatoric axial elastic strain and \( \varepsilon^{(\text{str})} \) and \( \varepsilon^{(\text{temp})} \) are the relative volume changes associated with the thermal stresses and the thermal expansion respectively, i.e.

\[
\varepsilon^{(\text{str})} = \frac{1-2\nu_m}{E_m} (\sigma_r + \sigma_\theta + \sigma_z),
\]

\[
\varepsilon^{(\text{temp})} = 3\alpha_m T_m.
\]
In equation (4) as well as in the following analysis \( \sigma_i, i = r, \theta, z \) denote the normal stresses within the plastic zone.

It will be further accepted that the stresses and the elastic strains are as usually related by the Hooke's law so that one has in particular

\[
\sigma_z = E_m e^e_z + \nu_m (\sigma_r + \sigma_\theta).
\]

In equation (6) as well as in the rest of the paper the notation

\[
e^e_z = e^e_z + \frac{1}{3} \varepsilon^{(n)},
\]

is used so that \( e^e_z \) means (cf. equation (3)) the part of the axial elastic strain due to the thermal stresses.

Let the matrix material obey the von Mises' yield condition, i.e. let the stresses \( \sigma_i, i = r, \theta, z \) satisfy the relation

\[
(a_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - a_r)^2 = 2a_r^2.
\]

Upon substituting for the \( \sigma_z \) stress from equation (6) into the latter relation one obtains

\[
\left(\frac{a_r - \sigma_\theta}{2}\right)^2 + \left(\frac{\sigma_r + \sigma_\theta}{2} - \frac{E_m e^e_z}{1-2\nu_m}\right)^2 \left(1-2\nu_m\right) = \frac{a_r^2}{3}.
\]

Now it is a matter of simple computation to show that equation (9) is identically satisfied provided the stresses are presented in the form

\[
\begin{align*}
\sigma_r &= \frac{E_m e^e_z}{1-2\nu_m} + \frac{\sigma_r}{\sqrt{3}} \cos(\omega \pm \phi), \\
\sigma_\theta &= \frac{\sigma_{\theta}}{\sqrt{3}} \sin(\omega \pm \phi),
\end{align*}
\]

where the notations are used

\[
\begin{align*}
\sin \omega &= \frac{a_r - a_\theta}{2} \sqrt{\frac{a_r}{3}}, \\
\cot \phi &= \frac{\sqrt{3}}{1-2\nu_m}.
\end{align*}
\]

Equations (10) reflect the implicit assumption that \( \sigma_\theta \geq \sigma_r \), which implies in accordance with equation (11) that \( 0 \leq \omega \leq \pi \).

Substituting now for the stresses \( \sigma_r \) and \( \sigma_\theta \) from equations (10) into the equilibrium equation

\[
\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0,
\]

one obtains the equation

\[
\frac{E_m}{1-2\nu_m} \frac{d\varepsilon^e_z}{dr} + \frac{\sigma_r}{\sqrt{3} \sin \phi} \sin(\omega + \phi) \frac{d\omega}{dr} - \frac{2\sigma_r}{\sqrt{3}} \frac{\sin \omega}{r} = 0,
\]

