JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
54, 3, pp. 743-755, Warsaw 2016
DOI: 10.15632/jtam-pl.54.3.743

GREEN'S FUNCTION FOR A MULTIFIELD MATERIAL WITH A HEAT SOURCE

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Green's functions for a multifield material subjected to a point heat source are presented in an explicit analytical form. The study concerns the steady-state thermal loading infinite region, half-space region and two-constituent magneto-electro-thermo-elastic material region. The new mono-harmonic potential functions, obtained by the author, are used in the analysis. The elastic displacement, electric potential, magnetic potential and induced by those coupled multifield physical quantities, caused by internal or external heat sources, are limited and presented in a very useful form, exactly and explicitly.

Keywords: Green's functions, point heat source, multifield material, magneto-electro-thermo--elastic fundamental solution, multifield composites, exact solution

1. Introduction

The basic solutions, related among others to multifield materials, are Green's functions, which were first proposed by George Green in 1828. There are two different analysis processes for solutions in scientific literature. One has focused on the displacement, electric potential and magnetic potential, constructing equilibrium equations. The second has emphasized equilibrium equations of stresses, electric displacements and magnetic inductions as well as compatibility equations for strains. There is Stroh's formalism (Stroh, 1958) and Lekhnitskii's approach (Lekhnitskii, 1963), for example. On the other hand, there are three commonly used methods in analyzing boundary effects: the theoretical solution, numerical solution and the experiment. But, appropriate Green's functions for a thermoelastic half-space is a specific task. This is due to the fact that the fundamental solution for the displacements is not limited at infinity, which is inconsistent with the mechanical sense. For example, Hou *et al.* (2008) derived a solution with a logarithmic singularity in the generalized displacement fields. Thus, the consideration of static equilibrium of the thermoelastic half-space, a quarter of the space, an octant, a wedge, and a half-wedge under the action of a unit point (as well as distributed) of the internal heat source and boundary temperature or heat flux is a special and important task. The importance is dictated by the fact that the computational scheme of many structural elements is reduced to those volume material regions.

In the context of multifield materials, the solutions depend on a large number of material parameters. For magneto-electro-thermo-elastic materials, it is twenty one, making any solution other than explicit analytical one impractical. The exact formulae, in terms of elementary functions for multifield materials, are presented in this study. The generalized displacements have been obtained with an accuracy up to arbitrary constants, which do not affect the value of stresses. This is the major motivation of the study presented in this paper. Although it sounds theoretically more reasonable, experiment based verification is still desired. It is mentioned here that mono-harmonic potential functions can be found in Chen *et al.* (2004), but some simpler results, obtained by the author of this paper, are presented for the reader's convenience.

The exact solutions related to crack and contact problems of multifield materials were recently presented by Rogowski (2012-2015) for instance.

2. The thermoelastic fundamental solution for magneto-electro-thermo-elastic multifield materials

2.1. The fundamental equations for a magneto-electro-thermo-elastic medium

We consider an axisymmetric problem. Assume that the field variables are functions of r and z in the cylindrical coordinate system (r,θ,z). Constitutive equations for a piezoelectric, piezomagnetic, electromagnetic and thermoelastic material polarized in the positive z-direction subjected to mechanical, thermal, magnetical and electrical fields can be written, in matrix representation, as

$$\begin{aligned}
&\begin{Bmatrix}\sigma_r\\ \sigma_\theta\\ \sigma_z\\ \sigma_{rz}\end{Bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{13}&0\\ c_{12}&c_{11}&c_{13}&0\\ c_{13}&c_{13}&c_{33}&0\\ 0&0&0&c_{44}\end{bmatrix}\begin{Bmatrix}u_{r,r}-\alpha_rT\\ u_r/r-\alpha_rT\\ u_{z,z}-\alpha_zT\\ u_{r,z}+u_{z,r}\end{Bmatrix}+\begin{bmatrix}0&e_{31}\\ 0&e_{31}\\ 0&e_{33}\\ e_{15}&0\end{bmatrix}\begin{Bmatrix}\phi_{,r}\\ \phi_{,z}\end{Bmatrix}+\begin{bmatrix}0&q_{31}\\ 0&q_{31}\\ 0&q_{33}\\ q_{15}&0\end{bmatrix}\begin{Bmatrix}\psi_{,r}\\ \psi_{,z}\end{Bmatrix}\\
&\begin{Bmatrix}D_r\\ D_z\end{Bmatrix}=\begin{bmatrix}0&0&0&e_{15}\\ e_{31}&e_{31}&e_{33}&0\end{bmatrix}\begin{Bmatrix}u_{r,r}+\alpha_rT\\ u_r/r+\alpha_rT\\ u_{z,z}+\alpha_zT\\ u_{r,z}+u_{z,r}\end{Bmatrix}-\begin{bmatrix}\varepsilon_{11}&0\\ 0&\varepsilon_{33}\end{bmatrix}\begin{Bmatrix}\phi_{,r}\\ \phi_{,z}\end{Bmatrix}-\begin{bmatrix}d_{11}&0\\ 0&d_{33}\end{bmatrix}\begin{Bmatrix}\psi_{,r}\\ \psi_{,z}\end{Bmatrix}\\
&\begin{Bmatrix}B_r\\ B_z\end{Bmatrix}=\begin{bmatrix}0&0&0&q_{15}\\ q_{31}&q_{31}&q_{33}&0\end{bmatrix}\begin{Bmatrix}u_{r,r}+\alpha_rT\\ u_r/r+\alpha_rT\\ u_{z,z}+\alpha_zT\\ u_{r,z}+u_{z,r}\end{Bmatrix}-\begin{bmatrix}d_{11}&0\\ 0&d_{33}\end{bmatrix}\begin{Bmatrix}\phi_{,r}\\ \phi_{,z}\end{Bmatrix}-\begin{bmatrix}\mu_{11}&0\\ 0&\mu_{33}\end{bmatrix}\begin{Bmatrix}\psi_{,r}\\ \psi_{,z}\end{Bmatrix}
\end{aligned}\tag{2.1}$$

where σ_{ij}, D_i, B_i are mechanical stresses, electric displacements and magnetic inductions, respectively; T is a temperature change; c_{11}, c_{12}, c_{13}, c_{33}, c_{44} denote elastic stiffness; ε_{11}, ε_{33}, and μ_{11}, μ_{33} denote dielectric permittivities and magnetic permeabilities, respectively; e_{kl}, q_{kl} and d_{ll} are piezoelectric, piezomagnetic and magnetoelectric coefficients, respectively, and u_r, u_z are mechanical displacements, while ϕ and ψ are electric and magnetic potentials, respectively; α_r and α_z are thermal expansion coefficients. The subscripts following a comma denote partial differentation with respect to the indicated variables. We mention that various uncoupled cases can be reduced by setting the appropriate coupling coefficients to zero.

