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SHALLOW SPHERICAL LATTICE-TYPE SHELL SUBJECTED TO NORMAL POINT LOAD-FUNDAMENTAL SOLUTION

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1. Introduction

Theoretical problems occuring in designing of the shell structures subjected to concentrated forces require special methods, which make it possible to examine stress concentrations around the loads. In particular, it is worth to consider specific problems of the lattice shells response, i.e. grid-, perforated-, or ribbed surface structures. In the present paper there will be presented: an approximate model of a lattice shell behaviour and its application to the special case of a spherical lattice-type elastic shell with an isotropic and centrosymmetric structure, being subjected to concentrated normal force. The proposed model is based on the lattice surface structure theory developed by C. Woźniak, [1]. Thus, the engineering problem considered here can be reduced to the analysis of one of the fundamental solutions of the theory. Before examining this solution an approximate set of governing equations is derived on the basis of the following assumptions often used when effect of local loads in the classical shell theory is considered, cf [2]:

— the boundary conditions have a neglecting effect on the local response of the shell,

— the deformations coresponding to the concentrated force vanish rapidly, hence the consideration can be confined to the small (comparing with the size of the shell) area around the point load.

The last assumption makes it possible to carry out the simplifications analogous to those used in Vlasov's shallow shell theory.

2. Fundamental equations and basic assumptions

Consider a point P and a vector $\boldsymbol{n}$ normal at this point to the mid-surface of the shell, referred to the coordinate system x^α, $\alpha = 1, 2$. The plane normal to the vector $\boldsymbol{n}$ containing P is denoted by π_P. A plane coordinate system $\bar{x}^\alpha$ (obtained via parallel projection of x^α in the direction $\boldsymbol{n}$ on π_P) is assumed to be i one-to-one correspondence with the parametric lines x^α. The metric tensors in plane and surface coordinate systems will be denoted by $\bar{g}_{\alpha\beta}$ and $g_{\alpha\beta}$, respectively. In the course of the procedure the coordinate system $\bar{x}^\alpha$ is assumed to be orthogonal.

An actual configuration of the shell is determined by functions: u^α, u, v^α, v, which stand for the tangent and normal displacements and rotations, respectively.

The state of strain is described by the tensors of plane and antiplane deformation $\gamma_{\alpha\beta}$, $\varkappa_{\alpha\beta}$, i.e. by the stretching and bending strain tensors; γ_α denotes antiplane transverse deformations while $\varkappa_\alpha$ — plane bending deformations.

The state of stress is determined by the membrane stress measures $p_{\alpha\beta}$, the bending (antiplane) moment tensor $m_{\alpha\beta}$, the transverse (antiplane) stresses p_α and the in-plane bending moment tensor (polar moments) m_α.

The mentioned quantities here satisfy the known system of equations due to Woźniak's lattice — type shell theory, [1].

The constitutive relations of the isotropic shell considered herein are assumed as follows

$$
\begin{aligned}
p_{\alpha\beta} &= \lambda_t \cdot g_{\alpha\beta}\gamma^{\sigma}_{.\sigma} + 2\mu_t \gamma_{(\alpha\beta)} + 2\alpha_t \cdot \gamma_{[\alpha\beta]}, \qquad & m_\alpha &= C_t \cdot \varkappa_\alpha, \\
m_{\alpha\beta} &= \lambda_p \cdot g_{\alpha\beta} \cdot \varkappa^{\sigma}_{.\sigma} + 2 \cdot \mu_p \cdot \varkappa_{(\alpha\beta)} + 2\alpha_p \cdot \varkappa_{[\alpha\beta]}, \qquad & p_\alpha &= C_p \cdot \gamma_\alpha.
\end{aligned}
\tag{2.1}
$$

Moduli λ_i, μ_i, α_i, C_i characterize elastic properties related to the plane ($i = t$) and antiplane ($i = p$) deformations.