where \( \varepsilon^e_z \) is an unknown function of the radial coordinate \( r \) and therefore the integration of equation (14) cannot be performed. But an approximate solution of equation (14) can be obtained which is valid at least within the immediate surrounding of the fibre.
This solution is based upon the assumption that within this region the elastic parts of the 
\( \varepsilon_r \) and \( \varepsilon_p \)-strain components due to the thermal stresses are negligible with respect to 
the corresponding plastic parts \( \varepsilon_p^r \) and \( \varepsilon_p^p \). This implies the relation 
\[
\varepsilon_p^r = \varepsilon_p^p.
\]  
(15)
Equation (15) together with equations (4), (6) and (10) gives now 
\[
\frac{\sigma_p(1+\nu_m)}{E_m} \cos \omega = -\varepsilon_p^p
\]  
(16)
The latter relation applies as assumed in a thin layer surrounding the fibre and over the 
fibre-matrix interface \( r = r_f \) in particular where the condition \( \varepsilon_p^r = \varepsilon_p^p \) is first achieved. 
The corresponding value \( \omega^* \equiv \omega(\varepsilon_p^p) \) of the angle \( \omega \) follows from equation (16) to be 
\[
\omega^* = \arccos \left[ -\frac{E_m \varepsilon_p^p}{\sigma_s(1+\nu_m)} \right]
\]  
(17)
According to the model of the plastic deformation process proposed in section 3 above 
a further increase in thermal loading which corresponds to a further decrease in the matrix 
temperature \( T_m \) results in the appearance of a second plastic zone \( r_f < r < R_c \) over the 
outer boundary of which equation (17) is valid. Finally, assuming that for the considered 
unit cell and the given scheme of loading the quantity \( \varepsilon_p^p \), respectively \( \omega^* \) (cf. equation (17)) 
is approximately constant and introducing the angle \( \omega_{R_c} \) as 
\[
\omega_{R_c} = \omega(R_c),
\]  
(18)
one obtains 
\[
\omega_{R_c} = \omega^*
\]  
(19)
where \( \omega^* \) is a constant now. 
Now the latter assumption makes equation (14) integrable within the whole second plastic 
zone \( r_f \leq r \leq R_c \). The result of this integration with the boundary condition \( \omega|_{r=r_c} = \omega_{R_c} \) reads 
\[
\frac{R_c^2}{r^2} = \frac{\sin \omega}{\sin \omega_{R_c}} \exp \left[ \frac{\sqrt{3}}{1-2\nu_m} (\omega - \omega_{R_c}) \right].
\]  
(20)
Moreover, it could be easily verified that the set of equations (6), (10), (17), (19) and (20) 
defines the stress state entirely within the second plastic zone \( r_f \leq r \leq R_c \), where \( R_c \) has 
still to be determined.

The stress state defined above allows certain important conclusions concerning the 
fracture behaviour of the considered unit cell. To this end we consider the shrinkage 
effect again. It is clear from most general positions that this effect is due to the difference 
in the lateral contraction of the fibre and matrix materials. Because of the plastic incompressibility of the matrix material this difference should be expected to increase in the 
course of the deformation process. In other words developing plastic deformations should 
further contribute to the shrinkage effect or, equivalently, the radial stress \( \sigma_r|_{r=r_f} \) acting 
over the fibre-matrix interface should decrease with increasing loading, i.e. with decreasing 
temperature of the matrix phase. The latter means in accordance with equations (10) 
that the angle \( \omega_{r_f} \) should increase in the course of the deformation process remaining
obviously larger than the angle \( \omega^* \). Moreover, equations (10) show that there exists a natural limitation of the shrinkage effect in the sense that this effect achieves its maximum at a value of \( \omega_{rf} = \pi - \phi \).

The value \( R^* \) of the radius \( R \) at this instant, that is
\[
R^*_c = R_{c\omega_{rf}} = \pi - \phi,
\]
follows from equation (20) to be
\[
R^*_c = r f^2 \frac{\sin \phi}{\sin \omega_{rf}} \exp \left[ \frac{\sqrt{3}}{1 - 2\mu_m} (\pi - \phi - \omega_{rf}) \right].
\]  

The model of the process applied here leads thus to the conclusion that further decrease in \( \alpha_{rf} = r_f \) as well as increase in \( R \) is impossible. Further, it would be of interest to examine the velocity field corresponding to this limiting state of the plastic deformation process within a unit cell.

To this regard the known concept of the associated flow rule will be applied with the yield function (9) serving as a plastic potential. Simple computations show that in accordance with this concept and the plastic incompressibility condition the plastic strain rates \( \xi, i = r, \theta, z \) satisfy the relation
\[
\xi_i = -\xi_\theta \frac{\Sigma_\theta + \Sigma_r}{\Sigma_\theta},
\]
within the second plastic zone where
\[
\frac{\Sigma_i}{\Sigma_\theta} = \frac{\sigma_y}{\sqrt{3}} \sin(\phi + \omega).
\]

It is easily observed from the latter equations that \( \xi_i|_{r=r_f} \to +\infty \) when \( \omega_{rf} \to \pi - \phi \). This result means physically that at this state free plastic flow tends to take place within a thin layer immediately surrounding the fibre. The behaviour of the composite at this state will obviously depend upon the interaction between this tendency and the strengthening effect of the fibre which tends itself to prevent the occurrence of such a singular velocity field. The very nature of these two competing effects implies the reasonable assumption that their interaction results in the occurrence of shearing stresses over the fibre-matrix interface. Moreover, these shearing stresses should be equal for obvious reasons to the shear yield stress \( \tau_y = \sigma_y/\sqrt{3} \) of the matrix material.