The equilibrium equations and the Maxwell equations, in the absence of body forces, electric and magnetic charge densities are given by

$$\begin{aligned}
&\frac{\partial\sigma_r}{\partial r}+\frac{\partial\sigma_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}=0 \qquad &&\frac{\partial\sigma_{rz}}{\partial r}+\frac{\partial\sigma_z}{\partial z}+\frac{\sigma_{rz}}{r}=0\\
&\frac{\partial D_r}{\partial r}+\frac{\partial D_z}{\partial z}+\frac{D_r}{r}=0 &&\frac{\partial B_r}{\partial r}+\frac{\partial B_z}{\partial z}+\frac{B_r}{r}=0
\end{aligned}\tag{2.2}$$

The temperature field in the medium without heat generation in a steady-state is governed by the following equation

$$\lambda_r\Big(\frac{\partial^2T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\Big)+\lambda_z\frac{\partial^2T}{\partial z^2}=0\tag{2.3}$$

where λ_r, λ_z are coefficients of thermal conductivity. Substituting constitutive equations (2.1) into equilibrium equations (2.2) yields the basic governing equilibrium equations for the displacements u_r and u_z, electric potential ϕ and magnetic potential ψ as follows

$$
\begin{aligned}
& c_{11}B_1u_r + c_{44}D^2u_r + (c_{13}+c_{44})D\frac{\partial u_z}{\partial r} + (e_{15}+e_{31})D\frac{\partial \phi}{\partial r} + (q_{15}+q_{31})D\frac{\partial \psi}{\partial r} - \beta_1\frac{\partial T}{\partial r} = 0 \\
& c_{44}B_0u_z + c_{33}D^2u_z + (c_{13}+c_{44})D\frac{\partial [ru_r]}{r\partial r} + (e_{15}B_0 + e_{33}D^2)\phi \\
& \qquad + (q_{15}B_0 + q_{33}D^2)\psi - \beta_3 DT = 0 \\
& (e_{15}+e_{31})D\frac{\partial [ru_r]}{r\partial r} + (e_{15}B_0 + e_{33}D^2)u_z - (\varepsilon_{11}B_0 + e_{33}D^2)\phi \\
& \qquad - (d_{11}B_0 + d_{33}D^2)\psi + p_3DT = 0 \\
& (q_{15}+q_{31})D\frac{\partial [ru_r]}{r\partial r} + (q_{15}B_0 + q_{33}D^2)u_z - (d_{11}B_0 + d_{33}D^2)\phi \\
& \qquad - (\mu_{11}B_0 + \mu_{33}D^2)\psi + \gamma_3DT = 0
\end{aligned}
\tag{2.4}
$$

where the following differential operators have been introduced

$$
B_k = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{k}{r^2} \qquad k = 0,1 \qquad D = \frac{\partial}{\partial z} \qquad D^2 = \frac{\partial^2}{\partial z^2}
\tag{2.5}
$$

In addition, β_i are the thermal moduli and p_3, γ_3 are pyroelectric and pyromagnetic constants, respectively, defined by

$$
\begin{aligned}
\beta_1 &= (c_{11}+c_{12})\alpha_r + c_{13}\alpha_z & \qquad \beta_3 &= 2c_{13}\alpha_r + c_{33}\alpha_z \\
p_3 &= 2e_{31}\alpha_r + e_{33}\alpha_z & \qquad \gamma_3 &= 2q_{31}\alpha_r + q_{33}\alpha_z
\end{aligned}
\tag{2.6}
$$

Equations (2.1) to (2.3) contain 13 equations and 13 unknowns. The 13 unknowns are: two elastic displacements, fourth stresses, two electric displacements and two magnetic inductions, one electric and one magnetic potential and temperature change of the body. Therefore, the 13 unknowns can be determined by solving the 13 equations (2.1) to (2.3).

The governing equations are generalized equilibrium equations (2.4) and heat conduction equation (2.3), which induces five unknowns. These are: two displacements, one electric and one magnetic potential and temperature change of the body.

The transversely isotropic multifield material is characterized by 17 material constants. If the effect of temperature change is taken into account then also four thermal constants appear in the analysis.

Based on the method named the Schmidt method (Morse and Feshbach, 1953) the general solution to the governing equations are obtained by the generalized Almansi theorem.

Then equations (2.4) can be further simplified to

$$
\begin{aligned}
u_r(r,z) &= \sum_{i=0}^{4}\alpha_{1i}\lambda_i\frac{\partial \varphi_i}{\partial r} & \qquad u_z(r,z) &= \sum_{i=0}^{4}\frac{1}{\lambda_i}\frac{\partial \varphi_i}{\partial z} \\
\phi(r,z) &= -\sum_{i=0}^{4}\frac{\alpha_{3i}}{\lambda_i}\frac{\partial \varphi_i}{\partial z} & \qquad \psi(r,z) &= -\sum_{i=0}^{4}\frac{\alpha_{4i}}{\lambda_i}\frac{\partial \varphi_i}{\partial z} \\
T(r,z) &= \frac{\alpha_{00}}{\lambda_0^2}\frac{\partial^2 \varphi_0}{\partial z^2}
\end{aligned}
\tag{2.7}
$$