Inserting the relationships (2.1) into the equations of equilibrium and utilizing the strain-displacement relations, the governing set of equations is obtained. Taking into account the spherical shape of the shell and introducing all simplifications yielding from the approximation $\bar{g}_{\alpha\beta} \approx g_{\alpha\beta}$, we have

$$
\begin{gathered}
[(\mu_t+\alpha_t)\nabla^2 - \underline{K^2 C_p}]\, u_\alpha + (\lambda_t+\mu_t-\alpha_t)\,\partial_\alpha\partial_\beta u_\beta + 2\cdot\alpha_t\cdot e_{\alpha\beta}\,\partial_\beta v - K e_{\alpha\beta} v_\beta - \\
- K\cdot[2(\mu_t+\lambda_t)+C_p]\,\partial_\alpha u + b_\alpha = 0, \\
2\alpha_t e_{\alpha\beta}\,\partial_\alpha u_\beta + [C_t\nabla^2 - 4\alpha_t - 4K^2\cdot(\mu_p+\alpha_p]v + K\cdot[C_t + 2(\mu_p+\lambda_p)]\,\partial_\alpha v_\alpha + h = 0, \\
C_p\cdot K e_{\alpha\beta} u_\beta - K\cdot[2(\mu_p+\lambda_p)+C_p]\,\partial_\alpha v + [(\mu_p+\alpha_p)\nabla^2 - C_p - \underline{K^2 C_t}]\, v_\alpha + \\
+ (\mu_p+\lambda_p-\alpha_p)\,\partial_\alpha\partial_\beta v_\beta + C_p e_{\alpha\beta}\,\partial_\beta u + h_\alpha = 0, \\
K\cdot[2(\mu_t+\lambda_t)+C_p]\,\partial_\alpha u_\alpha + C_p e_{\alpha\beta}\,\partial_\alpha v_\beta + [C_p\nabla^2 - 4K^2(\mu_t+\lambda_t)]u + b = 0,
\end{gathered}
\tag{2.2}
$$

where $\alpha, \beta, \sigma = 1, 2$, $\nabla^2 = \partial_1^2 + \partial_2^2$, $e_{\alpha\beta}$ — Ricci tensor, $K = 1/R$, R — the radius of curvature. The components b_α, b, h_α, h denote resultants due to the effective external forces and couples measured per unit area of the middle surface, tangent and normal to it, respectively.

In the course of the derivation we shall confine ourselves to such shells, whose elastic properties and response under local load satisfy the conditions

$$
\begin{gathered}
\text{i. } C_t \leqslant \delta^2 \cdot C_p \ll \delta^3 \cdot \mu_t, \\
\text{ii. } \delta/R \ll 1 \qquad \text{iii. } L/R \ll 1
\end{gathered}
\tag{2.3}
$$

where δ — a parameter characterizing a geometry of "microstructure" of the lattice. $[\delta] = m$, L — the wave length of the deformation pattern.

It can be proved, that the assumptions (2.3) allowus to neglect the underlined terms in (2.2), dependent on the second powers of the radius of curvature. The underlined terms in the first two equations result from an influence of transverse stresses p_α. The neglected quantities in the fourth and the fifth equations result from the effect of polar moments m_α.

The assumed simplifications are weaker than those of Vlasov's applied in the shallow shell theory.

3. Normal point load — fundamental solution

The fundamental solution will be found by Fourier transform technique. After carrying out the double Fourier transformation of the equations (2.2) with the right-hand sides $b_\alpha = h_\alpha = h = 0$, $b = P\cdot\delta(x^1)\cdot\delta(x^2)$ and then expressing the inverse transforms in the polar coordinates r, $\vartheta(x^1 = r\cos\vartheta,\ x^2 = r\sin\vartheta)$ and finally carrying out the ϑ — integration, we obtain

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{P\cdot K}{2\pi}\begin{Bmatrix} \cos\vartheta \\ \sin\vartheta \end{Bmatrix}\cdot\int_0^\infty \frac{\gamma^2[(2\mu_t+2\lambda_t+C_p)\cdot(\mu_p+\alpha_p)\gamma^2+2C_p(\mu_t+\lambda_t)]\cdot J_1(\gamma\cdot r)d\gamma}{M(\gamma)},$$

$$\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \frac{P\cdot C_p}{2\cdot\pi}\begin{Bmatrix} \sin\vartheta \\ -\cos\vartheta \end{Bmatrix}\cdot\int_0^\infty \frac{\gamma^2[-(2\mu_t+\lambda_t)\gamma^2+K^2(2\mu_t+2\lambda_t+C_p)]\cdot J_1(\gamma\cdot r)d\gamma}{M(\gamma)}, \tag{3.1}$$

$$v = 0,$$

$$u = \frac{P}{2\pi}\int_0^\infty \frac{\gamma[(2\mu_t+\lambda_t)(\mu_p+\alpha_p)\cdot\gamma^4+(2\mu_t+\lambda_t)C_p\cdot\gamma^2-K^2C_p^2]\cdot J_0(\gamma r)d\gamma}{M(\gamma)}$$

$$\begin{aligned} M(\gamma) = {} & (2\mu_t+\lambda_t)\cdot(\mu_p+\alpha_p)\cdot C_p\cdot\gamma^6+K^2(\mu_p+\alpha_p)\cdot[4\mu_t\cdot(\lambda_t+\mu_t)- \\ & -C_p(4\mu_t+4\lambda_t+C_p)]\gamma^4+4K^2\mu_t\cdot(\mu_t+\lambda_t)C_p\cdot\gamma^2- \\ & -4K^4C\cdot(\mu_t+\lambda_t), \end{aligned} \tag{3.2}$$

where J_ν are the Bessel functions of the first kind of order ν.