Let \( \tau_s \) be the shear strength of the fibre-matrix interface. If \( \tau_s \leq \tau_y \) then the very reaching of the considered critical state will obviously result in the immediate failure of the fibre-matrix interface by the so-called debonding effect. If, on the contrary, \( \tau_s > \tau_y \), then the known mechanism of fibres pull-out (see, for example [9]) will develop, most probably together with a process of fibre breaking.

### Plastic zone size and associated problems

In order to close the solution of the problem for the uncracked unit cell one should complete the results of the previous section with the temperature dependence of the radius of the plastic zone. Moreover, when dealing with a given composite material one should specify the actual value of \( \varepsilon^*_z \) which should be used in the computations.
The model of the plastic deformation process proposed above implies a simple approach to the latter problem. Starting point for this approach is the additional assumption that the first plastic zone presents itself a thin layer and thus one may consider the relation $R_c = r_c$ to hold approximately true. This assumption appears as acceptable one for the following reasons. Firstly, because of the local nature of the fibre concentration effect and since a low fibre volume fraction composite is considered. Therefore, both the $R_c$- and $r_c$-radii should be small compared with the value of $r_m$. Secondly, because of the low resistance of the matrix material with respect to the occurrence of intense plastic deformation such as the deformations within the second plastic zone are. Thus, one may expect that the transition zone between the elastically deformed matrix region and the second plastic zone is really a thin one. If so, then the first plastic zone could be simply considered to play the role of an elastic-plastic boundary, the latter having the form of a thin layer. Further, because of the thin layer shape of the elastic-plastic boundary a softened version of fulfillment of the standard elastic-plastic transition conditions of continuity of stresses and displacements could be applied, namely the following.

Firstly, because of the layer thinness one should not expect a substantial change of the radial stress within the layer itself which implies the relation

$$\sigma_{ir=Re} = \sigma_{ir=R_c}, \quad (25)$$

where $\sigma^i_r$, $i = r, \theta, z$ are the stresses acting within the elastic region $R_c \leq r \leq r_m$ of the matrix phase. Secondly, these stresses should satisfy the yield condition, equation (8), over the elastic-plastic boundary, that is

$$[(\sigma^r_r - \sigma^\theta_\theta)^2 + (\sigma^\theta_\theta - \sigma^z_z)^2 + (\sigma^z_z - \sigma^r_r)^2]_{r=R_c} = 2\sigma_0^2. \quad (26)$$

In accordance with the general form of the elastic solution of the problem [11, 18] and the results of the previous section one may present the latter equations in the form

$$\frac{E_m}{1 + \nu_m} C \left( \frac{1}{r_m} - \frac{1}{R_c} \right) = \frac{E_m \sigma^*_z}{1 - 2\nu_m} - \frac{\sigma^*_r}{\sqrt{3} \sin \phi} \cos (\omega_{R_c} + \phi), \quad (27)$$

$$3 \frac{C^2}{R_c^2} = \frac{\sigma^*_z (1 + \nu_m)^2}{E_m^2} - \left[ \frac{C (1 - 2\nu_m)}{r_m^2} - (1 + \nu_m) (\varepsilon_z - \alpha_m T_m) \right]^2, \quad (28)$$

where the constant $C$ has to be determined actually.

The remaining elastic-plastic transition conditions could be now viewed as satisfied as well in this way that the corresponding stresses and displacements change continuously within the layer between their values on the "elastic" and "plastic" surface. Thus, the equations (27) and (28), respectively, present the just mentioned softened version of the elastic-plastic transition conditions.