where

$$
\begin{aligned}
a_{1i} &= \frac{a_1\lambda_i^6 + b_1\lambda_i^4 + c_1\lambda_i^2 + d_1}{a_2\lambda_i^6 + b_2\lambda_i^4 + c_2\lambda_i^2 + d_2}\frac{1}{\lambda_i^2} & \qquad a_{3i} &= \frac{a_3\lambda_i^6 + b_3\lambda_i^4 + c_3\lambda_i^2 + d_3}{a_2\lambda_i^6 + b_2\lambda_i^4 + c_2\lambda_i^2 + d_2} \\
a_{4i} &= \frac{a_4\lambda_i^6 + b_4\lambda_i^4 + c_4\lambda_i^2 + d_4}{a_2\lambda_i^6 + b_2\lambda_i^4 + c_2\lambda_i^2 + d_2} & \qquad a_{00} &= \frac{a\lambda_0^8 + b\lambda_0^6 + c\lambda_0^4 + d\lambda_0^2 + e}{a_2\lambda_0^6 + b_2\lambda_0^4 + c_2\lambda_0^2 + d_2}
\end{aligned}
\tag{2.8}
$$

where $\lambda_0^2 = \lambda_r/\lambda_z$ and λ_i^2 are the roots of the following characteristic algebraic equation

$$a\lambda^8 + b\lambda^6 + c\lambda^4 + d\lambda^2 + e = 0 \tag{2.9}$$

whose parameters a, b, c, d and e and roots (eigenvalues) λ_i^2 $(i = 1, 2, 3, 4)$ are given in Appendix A.

The mono-harmonic functions satisfy the equations

$$\Big(\varDelta + \frac{1}{\lambda_i^2}\frac{\partial^2}{\partial z^2}\Big)\varphi_i(r,z) = 0 \qquad i = 0,1,2,3,4 \tag{2.10}$$

The parameters a_1, b_1, c_1, d_1, b_2, c_2, d_2 and coefficients a_{3i} and a_{4i}, which are defined by the coefficient a_{1i}, are listed in Appendix A for the reader's convenience.

The stresses are

$$\begin{aligned}
\sigma_r &= -\sum_{i=0}^4 \frac{\alpha_{5i}}{\lambda_i}\frac{\partial^2\varphi_i}{\partial z^2} - (c_{11} - c_{12})\frac{u_r}{r} - \beta_1 T \\
\sigma_\theta &= -\sum_{i=0}^4 \frac{\alpha_{5i}}{\lambda_i}\frac{\partial^2\varphi_i}{\partial z^2} - (c_{11} - c_{12})\frac{\partial u_r}{\partial r} - \beta_1 T \\
\sigma_z &= \sum_{i=0}^4 \frac{\alpha_{5i}}{\lambda_i^3}\frac{\partial^2\varphi_i}{\partial z^2} + \lambda_0^{-2}\beta_1 T \qquad \sigma_{zr} = \sum_{i=0}^4 \frac{\alpha_{5i}}{\lambda_i}\frac{\partial^2\varphi_i}{\partial r\partial z} + \lambda_0^{-2}\beta_1 a_{00}\frac{\partial^2\varphi_0}{\partial r\partial z}
\end{aligned} \tag{2.11}$$

where

$$a_{5i} = c_{11}a_{1i} - c_{13} + e_{31}a_{3i} + q_{31}a_{4i} \tag{2.12}$$

The components of the electric field vector E_r and E_z are

$$E_r = -\frac{\partial\phi}{\partial r} = \sum_{i=0}^4 \frac{\alpha_{3i}}{\lambda_i}\frac{\partial^2\varphi_i}{\partial r\partial z} \qquad E_z = -\frac{\partial\phi}{\partial z} = \sum_{i=0}^4 \frac{\alpha_{3i}}{\lambda_i}\frac{\partial^2\varphi_i}{\partial z^2} \tag{2.13}$$

The electric displacements are

$$\begin{aligned}
D_r &= e_{15}\Big(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\Big) + \varepsilon_{11}E_r + d_{11}H_r = \sum_{i=0}^4 a_{6i}\lambda_i\frac{\partial^2\varphi_i}{\partial r\partial z} \\
D_z &= e_{31}\Big(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\Big) + e_{33}\frac{\partial u_z}{\partial z} + \varepsilon_{33}E_z + d_{33}H_z + \beta_3 T = \sum_{i=0}^4 \frac{a_{6i}}{\lambda_i}\frac{\partial^2\varphi_i}{\partial z^2}
\end{aligned} \tag{2.14}$$

where

$$a_{6i} = e_{15}a_{1i} + \frac{e_{15} + \varepsilon_{11}a_{3i} + d_{11}a_{4i}}{\lambda_i^2} \tag{2.15}$$

The components of the magnetic field vector H_r and H_z are

$$H_r = -\frac{\partial\psi}{\partial r} = \sum_{i=0}^4 \frac{\alpha_{4i}}{\lambda_i}\frac{\partial^2\varphi_i}{\partial r\partial z} \qquad H_z = -\frac{\partial\psi}{\partial z} = \sum_{i=0}^4 \frac{\alpha_{4i}}{\lambda_i}\frac{\partial^2\varphi_i}{\partial z^2} \tag{2.16}$$

The magnetic inductions are

$$\begin{aligned}
B_r &= q_{15}\Big(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\Big) + \mu_{11}H_r + d_{11}E_r = \sum_{i=0}^4 a_{7i}\lambda_i\frac{\partial^2\varphi_i}{\partial r\partial z} \\
B_z &= q_{31}\Big(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\Big) + q_{33}\frac{\partial u_z}{\partial z} + \mu_{33}H_z + d_{33}E_z + \gamma_3 T = \sum_{i=0}^4 \frac{a_{7i}}{\lambda_i}\frac{\partial^2\varphi_i}{\partial z^2}
\end{aligned} \tag{2.17}$$

where

$$a_{7i} = q_{15}a_{1i} + \frac{q_{15} + \mu_{11}a_{4i} + d_{11}a_{3i}}{\lambda_i^2} \tag{2.18}$$

3. Thermal problems for multifield materials

Consider the problem of a point heat source placed within the multifield material. Introduce the following mono-harmonic functions, which are even functions with respect to the z-coordinate

$$\varphi_i(r,z_i) = A_i\Big[z_i \operatorname{arcsin} h\Big(\frac{z_i}{r}\Big) - R_i\Big] \qquad z_i=\lambda_i z \qquad R_i=\sqrt{r^2+z_i^2} \qquad i=0,1,2,3,4 \tag{3.1}$$

where A_i are constants to be determined.