Let us introduce the dimensionless variables $\bar{r} = r/l_0$, $\bar{\gamma} = \gamma\cdot l_0$ where

$$l_0 = [((2\mu_t+\lambda_t)(\mu_p+\alpha_p))/(4K^4C_p\cdot(\mu_t+\lambda_t))]^{\frac{1}{4}} \tag{3.3}$$

The quantity l_0 exists, provided the elastic moduli satisfy the inequalities: $\mu_t+\lambda_t > 0$, $\mu_t > 0$, $\mu_p > 0$, $\alpha_p > 0$, $C_p > 0$, resulting from the positive definiteness of the elastic potential of the shell. Equations (3.1) take the form

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{P\cdot l_0[2(\mu_t+\lambda_t)+C_p]\cdot K}{2\pi\cdot C_p\cdot(2\mu_t+\lambda_t)}\cdot\begin{Bmatrix} \cos\vartheta \\ \sin\vartheta \end{Bmatrix}\cdot\int_0^\infty \frac{\bar{\gamma}^2\cdot(\bar{\gamma}^2+c)\cdot J_1(\bar{\gamma}\cdot\bar{r})d\bar{\gamma}}{W(\bar{\gamma})},$$

$$\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \frac{P\cdot l_0}{2\pi\cdot(\mu_p+\alpha_p)}\cdot\begin{Bmatrix} -\sin\vartheta \\ \cos\vartheta \end{Bmatrix}\cdot\int_0^\infty \frac{(\bar{\gamma}^4-\bar{b}_0\cdot\bar{\gamma}^2)\cdot J_1(\bar{\gamma}\cdot\bar{r})d\bar{\gamma}}{W(\bar{\gamma})}, \tag{3.4}$$

$$u = \frac{P}{2\cdot\pi\cdot C_p}\cdot\int_0^\infty \frac{(\bar{\gamma}^4+\bar{d}_2\bar{\gamma}^2+\bar{d}_0)\cdot\bar{\gamma}\cdot J_0(\bar{\gamma}\cdot\bar{r})d\bar{\gamma}}{W(\bar{\gamma})},$$

where

$$W(\gamma) = \omega(\gamma^2), \quad \omega(z) \equiv z^3 + \bar{a}_4 \cdot z^2 + \bar{a}_2 z - 1, \tag{3.5}$$

$$c = \frac{2C_p \cdot (\mu_t + \lambda_t) \cdot l_0^2}{(\mu_p + \alpha_p) \cdot (2\mu_t + 2\lambda_t + C_p)}, \quad \bar{b}_0 = \frac{[2\mu_t + 2\lambda_t + C_p] \cdot K^2 \cdot l_0^2}{(2\mu_t + \lambda_t)},$$

$$\bar{a}_4 = \frac{K^2 \cdot [4\mu_t(\mu_t + \lambda_t) - C_p \cdot (4\mu_t + 4\lambda_t + C_p)] \cdot l_0^2}{(2\mu_t + \lambda_t) \cdot C_p}, \quad \bar{d}_2 = \frac{C_p \cdot l_0^2}{(\mu_p + \alpha_p)}, \tag{3.6}$$

$$\bar{a}_2 = \frac{4 \cdot K^2 \cdot \mu_t \cdot l_0^4 \cdot (\mu_t + \lambda_t)}{(2\mu_t + \lambda_t) \cdot (\mu_p + \alpha_p)}, \quad \bar{d}_0 = \frac{-K^2 \cdot C_p^2 \cdot l_0^4}{(2\mu_t + \lambda_t) \cdot (\mu_p + \alpha_p)}.$$

The series expansions of the integrals (3.4) with respect to the variable $\bar{r}$ depend on the roots of the polynomial ω. In view of the obvious fact, that one of the real roots is positive, the polynomial ω can be written in the form

$$\omega(z) = (z^2 - a^2) \cdot (z^2 + c_2 \cdot z + c_0), \quad c_0 > 0.$$

Let us consider the three cases[1])