If solved for the unknowns $R_c$ and C the set of equations (27) and (28) implies as a matter of fact the temperature dependence of both $R_c$ and $C$, respectively, in the form

$$R_c = R_c(T_m, \varepsilon_z, \varepsilon^*_z, E_m, \nu_m, \alpha_m, r_m, \sigma_y), \quad (29)$$

$$C = C(T_m, \varepsilon_z; \varepsilon^*_z, E_m, \nu_m, \alpha_m, r_m, \sigma_y), \quad (30)$$

where $\varepsilon_z$ itself is a still unknown function of the matrix temperature $T_m$. 

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It is important to mention at this point that with $R_{e}$ and $C$ once determined from the set of equations (27) and (28) one may consider the axial stresses $\sigma_{z}$ and $\sigma_{z}^{e}$ acting within the plastic and elastic regions of the matrix phase, respectively, to be known functions of the same parameters as those in the presentations (29) and (30). The corresponding expressions for these stresses can be easily given by

$$\sigma_{z} = \frac{1}{1-2\nu_{m}} (E_{m}^{e}e_{z}^{e} + 2\sigma_{y}\cos\omega), \quad r_{f} \leq r \leq R_{e}, \quad (31)$$

$$\sigma_{z}^{e} = E_{m}e_{z} + \frac{2\nu_{m}E_{m}C}{(1+\nu_{m})r_{e}^{2}}, \quad R_{e} \leq r \leq r_{m}, \quad (32)$$

where both $R_{e}$ and $C$ can be considered now as known functions of the form given by the equations (29) and (30), respectively.

Further, it is a matter of a simple verification that upon satisfying the continuity condition for the radial stresses over the fibre-matrix interface, i.e. the equation

$$\sigma^{m}_{r}|_{r=r_{f}} = \sigma^{f}_{r}|_{r=r_{f}}, \quad (33)$$

one may construct the expressions for the stresses $\sigma_{i}, i = r, \theta, z$ acting within the fibre. Thereby the expression for the axial stress $\sigma_{r}^{e}$ reads

$$\sigma_{r}^{e} = E_{f}e_{z} + 2\nu_{f} \left[ \frac{E_{m}^{e}e_{z}^{e}}{1-2\nu_{m}} + \frac{\sigma_{y}}{\sqrt{3}\sin\phi} \cos(\omega_{r} + \phi) \right], \quad 0 \leq r \leq r_{f}, \quad (34)$$

where the value of $\omega_{r}$ follows from equation (20) with $r = r_{f}$ and with the quantity $R_{e}$ given in the form of equation (29).

It is easily observed that the axial stresses as presented by the equations (31), (32) and (34) can be considered now as known functions of $T_{m}$ and $e_{z}$ and the remaining parameters of the problem, i.e. $e_{z}^{e}, \sigma_{y}$ and $E_{i}, v_{i}, \alpha_{i}, r_{i}$ where $i = f, m$. These stresses have to satisfy the equilibrium condition for the forces, acting in the axial direction, which in our self-stress problem is given by the following condition of self—equilibrium

$$\sigma_{z}^{e} \cdot r_{f}^{2} = 2 \int_{r_{f}}^{R_{e}} \sigma_{z}rdr + (r_{f}^{2} - R_{e}^{2}) \sigma_{z}^{e}. \quad (35)$$

By substituting for $R_{e}$ from equation (29) into equation (35) leads to a relation of the form

$$e_{z} = e_{z}(T_{m}; e_{z}^{e}, E_{i}, v_{i}, \alpha_{i}, r_{i}, \sigma_{y}), \quad (36)$$

where $i = f, m$. Equation (36) represents in fact the equation of the theoretical $e_{z}$ versus $T_{m}$ curve in the framework of the proposed model for the considered two-phase material.

Upon substituting for $e_{z}$ from equation (36) into equation (29) one obtains the desired dependence of the plastic zone radius $R_{e}$ on the matrix temperature $T_{m}$. This dependence is obviously of the form

$$R_{e} = R_{e}(T_{m}; e_{z}^{e}, E_{i}, v_{i}, \alpha_{i}, r_{i}, \sigma_{y}), \quad (37)$$

where again $i = f, m$.