The derivatives φ_i are as follows

$$\begin{aligned} &\frac{\partial\varphi_i}{\partial r} = -\frac{R_i}{r} \qquad \frac{\partial\varphi_i}{\partial z_i} = \operatorname{arcsin} h\Big(\frac{z_i}{r}\Big) \qquad \frac{\partial^2\varphi_i}{\partial r\partial z_i} = -\frac{z_i}{rR_i}\\ &\frac{\partial^2\varphi_i}{\partial z_i^2} = \frac{1}{R_i} \qquad \frac{\partial^2\varphi_i}{\partial r^2} = \frac{z_i^2}{r^2R_i} \qquad \frac{\partial^2\varphi_i}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi_i}{\partial r} + \frac{\partial^2\varphi_i}{\partial z_i^2} = 0 \end{aligned} \tag{3.2}$$

The physical multifields are as follows

$$\begin{aligned} &u_r = -\sum A_i a_{1i}\lambda_i\frac{R_i}{r} \qquad (u_z,\phi,\psi) = \sum A_i(1,-a_{3i},-a_{4i}) \operatorname{arcsin} h\Big(\frac{z_i}{r}\Big)\\ &T = A_0a_{00}\frac{1}{R_0} \qquad \frac{\partial T}{\partial z} = -A_0a_{00}\frac{\lambda_0^2 z}{R_0^3} \qquad \frac{\partial T}{\partial r} = -A_0a_{00}\frac{r}{R_0^3}\\ &\sigma_r = -\sum A_i a_{5i}\lambda_i\frac{1}{R_i} + (c_{11}-c_{12})\sum A_i a_{1i}\lambda_i\frac{R_i}{r^2} - \beta_1 T\\ &\sigma_\theta = -\sum A_i a_{5i}\lambda_i\frac{1}{R_i} - (c_{11}-c_{12})\sum A_i a_{1i}\lambda_i\Big(\frac{R_i}{r^2} - \frac{1}{R_i}\Big) - \beta_1 T\\ &\sigma_z = \sum A_i\frac{a_{5i}}{\lambda_i}\frac{1}{R_i} + \frac{\beta_1}{\lambda_0^2}T \qquad \sigma_{zr} = -\sum A_i a_{5i}\frac{z_i}{rR_i} - \frac{\beta_1}{\lambda_0}\frac{z}{r}T\\ &E_r = -\sum A_i a_{3i}\frac{z_i}{rR_i} \qquad E_z = \sum A_i a_{3i}\lambda_i\frac{1}{R_i}\\ &H_r = -\sum A_i a_{4i}\frac{z_i}{rR_i} \qquad H_z = \sum A_i a_{4i}\lambda_i\frac{1}{R_i}\\ &D_r = \sum A_i a_{6i}\lambda_i^2\frac{z_i}{rR_i} \qquad D_z = \sum A_i a_{6i}\lambda_i\frac{1}{R_i}\\ &B_r = \sum A_i a_{7i}\lambda_i^2\frac{z_i}{rR_i} \qquad B_z = \sum A_i a_{7i}\lambda_i\frac{1}{R_i} \end{aligned} \tag{3.3}$$

where the following abbreviation notation is used $\sum = \sum_{i=0}^4$.

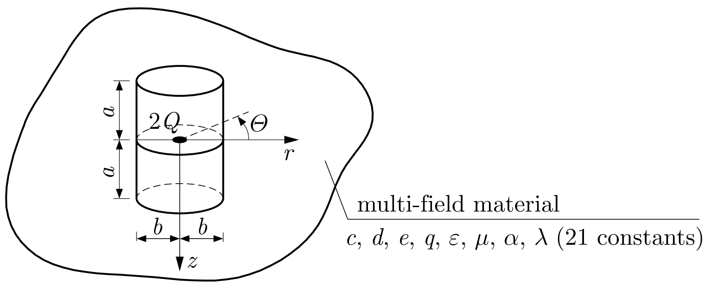

Fig. 1. A point heat source $2Q$ in an infinite multifield material

When we use the physical consideration that the total heat flux transmitted through a cylinder $0 \leqslant z \leqslant a$, $r \leqslant b$ must be equal to a point heat source Q (see Fig. 1), the following equation can be written

$$-2\pi\lambda_z \int\limits_0^b \frac{\partial T}{\partial z}(r,a)r\,dr - 2\pi\lambda_r \int\limits_0^a \frac{\partial T}{\partial r}(b,z)\,dz = Q \tag{3.4}$$

The substitution of Eq$(3.3)_{4,5}$ and integrations yields

$$\begin{aligned}\frac{Q}{2\pi} &= \lambda_z\lambda_0^2 a a_{00} A_0 \int\limits_0^b \frac{r}{\sqrt{(r^2+\lambda_0^2 a^2)^3}}\,dr + \lambda_r b^2 a_{00} A_0 \int\limits_0^a \frac{1}{\sqrt{(b^2+\lambda_0^2 z^2)^3}}\,dz \\ &= \lambda_r a a_{00} A_0 \Big(-\frac{1}{\sqrt{r^2+\lambda_0^2 a^2}}\Big)\Big|_0^b - \lambda_r a_{00} A_0 \Big(1 - \frac{z}{\sqrt{b^2+\lambda_0^2 z^2}}\Big)\Big|_0^a \\ &= \lambda_r a_{00} A_0 \Big(-\frac{a}{\sqrt{b^2+\lambda_0^2 a^2}} + \frac{1}{\lambda_0} + \frac{a}{\sqrt{b^2+\lambda_0^2 a^2}}\Big) = a_{00} A_0 \sqrt{\lambda_r\lambda_z}\end{aligned} \tag{3.5}$$

that is

$$A_0 = \frac{Q}{2\pi a_{00}\sqrt{\lambda_r\lambda_z}} \tag{3.6}$$

Note that in an infinite medium with the point heat source $2Q$, the constant A_0 assumes the same value. The temperature and heat fluxes are

$$T = \frac{Q}{2\pi\sqrt{\lambda_r\lambda_z}}\frac{1}{R_0} \qquad \lambda_z\frac{\partial T}{\partial z} = -\frac{Q}{2\pi}\frac{\lambda_0 z}{R_0^3} \qquad \lambda_r\frac{\partial T}{\partial r} = -\frac{Q}{2\pi}\frac{\lambda_0 r}{R_0^3} \tag{3.7}$$