1. The polynomial $p(z) = z^2 + c_2 z + c_0$ has no real roots, so that the following inequality hold true

$$\left(\frac{1}{3}\bar{a}_2 - \frac{1}{9}\bar{a}_4^2\right)^3 + \left[\frac{1}{6}(\bar{a}_2\bar{a}_4 + 3) - \frac{1}{27}\bar{a}_4^3\right]^2 > 0. \tag{3.7}$$

Thus the polynomial W can be written in the form

$$W(\bar{\gamma}) = (\bar{\gamma}^2 - a^2) \cdot (\bar{\gamma}^2 + s\bar{\gamma} + q) \cdot (\bar{\gamma}^2 - s \cdot \bar{\gamma} + q), \quad s, q \in R$$

2. The polynomial $p(z)$ has positive real roots. $W(\gamma)$ takes the form

$$W(\bar{\gamma}) = (\bar{\gamma}^2 - a^2) \cdot (\bar{\gamma}_2 - b^2) \cdot (\bar{\gamma}^2 - c^2), \quad b, c > 0.$$

3. The polynomial $p(z)$ has negative real roots. Hence, we have

$$W(\gamma) = (\gamma^2 - a^2) \cdot (\bar{\gamma}^2 + b^2) \cdot (\bar{\gamma}^2 + c^2), \quad b, c > 0.$$

To carry out a complete analysis of the behaviour of functions u^α, u, v^α, the cases 1 - 3 will be considered separately.

Ad. 1. Decomposing the integrands in (3.4) into a sum of simple fractions and using the definitions (1, 2) of special functions examined in the Appendix, we find

$$\begin{aligned} &\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}(\bar{r}, \vartheta) = \mathscr{L}(g_1(\bar{r}, a); h_{1,1}^{(R)}(\bar{r}; s, q); h_{1,0}^{(S)}(\bar{r}; s, q)) \cdot \begin{Bmatrix} \cos\vartheta \\ \sin\vartheta \end{Bmatrix}, \\ &\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix}(\bar{r}, \vartheta) = \mathscr{L}(g_1(\bar{r}, a); h_{1,1}^{(R)}(\bar{r}; s, q); h_{1,0}^{(S)}(\bar{r}; s, q)) \cdot \begin{Bmatrix} -\sin\vartheta \\ \cos\vartheta \end{Bmatrix}, \\ &u(r, \vartheta) = \mathscr{L}(g_0(\bar{r}, a); h_{0,1}^{(S)}(\bar{r}; s, q); h_{0,0}^{(R)}(r, s, q)), \quad v(r, \vartheta) = 0. \end{aligned} \tag{3.8}$$

The expression $\mathscr{L}(f_1; \ldots; f_n)$ means the linear combination of the functions $f_1, \ldots, f_n$.

Ad 2. Proceeding similarly as in the first case, the equations

[1]) The case of double roots is omitted.

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}(r, \vartheta) = \mathscr{L}(g_1(\bar{r}, a); g_1(\bar{r}, b); g_1(r, c)) \cdot \begin{Bmatrix} \cos\vartheta \\ \sin\vartheta \end{Bmatrix},$$

$$(3.9) \qquad \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix}(\bar{r}, \vartheta) = \mathscr{L}(g_1(\bar{r}, a); g_1(\bar{r}, b); g_1(\bar{r}, C)) \cdot \begin{Bmatrix} -\sin\vartheta \\ \cos\vartheta \end{Bmatrix},$$

$$u(\bar{r}, \vartheta) = \mathscr{L}(g_0(\bar{r}, a); g_0(\bar{r}, b); g_0(\bar{r}, c)), \quad v(\bar{r}, \vartheta) = 0,$$

are obtained.

Ad 3. In this case, we have

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}(\bar{r}, \vartheta) = \mathscr{L}(g_1(\bar{r}, a); f_1(\bar{r}, b); f_1(\bar{r}, c)) \begin{Bmatrix} \cos\vartheta \\ \sin\vartheta \end{Bmatrix},$$

$$(3.10) \qquad \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix}(\bar{r}, \vartheta) = \mathscr{L}(g_1(\bar{r}, a); f_1(r, b); f_1(\bar{r}, c)) \cdot \begin{Bmatrix} -\sin\vartheta \\ \cos\vartheta \end{Bmatrix},$$

$$u(\bar{r}, \vartheta) = \mathscr{L}(g_0(\bar{r}, a); f_0(\bar{r}, b); f_0(\bar{r}, c)), \quad v(\bar{r}, \vartheta) = 0.$$

Strains and stresses can be found with the aid of the straindisplacement and constitutive equations.