Note that by applying equation (37) to the critical state of the unit cell, i.e. if $R_{e} = R_{e}^{\ast}$ (cf. equation (22)), one obtains the critical temperature $T_{m}^{\ast}$ at which one of the
failure modes of the fibre-matrix interface described in section 4 occurs. This critical
temperature appears to be of the form

\[ T^*_m = T^*_m(R_e, \varepsilon_e, E_i, \nu_i, \alpha_i, r_t, \sigma_y); \quad i = f, m. \]  

Equation (38) implies itself a simple criterion of failure of the fibre-matrix interface of
the form

\[ T_m = T^*_m \]  

Now it is easily observed that the actual value of \( \varepsilon_e \) can be determined by means of
a comparison of the theoretically predicted \( \varepsilon_e(T_m) \)-curve, equation (36), with a correspond-
ing curve obtained experimentally for the considered composite material.

A simpler approach to the problem consists in the determination of \( \varepsilon_e \) from equation
(38) provided the \( T^*_m \) and \( R^*_e \)-values in this equation are the values which have been ob-
served experimentally.

The scheme described above does not imply a closed form solution for the quantities
\( R_e \) and \( \varepsilon_e \) but the associated numerical treatment of the problem is not very complicated.
With this in mind one may consider that the whole problem concerning the elastic-plastic
behaviour of the uncracked unit cell has been solved completely.

The cracked unit cell

It should be remembered in the following that as accepted in section 2 the thermal stress
field within the cracked matrix phase can be considered as a superposition of an elastic-
plastic stress field for the uncracked unit cell and a corrective stress field caused by the
presence of a crack situated along the segment \( R_t \leq r \leq R_r, \theta = 0 \) of the symmetry
line of the cross section of the unit cell. Now the first stress field is known from the preced-
ing analysis, sections 4 and 5. The second stress field will be examined as already mentioned
in the framework of the Dugdale crack model [17].

The thermal stress field in the uncracked elastically deformed matrix region is given by
the following expressions

\[ \sigma^e_{ij} = \frac{\varepsilon_e}{1 + \nu_m} C \left( \frac{1}{r_m^2} \mp \frac{1}{r^2} \right), \quad R_e \leq r \leq R_m, \]  

\[ \sigma^e_z = E_m(\varepsilon_z - \alpha_m T_m) + \frac{2\nu_m E_m}{1 + \nu_m} C \frac{1}{r_m^2}, \]

where \( C = C(T_m, \ldots) \) and \( R_e = R_e(T_m, \ldots) \) can be considered as known functions of the
temperature \( T_m \) in accordance with the results of the previous section.

The corrective stress field \( \sigma^e_{ij}, i, j = r, \theta, z \) can be obtained from the solution of the following mixed boundary-value problem for a Dugdale type crack with an actual length
\( 2L = r_r - r_t \) and a fictitious length \( 2L = R_r - R_t \) (cf. Fig. 1 for notation)

\[ \sigma^e_{ij}(r, 0) = \begin{cases} -\sigma_e^e, & r_t \leq r \leq r_r, \\ -\sigma_e^e + \sigma_y, & R_t \leq r \leq r_t \quad \text{and} \quad r_r \leq r \leq R_r, \end{cases} \]  

\[ \sigma^e_{ij}(r, \theta) = \sigma^e_{ij}(r, 0), \]

\[ \sigma^e_{ij}(R_r, \theta) = 0. \]
\[ \sigma_{0}(r, 0) = 0, \quad \theta = 0, \forall r, \]
\[ u_{0}(r, 0) = 0, \quad \theta = 0, \quad R_{r} \leq r \leq R_{l}. \]

The boundary-value problem (40) - (42) has been analyzed in detail by Herrmann [11]. Applying the results obtained in [11] together with those presented above one immediately obtains the desired plastic zone lengths \( s_{l} \) and \( s_{r} \) (cf. Fig. 1) for our case from the solutions of the equation

\[ \frac{E_{m}}{2(1+v_{m})r_{m}^{2}} \left\{ 1 + \frac{\sigma_{0}^{2}[r_{0} \pm (l+s)]}{[\sigma_{0}^{2} - (l+s)^{2}]^{3/2}} \right\} - \frac{\sigma_{0}^{2}}{\pi} \arccos \frac{l}{l+s} = 0, \]

where \( s \) is the unknown and \( r_{0} = (r_{r} + r_{l})/2 \). The quantities \( s_{l} \) and \( s_{r} \) correspond to the upper and lower signs, respectively, in the brackets in equation (43).