Of course

$$\lambda_r\Big(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\Big) + \lambda_z\frac{\partial^2 T}{\partial z^2} = 0 \tag{3.8}$$

3.1. The half-space problem

The boundary conditions and the corresponding equations for A_i are

$$\begin{array}{lll}
\text{(a)} & \sigma_{zr}(r,0)=0 & \text{is identically satisfied} \\
\text{(b)} & \sigma_z(r,0)=0 & \displaystyle\sum_{i=1}^4 A_i\frac{a_{5i}}{\lambda_i} + \frac{Q}{2\pi\lambda_z a_{00}}\Big(\frac{\beta_1 a_{00}}{\lambda_0} + a_{50}\Big)\frac{1}{\lambda_0^2} = 0 \\
\text{(c)} & D_z(r,0)=0 & \displaystyle\sum_{i=1}^4 A_i a_{6i}\lambda_i + \frac{Q}{2\pi\lambda_z a_{00}}a_{60} = 0 \\
\text{(d)} & B_z(r,0)=0 & \displaystyle\sum_{i=1}^4 A_i a_{7i}\lambda_i + \frac{Q}{2\pi\lambda_z a_{00}}a_{70} = 0 \\
\text{(e)} & u_r \ \text{ is finite at } \ r=0 & \displaystyle\sum_{i=1}^4 A_i a_{1i}\lambda_i^2 + \frac{Q}{2\pi\lambda_z a_{00}}a_{10}\lambda_0 = 0
\end{array} \tag{3.9}$$

Thus, the coupled field in a semi-infinite transversely isotropic multifield material is determined by solution (3.3) and the following constants $A_i\lambda_i$

$$\begin{bmatrix} A_1\lambda_1 \\ A_2\lambda_2 \\ A_3\lambda_3 \\ A_4\lambda_4 \end{bmatrix} = -\frac{Q}{2\pi\lambda_z a_{00}} \begin{bmatrix} \dfrac{a_{51}}{\lambda_1^2} & \dfrac{a_{52}}{\lambda_2^2} & \dfrac{a_{53}}{\lambda_3^2} & \dfrac{a_{54}}{\lambda_4^2} \\ a_{61} & a_{62} & a_{63} & a_{64} \\ a_{71} & a_{72} & a_{73} & a_{74} \\ a_{11}\lambda_1 & a_{12}\lambda_2 & a_{13}\lambda_3 & a_{14}\lambda_4 \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{1}{\lambda_0^2}\Big(\dfrac{\beta_1 a_{00}}{\lambda_0} + a_{50}\Big) \\ a_{60} \\ a_{70} \\ a_{10}\lambda_0 \end{bmatrix} \tag{3.10}$$

Generally, the permittivity and permeability of air or vacuum is about 680 and 475 times smaller, respectively, than that of commercial multifield materials. In reality, D_z and B_z do not transmit through the free boundary of half-space as assumed in conditions (c) and (d) of equations (3.9). It can be seen from equations (3.10) and (3.3) that Green's functions for point heat sources applied on the boundary of the half-space are expressed exactly and explicitly in terms of elementary functions. This will be greatly beneficial to the succeeding analysis of thermoelastic problems of magneto-electro-thermo-elastic materials. Note that the total heat flux transmitted through the free boundary $z = 0$ is

$$Q + 2\pi\lambda_z \int_0^\infty \frac{\partial T}{\partial z} r\, dr = Q + Q\frac{\lambda_0 z}{R_0}\bigg|_0^\infty = Q - Q = 0 \tag{3.11}$$

This is a confirmation of the correctness of the obtained result. Note again that

$$\begin{aligned} a_{5i} &= c_{11}a_{1i} - c_{13} + e_{31}a_{3i} + q_{31}a_{4i} \qquad & a_{6i} &= e_{15}a_{1i} + \frac{e_{15} + \varepsilon_{11}a_{3i} + d_{11}a_{4i}}{\lambda_i^2} \\ a_{7i} &= q_{15}a_{1i} + \frac{q_{15} + \mu_{11}a_{4i} + d_{11}a_{3i}}{\lambda_i^2} \end{aligned} \tag{3.12}$$

and a_{1i}, a_{3i}, a_{4i} are defined by equations (3.15), see also very coupled but alternative equations (A2) and (A3) in Appendix A and equations (3.22) in special cases.

3.2. An infinite body containing a point heat source $2Q$

If

$$\sum_{i=0}^{4} A_i a_{1i}\lambda_i^2 = 0 \qquad \text{and} \qquad \sum_{i=0}^{4} A_i(1, -a_{3i}, -a_{4i}) = 0 \tag{3.13}$$

then the generalized displacements u_r, u_z, ϕ and ψ caused by the internal heat source $2Q$ are limited, but they cannot be calculated in the neighborhood of the z-axis, that is when $r \to 0$. The displacements which are obtained with an accuracy up to an arbitrary constant do not affect the value of stresses. Arbitrary constants can be treated as linear displacements of the medium as a rigid body in the axial direction without rotation.