The results obtained in the Appendix enable us to prove, that in each of the cases considered above functions u^α, u, v^α in the vicinity of point P can be approximated by

$$(3.11) \qquad \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \sim \begin{Bmatrix} U_1 \cdot \cos\vartheta \\ U_2 \cdot \sin\vartheta \end{Bmatrix} \cdot r \cdot \ln r, \quad \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} \sim \begin{Bmatrix} -V_1 \sin\vartheta \\ V_2 \cos\vartheta \end{Bmatrix} r \cdot \ln r, \quad u \sim U \cdot \ln r.$$

Hence, the following singularities of the components of the state of stress can be found

$$p_{\alpha\alpha} \sim P_{\alpha\alpha} \cdot \ln r, \quad p_1 \sim P_1 \cdot \frac{1}{r}, \quad p_{12} = p_{21} = p_2 = 0,$$

$$(3.12)$$

$$m_{\alpha\beta} \sim M_{\alpha\beta} \cdot \ln r, \quad m_2 \sim M_2 \cdot r \cdot \ln r, \quad m_1 = 0.$$

4. Grid shell

The results obtained at Secs. 2 and 3 can be applied to the analysis of a grid shell of isotropic structure. The considerations will be confined to the so called geodesic lattice domes constructed by three families of bars, formed on the basis of icosahedron by means of the known methods, due to Fuller [3] or Tarnai [4], cf. [5]. The desired properties, namely the isotropy and centrosymmetry are satisfied with the sufficient accuracy for the engineering practice. Effective elastic moduli of such structures have been given in [1]; it is worth mentioning, that the relations $\lambda_i = \mu_i - \alpha_i$, $i = p, t$, hold true. In the case of slender bars, conditions $(2.3)_1$ are satisfied. Furthermore it can be proved, that for all real grid shells of this structure the inequality (3.7) is valid; thus displacements and rotations of nodes are approximated by means of the formulae (3.8).

A quantitative analysis of the response of geodesic grid shell subjected to normal point load will be presented in a separate paper.

5. Concluding remarks

In the paper one version of simplifications of the governing system of equations o Woźniak's lattice-type shell theory has been proposed. The aim of the model is to describe special type of deformations occuring in shells subjected to local loads. An analysis of the response of the spherical lattice shell to the normal point load confirmed, that the model is useful to both quantitative and qualitative considerations. In particular, it is possible to prove, that in the case considered, the singularities of the displacements and stresses are of the same order as those in the classical Reissner's type theory of shells, cf (3.11 - 3.12) and [2].

Nevertheless, the proposed model can not be used when boundary value problems of shallow lattice shells are considered since there the governing set of equations does not satisfy the strong ellipticity condition.

6. References

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Appendix

Evaluate the improper integrals

$$f_\nu(y, a) = \int_0^\infty \frac{J_\nu(x \cdot y) \cdot x^{1-\nu} dx}{x^2+a^2}, \quad g_\nu(y, a) = \int_0^\infty \frac{J_\nu(x \cdot y) \cdot x^{1-\nu} dx}{x^2-a^2}$$

$$(1) \qquad h_{\nu,\mu}(y; r, q) = \int_0^\infty \frac{J_\nu(x \cdot y) \cdot x^\mu dx}{x^2+r \cdot x+q}, \quad \begin{array}{ll} \nu, \mu = 0, 1, & a, q, r \in R \\ -4q+r^2 < 0, & q > 0. \end{array}$$

In the paper the following functions are also used

$$(2) \qquad \begin{aligned} h^{(S)}_{\nu,\mu}(y; r, q) &= h_{\nu,\mu}(y; r, q)+h_{\nu,\mu}(y; -r, q), \\ h^{(R)}_{\nu,\mu}(y; r, q) &= h_{\nu,\mu}(y; r, q)-h_{\nu,\mu}(y; -r, q). \end{aligned}$$

1. Integrals f_ν, g_ν

According to the tables [6], we have

(3)
$$f_0(y,a) = K_0(a\cdot y), \quad g_0(y,a) = -\frac{\pi}{2}\cdot Y_0(a\cdot y),$$
$$f_1(y,a) = \frac{1}{a}\left[\frac{1}{a\cdot y} - K_1(a\cdot y)\right], \quad g_1(y,a) = -\frac{1}{a}\left[\frac{\pi}{2}\cdot Y_1(a,\cdot y)+\frac{1}{a\cdot y}\right],$$

where Y_0, Y_1, K_0, K_1 are Bessel functions and modified Bessel functions of order zero and one, [6, 7]. Expansions of f_ν, g_ν in the vicinity of $y = 0$, take the form