Upon solving equation (43) one obtains the plastic zone lengths as functions of the temperature \( T_{m} \) and the actual crack length \( 2l \), i.e. the relations

\[ s_{i} = s_{i}(T_{m}, l), \quad i = r, l. \]

It is well understood that each of the quantities \( s_{i} \), \( i = r, l \) depends in addition on the remaining parameters of the problem as well so that the representations (44) are actually schematic ones. They imply in an obvious way the relations

\[ R_{i} = R_{i}(T_{m}, l), \quad i = r, l. \]

The equations (45) present the dependence of the positions of the left and the right tip of the imaginary crack on the current matrix temperature. That dependence could be now considered as known from the solution of equation (43). The latter solution could be itself obtained by means of a numerical treatment [11].

Crack and cell behaviour

As already accepted (cf. equation (38)) let \( T_{m}^{*} \) be the value of the matrix temperature at which one of the two modes of failure of the fibre-matrix interface (cf. section 4) occurs and let \( R_{l}^{*} \) be the value of \( R_{l} \) corresponding to this value of the temperature (cf. equation (45)). Then \( R_{l}^{*} \) defines in fact the position of the left plastic zone tip at the instant of failure of the fibre-matrix interface. Further, let \( R_{l} \) be the value of \( R_{l} \) at which the crack begins to grow in accordance with a certain crack growth criterion and let \( T_{m} \) be the corresponding matrix temperature, i.e.

\[ \bar{R}_{l} = R_{l}(T_{m}, l). \]

Then equation (46) corresponds to the equation (45) but now applied to the instant of crack growth initiation. It is assumed implicitly that due to the fibre concentration effect the crack will start to propagate from its left tip toward the fibre.

It follows from the whole scheme of analysis that the values of both quantities \( R_{l}^{*} \) and \( \bar{R}_{l} \), respectively, appear as specific ones for each given cracked unit cell or, equivalently, for a given composite material. Thereby both values could be defined from the present analysis provided the corresponding crack growth criterion is given. The latter concerns obviously the determination of \( R_{l} \). Once evaluated for a certain unit cell with
a given crack configuration these \( R^*_f \)- and \( \overline{R}_f \)-values imply immediately the following
evident conclusions concerning the behaviour of the cracked unit cell. If \( R^*_f > \overline{R}_f \) then
the fibre-matrix interface fails while the crack is still in rest. If on the contrary \( R^*_f < \overline{R}_f \)
then the crack growth initiation precedes the failure of the fibre-matrix interface. In this
case the whole scheme of analysis remains further valid (up to the possible failure of the
interface) provided the crack propagates quasi-statically and \( 2l \) is its current length upon
which both \( R^*_f \) and \( \overline{R}_f \) depend. Finally, if \( R^*_f = \overline{R}_f \) then the fibre-matrix interface fails
simultaneously with the initiation of the crack propagation. The further behaviour of
the crack and the unit cell in this case as well as in the first one where \( R^*_f > \overline{R}_f \) needs
a new approach since the present considerations are based upon the assumption of a per-
fact fibre-matrix contact.

It should be mentioned that these simple conclusions are valid under the assumptions
made earlier that both the crack length \( 2l \) and the plastic zone radius \( R_c \) are small compared
with the geometrical quantities \( r_m \) and \( r_o \), respectively. That means the crack should not
influence the stress state within the plastic zone \( r_o \leq r \leq R_c \) where the solution of section 4
is thus expected to apply. Moreover, it has been shown in [19] that the approximate ana-
ytical elastic solution of the considered thermal crack problem in a composite unit cell
obtained in [11] remains still valid even if the restrictions concerning the quantities \( 2l, r_o, r_m \)
and \( r_f \) (the latter quantity plays the role of \( R_c \) in the elastic case) are somehow violated.
One may consequently expect that the results of the present elastic-plastic analysis will
also remain valid when the restrictions mentioned above are somehow softened since the
values of \( R_t, i = r, l \) used in our considerations are actually obtained from the same
elastic solution [11]. If so, then one comes easily up with a couple of further implications
of the analysis concerning the crack and cell behaviour.