The solution to algebraic system of equations (3.13) is

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = -\frac{Q}{2\pi a_{00}\sqrt{\lambda_r\lambda_z}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{11}\lambda_1^2 & a_{12}\lambda_2^2 & a_{13}\lambda_3^2 & a_{14}\lambda_4^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ a_{30} \\ a_{40} \\ a_{10}\lambda_0^2 \end{bmatrix} \tag{3.14}$$

where the coefficients a_{1i}, a_{3i}, a_{4i} for $i = 0, 1, 2, 3, 4$ are as follows

$$\begin{Bmatrix} a_{1i} \\ a_{3i} \\ a_{4i} \end{Bmatrix} = \begin{bmatrix} (e_{31}+e_{15})\lambda_i^2 & \varepsilon_{11} - \varepsilon_{33}\lambda_i^2 & d_{11} - d_{33}\lambda_i^2 \\ (q_{31}+q_{15})\lambda_i^2 & d_{11} - d_{33}\lambda_i^2 & \mu_{11} - \mu_{33}\lambda_i^2 \\ c_{11} + c_{13}\lambda_i^2 & e_{31} + e_{33}\lambda_i^2 & q_{31} + q_{33}\lambda_i^2 \end{bmatrix}^{-1} \begin{Bmatrix} e_{33}\lambda_i^2 - e_{15} + p_3 a_{00}\lambda_i\delta_{i0} \\ q_{33}\lambda_i^2 - q_{15} + \gamma_3 a_{00}\lambda_i\delta_{i0} \\ c_{33}\lambda_i^2 + c_{13} - (\beta_3 + \beta_1\lambda_i^{-2})a_{00}\lambda_i\delta_{i0} \end{Bmatrix} \tag{3.15}$$

This is an alternative and simpler form of parameters defined by equations (A2) and (A3).

Note that the units of the elements of the last matrix are for typical multifield materials

$$\begin{aligned} &[e] = \mathrm{C/m^2} \qquad &&[p_3] = 10^{-6}\mathrm{C/(m^2K)} \qquad &&[a_{00}] = 10^6\mathrm{K} \qquad &&[p_3a_{00}] = \mathrm{C/m^2} \\ &[q] = 10^2\mathrm{N/(Am)} &&[\gamma_3] = 10^{-4}\mathrm{N/(AmK)} &&[\gamma_3 a_{00}] = 10^2\mathrm{N/(Am)} \\ &[c] = 10^{10}\mathrm{N/m^2} &&[\beta_1, \beta_3] = 10^4\mathrm{N/(m^2K)} &&[(\beta_1, \beta_3)a_{00}] = 10^{10}\mathrm{N/m^2} \end{aligned} \tag{3.16}$$

with the multiplier $\in \langle 1, 10\rangle$.

This states that the constituents of the sums are of the same order in each row of the last matrix in (3.15).

Green's functions for the internal heat source applied in multifield materials are determined by equations (3.3), (3.14) and (3.15). All physical components of multifield materials are expressed in forms of elementary functions. It is very simple and straightforward to give numerical results. The results may help the understanding of behaviour of "smart" devices and "intelligent" structures made by multifield materials.

3.3. Single phase materials and multifield composite materials

Multifield composite materials usually comprise alternating piezoelectric and piezomagnetic materials. If the material is piezoelectric then we define the matrix

$$\mathbf{C}_E = \begin{bmatrix} (e_{31}+e_{15})\lambda_i^2 & \varepsilon_{11}-\varepsilon_{33}\lambda_i^2 & 0 \\ 0 & 0 & -\infty \\ c_{11}+c_{13}\lambda_i^2 & e_{31}+e_{33}\lambda_i^2 & 0 \end{bmatrix} \tag{3.17}$$

and its inverse matrix

$$\begin{aligned} \mathbf{C}_E^{-1} &= \frac{1}{\Delta_E}\begin{bmatrix} e_{31}+e_{33}\lambda_i^2 & 0 & -(\varepsilon_{11}-\varepsilon_{33}\lambda_i^2) \\ -(c_{11}+c_{13}\lambda_i^2) & 0 & (e_{31}+e_{15})\lambda_i^2 \\ 0 & 0 & 0 \end{bmatrix} \\ \Delta_E &= (e_{31}+e_{15})\lambda_i^2(e_{31}+e_{33}\lambda_i^2)-(c_{11}+c_{13}\lambda_i^2)(\varepsilon_{11}-\varepsilon_{33}\lambda_i^2) \end{aligned} \tag{3.18}$$

Of course

$$\mathbf{C}_E\mathbf{C}_E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.19}$$

For piezomagnetic material, it is

$$\mathbf{C}_H = \begin{bmatrix} 0 & -\infty & 0 \\ (q_{31}+q_{15})\lambda_i^2 & 0 & \mu_{11}-\mu_{33}\lambda_i^2 \\ c_{11}+c_{13}\lambda_i^2 & 0 & q_{31}+q_{33}\lambda_i^2 \end{bmatrix} \tag{3.20}$$

Then we obtain

$$\begin{aligned} \mathbf{C}_H^{-1} &= \frac{1}{\Delta_H}\begin{bmatrix} 0 & q_{31}+q_{33}\lambda_i^2 & -(\mu_{11}-\mu_{33}\lambda_i^2) \\ 0 & 0 & 0 \\ 0 & -(c_{11}+c_{13}\lambda_i^2) & (q_{31}+q_{15})\lambda_i^2 \end{bmatrix} \\ \Delta_H &= (q_{31}+q_{15})\lambda_i^2(q_{31}+q_{33}\lambda_i^2)-(c_{11}+c_{13}\lambda_i^2)(\mu_{11}-\mu_{33}\lambda_i^2) \end{aligned} \tag{3.21}$$

where $\mathbf{C}_H\mathbf{C}_H^{-1} = \mathbf{I}$; $\mathbf{I}$ is the square unit matrix.