(4)
$$f_0, g_0 \sim -\ln y, \quad f_1, g_1 \sim -\frac{1}{2}\cdot y\cdot \ln y.$$

2. The integrals $h_{\nu,\mu}$

The author has not fiund expansions of the integrals $h_{\nu,\mu}$ in the available monographs on the special functions. A method of evaluating of the integrals based on the Poisson's integral representation of Bessel functions is presented briefly. Some ideas of the procedure has been taken from the paper of Simmonds and Bradley, [8], where the integral

$$\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \frac{\exp(-i(\alpha\cdot x+\beta\cdot y))d\alpha d\beta}{(\alpha^2+\beta^2)^2+i(\alpha^2+k\beta^2)}, \quad k\in(0,1),$$

has been examined.

Starting from the identity

(5)
$$x^\mu\cdot(x^2+r\cdot x+q)^{-1} = \frac{a^\mu}{2bi}\cdot\left(\frac{1}{x-\zeta_1}-\frac{1}{x-\zeta_2}\right)+\frac{\delta_{1\mu}}{2}\cdot\left(\frac{1}{x-\zeta_1}-\frac{1}{x-\zeta_2}\right)$$

where:

$$\mu = 0, 1, \quad \zeta_1 = a+bi, \quad \zeta_2 = a-bi, \quad a = \frac{-r}{2}, \quad b = \left(q-\frac{r^2}{4}\right)^{1/2},$$

one obtains

(6)
$$h_{\nu,\mu}(y;r,q) = \frac{a^\mu}{2bi}(H^{(1)}_{\nu,\mu}-H^{(2)}_{\nu,\mu})+\frac{1}{2}\cdot\delta_{1\mu}\cdot(H^{(1)}_{\nu,\mu}+H^{(2)}_{\nu,\mu}),$$
$$H^{(\lambda)}_{\nu,\mu} = \int\limits_0^\infty \frac{J_\nu(xy)dx}{x-\zeta_\lambda}.$$

Recalling the integral representation of Bessel functions J_ν

(7)
$$J_\nu(x\cdot y) = \frac{i^\nu}{2\cdot\pi}\int\limits_0^{2\pi} e^{-ixy\cdot\cos\varphi}\cdot\cos\nu\varphi d\varphi, \quad \nu = 0, 1,$$

and interchanging of orders of integrations, we find

(8)
$$H^{(\lambda)}_{\nu,\mu} = \frac{i^\nu}{2\pi}\int\limits_0^{2\pi}\cos\nu\varphi\cdot F^\lambda_\mu(\varphi)d\varphi, \quad F^\lambda_\mu(y,\varphi) = \int\limits_0^\infty \frac{\exp(i\cdot\omega(\varphi)\cdot x)dx}{x-\zeta_\lambda},$$

where $\omega(\varphi) = -y\cos\varphi$. The function F_μ^λ can be expressed by means of the complex exponential integral E_1

$$(9)\qquad F_\mu^\lambda(y,\varphi) = e^{i\omega\zeta_\lambda}\left\{E_1(i\cdot\omega\cdot\zeta_\lambda)+\frac{\pi\cdot i}{2}\cdot(1+\mathrm{sgn}(\mathrm{Re}\,\zeta_\lambda))\cdot(\mathrm{sgn}\,\omega+\mathrm{sgn}(\mathrm{Im}\,\zeta_\lambda))\right\}.$$

For the sake of brevity the proof is omitted. Expanding E_1 into power series, inserting (9) to $(8)_1$ and carrying out the φ — integration with the aid of the equations

$$(10)\qquad s_n \equiv \frac{1}{2\pi}\int_0^{2\pi}\cos^n x\,dx = \begin{cases} 2^{-2k}\cdot\binom{2k}{k}, & n=2k,\\ 0 & n=2k+1,\ k=0,1,\ldots,\end{cases}$$

$$(11)\qquad r_n \equiv \frac{1}{2\pi}\cdot\int_0^{2\pi}\cos^n x\cdot\ln|\cos x|\,dx = $$

$$= \begin{cases} \sum\limits_{j=-k}^{k} c_j^k\cdot\tau_j,\ c_j^k = \dfrac{2^{-2k}\cdot(2k)!}{(k+j)!\cdot(k-j)!} & n=2k\\ 0 & n=2k+1, \quad k=0,1,\ldots,\end{cases}$$