Let \( T_m \) be the temperature at which both plastic zones, i.e. the annulus \( r_f \leq r \leq R_c \) and
the segment \( R_t \leq r \leq r_l \) join each other, and let \( \overline{R}_c = R_c(T_m) \) and \( \overline{R}_l = R_l(T_m, l) \) be the
corresponding values of the quantities \( R_c \) and \( R_l \), respectively. Further, let us assume the
validity of the relations \( T_m > T^*_m \) and \( \tilde{T}_m > \tilde{T}_t \) (note that the temperatures \( T^*_m, \tilde{T}_m \)
and \( \tilde{T}_t \) are negative) for the considered unit cell. That means that the two plastic zones meet
each other before the conditions for the failure of the fibre-matrix interface and for crack
growth initiation are fulfilled. Upon reaching this instant, that is with the two plastic
zones adjoined, the behaviour of the cracked unit cell will depend essentially on the in-
teraction between the plastic mechanism of failure of the entirely plastificated segment
\( r_f \leq r \leq r_l, \theta = 0 \) and the brittle mechanism of crack growth at the right crack tip \( r =
\approx r_r \). A possible approach to this problem could be based upon the application of the
rigid-plastic body model and the limit load concept associated with this model (see, for
example [20]). When applying the latter concept to the plastificated segment \( r_f \leq r \leq r_l, \theta = 0 \) one may expect that further thermal loading, i.e. further decrease of \( T_m \), will re-
sult in the activation of the crack propagation mechanism at the right crack tip.

Another case of interest is the one for which \( \tilde{T}_m < \tilde{T}_t \) and \( \tilde{T}_m > T^*_m \). In this case both
plastic zones join each other again but the segment \( R_t \leq r \leq r_l, \theta = 0 \) presents now
the plastic zone at the left tip of the running crack. Depending upon the plastic zone
thickness \( d = R_c - r_f \) and the crack velocity the crack may stop before reaching the elastic-
plastic boundary or at the boundary or may traverse partially or entirely the plastificated
annulus around the fibre. Depending in addition on the fibre-matrix contact the crack may stop at the fibre-matrix interface in order to traverse it or to create an interface crack. The investigation of all these possibilities, being a problem of definite interest, is associated with considerable difficulties arising from the necessity of solving boundary-value problems for cracks partially situated within plastificated regions as well as of applying reasonable criteria of crack propagation and arrest.

Concluding remarks

The analysis presented above implies certain definite conclusions concerning the behaviour of a cracked unit cell of a fibre-reinforced composite material under the conditions of thermal loading. The analysis could be easily transformed to the more general case $T_f \neq 0$ if the temperature difference $\Delta T = T_m - T_f$ will actually play the role of the quantity $T_m$ used in the preceding calculations. In that case the linear coefficient of thermal expansion $\alpha_f$ will influence the processes of plastification and fracture as well.

The model of the plastic deformation process proposed leads to both closed form results and to a relatively simple procedure concerning the numerical treatment of the problem on the whole. The analysis shows that the entire solution of the considered problem is associated with the necessity of directed experimental investigations concerning the determination of the specific measure of elastic response $\varepsilon^e$ for the fibrous composite materials.

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References


Резюме

Размер пластической зоны трещины, типа Дугдэла, в предварительно напряженной двухфазной среде, с частично пластическим матрицей

Предлагается модель пластической деформации насаженной одночной ячейки композита усиленного волокон. Мы исследовали процесс деформации ячейки при учете возможного разрушения в месте соединения матрицы и волокна. При учете модели Дугдэла мы сделали выводы насаженного термического возраста радиальных трещин в матрице.

Streszczenie

Zasięg strefy uplastycznienia szczelin typu Dugdale’a we wstępnie naprężonym ośrodku dwufazowym z częściowo uplastycznionym materiałem matrycy

Proponujemy model odkształcenia plastycznego dotyczący pojedynczej komórki kompozytu wzmocnionego włożonymi włóknami. Zbadany został proces odkształcenia komórki przy uwzględnieniu możliwego mechanizmu pękania w miejscu połączenia matrycy i włośnka.

Przy uwzględnieniu modelu Dugdale’a wyciągnięto wnioski dotyczące termicznego wzrostu szczelin promieniowych w matrycy.

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