Thus

$$\begin{aligned} \begin{Bmatrix} a_{1i} \\ a_{3i} \end{Bmatrix}^E &= \frac{1}{\Delta_E}\begin{bmatrix} e_{31}+e_{33}\lambda_i^2 & -(\varepsilon_{11}-\varepsilon_{33}\lambda_i^2) \\ -(c_{11}+c_{13}\lambda_i^2) & (e_{31}+e_{15})\lambda_i^2 \end{bmatrix}\begin{Bmatrix} e_{33}\lambda_i^2-e_{15}+p_3a_{00}\lambda_i\delta_{i0} \\ c_{33}\lambda_i^2+c_{13}-(\beta_3+\beta_1\lambda_i^{-2})a_{00}\lambda_i\delta_{i0} \end{Bmatrix} \\ \begin{Bmatrix} a_{1i} \\ a_{4i} \end{Bmatrix}^H &= \frac{1}{\Delta_H}\begin{bmatrix} q_{31}+q_{33}\lambda_i^2 & -(\mu_{11}-\mu_{33}\lambda_i^2) \\ -(c_{11}+c_{13}\lambda_i^2) & (q_{31}+q_{15})\lambda_i^2 \end{bmatrix}\begin{Bmatrix} q_{33}\lambda_i^2-q_{15}+\gamma_3a_{00}\lambda_i\delta_{i0} \\ c_{33}\lambda_i^2+c_{13}-(\beta_3+\beta_1\lambda_i^{-2})a_{00}\lambda_i\delta_{i0} \end{Bmatrix} \end{aligned} \tag{3.22}$$

respectively for the piezoelectric and piezomagnetic thermoelastic materials.

Note that for the piezoelectric material is $a_{4i} = 0$, but a_{3i} defines a_{7i}, that is also B_z by the electromagnetic constant d_{11}. Similarly is for the piezomagnetic material where $a_{3i} = 0$, but a_{4i} defines a_{6i}, that is also D_z as a consequence of the electromagnetic effect (see equations (3.12)).

Fore the two-phase multifield material, the constant A_0 will be

$$A_0 = \frac{Q}{\pi\Big(a_{00}\sqrt{\lambda_r\lambda_z} + a'_{00}\sqrt{\lambda'_r\lambda'_z}\Big)} \tag{3.23}$$

where the material parameter of the second material is denoted by prime.

The inverse matrices are obtained as arithmetically average values in this case. Since the plane $z = 0$ is a plane of symmetry ($\sigma_{zr} = 0$, $H_r = 0$, $E_r = 0$, $u_z = 0$, $\phi = 0$ and $\psi = 0$ on this plane), the solutions may be used for the two-phase multifield composite material.

3.4. Solution for a purely thermoelastic material

For a transversely isotropic thermoelastic medium, the temperature field is the same as that obtained in Section 3 and described by equations (3.7). The thermoelastic solution for the purely elastic problem can be easily derived from that of the piezoelectric material (on assumption that $\varepsilon_{11} - \varepsilon_{33}\lambda_i^2 \to \infty$ and $e_{31} = e_{33} = e_{15} = 0$) or the piezomagnetic material (by assuming $\mu_{11} - \mu_{33}\lambda_i^2 \to \infty$ and $q_{31} = q_{33} = q_{15} = 0$). Both formulae (3.22) give the same result

$$a_{1i} = \frac{c_{33}\lambda_i^2 + c_{13} - (\beta_3 + \beta_1\lambda_i^{-2})a_{00}\lambda_i\delta_{i0}}{c_{11} + c_{13}\lambda_i^2} \qquad i = 0, 1, 2 \tag{3.24}$$

and equations (2.12) and (2.8) yield

$$a_{5i} = c_{11}a_{1i} - c_{13} \qquad a_{00} = \frac{c_{44}c_{33}(\lambda_0^2 - \lambda_1^2)(\lambda_0^2 - \lambda_2^2)}{\beta_1(c_{33}\lambda_0^2 - c_{44}) - \beta_3\lambda_0^2(c_{13} + c_{44})} \tag{3.25}$$

The remaining material parameters a_{3i}, a_{4i}, a_{6i} and a_{7i} vanish.

The constants A_1 and A_2 are obtained as follows

$$\begin{aligned} A_1 &= -\frac{Q\lambda_1}{2\pi\lambda_r a_{00}} \frac{\Big(\frac{\beta_1 a_{00}}{\lambda_0} + a_{50}\Big)a_{12}\lambda_2^3 - a_{10}a_{52}\lambda_0^3}{a_{51}a_{12}\lambda_2^3 - a_{52}a_{11}\lambda_1^3} \\ A_2 &= \frac{Q\lambda_2}{2\pi\lambda_r a_{00}} \frac{\Big(\frac{\beta_1 a_{00}}{\lambda_0} + a_{50}\Big)a_{11}\lambda_1^3 - a_{10}a_{51}\lambda_0^3}{a_{51}a_{12}\lambda_2^3 - a_{52}a_{11}\lambda_1^3} \end{aligned} \tag{3.26}$$

for the half-space problem and

$$A_1 = -\frac{Q}{2\pi a_{00}\sqrt{\lambda_r\lambda_z}} \frac{a_{12}\lambda_2^2 - a_{10}\lambda_0^2}{a_{12}\lambda_2^2 - a_{11}\lambda_1^2} \qquad A_2 = \frac{Q}{2\pi a_{00}\sqrt{\lambda_r\lambda_z}} \frac{a_{11}\lambda_1^2 - a_{10}\lambda_0^2}{a_{12}\lambda_2^2 - a_{11}\lambda_1^2} \tag{3.27}$$

for an infinite body.

The parameters λ_1 and λ_2 are the roots of the following equation

$$c_{33}c_{44}\lambda^4 - [c_{11}c_{33} - c_{13}(c_{13} + 2c_{44})]\lambda^2 + c_{11}c_{44} = 0 \tag{3.28}$$

These parameters are the eigenvalues of the transversely isotropic material.