where:

$$\tau_k = \begin{cases} -\ln 2 & k=0\\ \dfrac{(-1)^{k+1}}{2\cdot|k|} & k=\pm1,\ \pm2,\ldots,\end{cases}$$

$$(12)\qquad v_n \equiv \frac{1}{2\pi}\int_0^{2\pi}\mathrm{sgn}(-\cos x)\cdot\cos^n x\,dx =$$

$$= \begin{cases} 0 & n=2k,\\ -\dfrac{2^{2k}}{2\pi}\cdot\dfrac{1}{k\cdot\binom{2k-1}{k-1}} & n=2k+1 \quad k=0,1,\ldots,\end{cases}$$

the expansions of the complex functions $\mathrm{H}_{\nu,\mu}^{(\lambda)}$ and real functions $h_{\nu,\mu}$ are found. Finally, we obtain

$$(13)\qquad h_{0,0}(y;r,q) = \frac{1}{b}\cdot\ln y\cdot\sum_1^\infty s_{2n}\cdot f_{S_s}^{(n)}(\vartheta)\cdot y^{2n}+\frac{\pi}{2b}\cdot\sum_0^\infty\cdot v_{2n+1}\cdot\cdot f_{S_c}^{(n)}(\vartheta)\cdot y^{2n+1}+$$

$$+\sum_0^\infty h_{0,0}^{(n)}(\vartheta)\cdot s_{2n}\cdot y^{2n},$$

$$h_{1,0}(y;r,q) = -\frac{1}{b}\cdot\ln y\cdot\sum_0^\infty f_{\delta s}^{(n)}(\vartheta)\cdot s_{2n+2}\cdot y^{2n+1}+\frac{\pi}{2\cdot b}\cdot\sum_0^\infty\cdot f_{S_s}^{(n)}(\vartheta)\cdot v_{2n+1}\cdot y^{2n}+$$

$$+\sum_0^\infty h_{1,0}^{(n)}(\vartheta)\cdot s_{2n+2}\cdot y^{2n+1},$$

$$h_{01}(y;r,q) = \ln\bar{y}\cdot\sum_0^\infty\left(\frac{a}{b}\cdot f_{S_s}^{(n)}(\vartheta)-f_{C_c}^{(n)}(\vartheta)\right)\cdot s_{2n}\cdot\bar{y}^{2n}+$$

(13) [cont.]
$$+\frac{\pi}{2}\cdot\sum_0^\infty\left(\frac{a}{b}f_{S_c}^{(n)}(\vartheta)+f_{C_s}^{(n)}(\vartheta)\right)\cdot v_{2n+1}\cdot\bar{y}^{2n+1}+\sum_0^\infty h_{0,1}^{(n)}\cdot s_{2n}\cdot\bar{y}^{2n},$$

$$h_{1,1}(y;r,q) = -\ln\bar{y}\cdot\sum_0^\infty\left(\frac{a}{b}f_{S_c}^{(n)}(\vartheta)+f_{C_s}^{(n)}(\vartheta)\right)\cdot s_{2n+2}\cdot\bar{y}^{2n+1}+$$

$$+\frac{\pi}{2}\cdot\sum_0^\infty\left(\frac{a}{b}\cdot f_{S_s}^{(n)}(\vartheta)-f_{C_c}^{(n)}(\vartheta)\right)v_{2n+1}\cdot\bar{y}^{2n}+\sum_0^\infty h_{1,1}^{(n)}\cdot s_{2n+2}\cdot\bar{y}^{2n+1},$$

where $\bar{y} = y\cdot\varrho$, $\varrho = q^{1/2}$, $\cos\vartheta = -r/2\sqrt{q}$, $\sin\vartheta > 0$.

The functions $f_{ij}^{(n)}(\vartheta)$, $i = S, C$, $j = s, c$ are the coefficients in the series expansions

(14)
$$\mathrm{Sh}(x\cdot\sin\vartheta)\cdot\begin{Bmatrix}\sin(x\cdot\cos\vartheta)\\ \cos(x\cdot\cos\vartheta)\end{Bmatrix} = \sum_{n=0}^\infty\begin{Bmatrix}f_{S_s}^{(n)}(\vartheta)\\ f_{S_c}^{(n)}(\vartheta)\end{Bmatrix}\cdot x^{2n},$$

$$\mathrm{Ch}(x\cdot\sin\cdot\vartheta)\cdot\begin{Bmatrix}\sin(x\cdot\cos\vartheta)\\ \cos(x\cdot\cos\vartheta)\end{Bmatrix} = \sum_{n=0}^\infty\begin{Bmatrix}f_{C_s}^{(n)}(\vartheta)\cdot x\\ f_{C_c}^{(n)}(\vartheta)\end{Bmatrix}\cdot x^{2n}.$$

In order to save space, the complex definitions of the coefficients $h_{\alpha,\beta}^{(n)}$ will not be given here.

In the neighbourhood of point $y = 0$ the series expansions (13) with the first few terms written out explicitly have the form

$$h_{0,0}(y;r,q)\sim\frac{\sin 2\vartheta}{4\varrho\cdot\sin\vartheta}\cdot\bar{y}^2\cdot\ln\bar{y}+\frac{\pi-\vartheta}{\varrho\cdot\sin\vartheta}-\frac{1}{2\varrho}\cdot\bar{y},$$

$$h_{1,0}(y;r,q)\sim-\frac{1}{2\varrho}\cdot\ln\bar{y}\cdot\bar{y}+\frac{1}{2\varrho}\cdot[\mathrm{ctg}\,\vartheta\cdot(\pi-\vartheta)+(\ln 2-\gamma_E-1)]\cdot\bar{y}-\frac{\sin 2\vartheta}{3\varrho\cdot\sin\vartheta}\cdot\bar{y}^2,$$

(15)
$$h_{0,1}(y;r,q)\sim-\ln\bar{y}+\frac{\sin 3\vartheta}{4\cdot\sin\vartheta}\cdot\bar{y}^2\cdot\ln\bar{y}-\bar{y}\cdot\cos\vartheta,$$

$$h_{1,1}(y;r,q)\sim\cos\vartheta\cdot\bar{y}\cdot\ln\bar{y}+\frac{1}{2}+\frac{1}{2\sin\vartheta}[(\pi-\vartheta)\cdot\cos 2\vartheta+(\ln 2-\gamma_E-0{,}5)\cdot\sin 2\vartheta]\cdot\bar{y}$$

where $\gamma_E\cong 0.5772$ — Euler's constant. Hence, for $y = 0$

$$h_{0,0} = (\pi-\vartheta)/(\varrho\cdot\sin\vartheta),\quad h_{1,0} = 0,\quad h_{1,1} = 0{,}5.$$

The function $h_{0,1}$ has a logarithmic singularity at point $y = 0$.

Резюме

ДЕЙСТВИЕ СОСРЕДОТОЧЕННОЙ НОРМАЛЬНОЙ СИЛЫ НА ПОЛОГУЮ, СФЕРИЧЕСКУЮ СЕТЧАТУЮ ОБОЛОЧКУ-ФУНДАМЕНТАЛЬНОЕ РЕШЕНИЕ

В работе приводится статический анализ передачи нормальной сосредоточенной силы на сферическую сетчатую оболочку о изотропной и центросимметрической структуре. Применено теорию сетчатых оболочек Возняка. Представлена проблема приводится к анализу одного из фундаментальных решений этой теории.

В первой части получены приближенные уравнения равновесия (в омещениях) базируя на предположениях относительно слабших од тих, при помощи которых В. З. Власов сформулировал теорию пологих, однородных оболочек.

Фундаментальное решение найдено с помощью интегральной трансформации Фурье. Проаналиризовано сингурярности перемещений, деформаций и напряжений.

Streszczenie

DZIAŁANIE NORMALNEJ SIŁY SKUPIONEJ NA MAŁO WYNIOSŁĄ SIATKOWĄ POWŁOKĘ KULISTĄ — ROZWIĄZANIE PODSTAWOWE

Przedmiotem pracy jest analiza statyczna niewielkiego obszaru sferycznej powłoki siatkowej o strukturze izotropowej i centrosymetrycznej wokół punktu przyłożenia normalnej siły skupionej. Wykorzystano teorię powłok siatkowych Woźniaka. Postawiony problem sprowadza się do analizy jednego z rozwiązań podstawowych tej teorii.

W pierwszej części pracy wyprowadzono równania przemieszczeniowe mało wyniosłej, sferycznej powłoki siatkowej. Przyjęto założenia relatywnie słabsze od uproszczeń W. Z. Własowa będących podstawą teorii jednorodnych powłok połogich.

Rozwiązanie podstawowe znaleziono za pomocą całkowej transformacji Fouriera. Zbadano rzędy osobliwości składowych stanu przemieszczenia, odkształcenia i napięcia.

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