By defining

$$\begin{aligned} \alpha &= \sqrt{\frac{c_{11}c_{33} - c_{13}(2c_{44} + c_{13}) + 2c_{44}\sqrt{c_{11}c_{33}}}{c_{33}c_{44}}} \\ \beta &= \sqrt{\frac{c_{11}c_{33} - c_{13}(2c_{44} + c_{13}) - 2c_{44}\sqrt{c_{11}c_{33}}}{c_{33}c_{44}}} \end{aligned} \tag{3.29}$$

the eigenvalues λ_1 and λ_2 can be written as

$$\lambda_1 = \frac{1}{2}(\alpha + \beta) \qquad \lambda_2 = \frac{1}{2}(\alpha - \beta) \tag{3.30}$$

It is noted that λ_1 and λ_2 can be either two positive real numbers or complex conjugate with a positive real part. In other words, $\alpha = \lambda_1 + \lambda_2$ and it is always real. The results are valid even for the degenerate case of $\beta = \lambda_1 - \lambda_2 = 0$, including the isotropic material where $\lambda_1 = \lambda_2 = 1$. In this case, the limiting calculations with the use of de l'Hospital's rule give the solution.

4. Conclusions

- In comparison with the traditional methods applied to the solution of boundary value problems of thermoelasticity, in the proposed method, there is no need to solve boundary value problems of heat conduction for preliminary determination of the temperature field (the first stage of solving the problem) and then to solve the equations of thermoelasticity (the second stage of solving the problem).
- Green's functions for the half-space, infinite space made by multifield materials are obtained in an exact analytical form; the solutions are regular.
- For the temperature and heat flux applied along the circumference on an arbitrary plane, the thermal loading conditions may be written by means of the Dirac delta function. Then integration and/or superposition of Green's functions gives the multi-field result.

Appendix A. The material coefficients for mulifield materials

A1. The material parameters in characteristic equation (2.9) are as follows

$$a = c_{44}[\mu_{33}e_{33}^2 + \varepsilon_{33}q_{33}^2 + c_{33}\mu_{33}\varepsilon_{33} - d_{33}(c_{33}d_{33} + 2e_{33}q_{33})]$$

$$\begin{aligned}
b = {}& \mu_{33}\{(e_{31} + e_{15})[2c_{13}e_{33} - c_{33}(e_{31} + e_{15})] + 2c_{44}e_{33}e_{31} - c_{11}e_{33}^2 - c_{33}c_{44}\varepsilon_{11}\} \\
& + \varepsilon_{33}\{(q_{31} + q_{15})[2c_{13}q_{33} - c_{33}(q_{31} + q_{15})] + 2c_{44}q_{33}q_{31} - c_{11}q_{33}^2 - c_{33}c_{44}\mu_{11}\} \\
& - \mu_{33}\varepsilon_{33}\widetilde{c}^2 - (e_{31} + e_{15})^2 q_{33}^2 - (q_{31} + q_{15})^2 e_{33}^2 - c_{44}\mu_{11}e_{33}^2 - c_{44}\varepsilon_{11}q_{33}^2 \\
& + 2e_{33}q_{33}(q_{31} + q_{15})(e_{31} + e_{15}) + d_{33}^2\widetilde{c}^2 + 2c_{33}d_{33}(e_{31} + e_{15})(q_{31} + q_{15}) \\
& + 2c_{44}c_{33}d_{11}d_{33} + 2e_{33}q_{33}(c_{44}d_{11} + c_{11}d_{33}) - 2d_{33}(c_{13} + c_{44})[e_{33}(q_{31} + q_{15}) \\
& + q_{33}(e_{31} + e_{15})]
\end{aligned}$$

$$\begin{aligned}
c = {}& \mu_{33}\{2e_{15}[c_{11}e_{33} - c_{13}(e_{31} + e_{15})] + c_{44}e_{31}^2 + \varepsilon_{11}\widetilde{c}^2\} \\
& + \varepsilon_{33}\{2q_{15}[c_{11}q_{33} - c_{13}(q_{31} + q_{15})] + c_{44}q_{31}^2 + \mu_{11}\widetilde{c}^2\} \\
& + c_{33}c_{44}\mu_{11}\varepsilon_{11} + c_{11}c_{44}\mu_{33}\varepsilon_{33} + 2(c_{13} + c_{44})(q_{31} + q_{15})(d_{11}e_{33} + d_{33}e_{15} - q_{33}\varepsilon_{11}) \\
& + 2(c_{13} + c_{44})(e_{31} + e_{15})(d_{11}q_{33} + d_{33}q_{15} - e_{33}\mu_{11}) \\
& + (q_{31} + q_{15})^2(c_{33}\varepsilon_{11} + 2e_{33}e_{15}) + (e_{31} + e_{15})^2(c_{33}\mu_{11} + 2q_{33}q_{15}) \\
& - 2(q_{31} + q_{15})(e_{31} + e_{15})(e_{33}q_{15} + q_{33}e_{15} + c_{33}d_{11} + c_{44}d_{33}) \\
& - 2c_{11}d_{33}(e_{33}q_{15} + q_{33}e_{15}) - 2c_{44}d_{11}(q_{33}e_{15} + e_{33}q_{15}) \\
& - 2c_{11}d_{11}q_{33}e_{33} - 2c_{44}d_{33}q_{15}e_{15} + 2c_{44}q_{15}q_{33}\varepsilon_{11} \\
& + 2c_{44}e_{15}e_{33}\mu_{11} + c_{11}q_{33}^2\varepsilon_{11} + c_{11}e_{33}^2\mu_{11} - 2\widetilde{c}^2 d_{33}d_{11} - c_{11}c_{44}d_{33}^2 - c_{44}c_{33}d_{11}^2
\end{aligned}$$

$$
\begin{aligned}
d &= -c_{11}\mu_{33}(c_{44}\varepsilon_{11} + e_{15}^2) - c_{11}\varepsilon_{33}(c_{44}\mu_{11} + q_{15}^2) - c_{44}(e_{31}^2\mu_{11} + q_{31}^2\varepsilon_{11}) - e_{31}^2 q_{15}^2 - q_{31}^2 e_{15}^2 \\
&\quad - \mu_{11}\varepsilon_{11}\widetilde{c}^2 + d_{11}\widetilde{c}^2 + 2c_{11}c_{44}d_{11}d_{33} + 2c_{13}q_{15}q_{31}\varepsilon_{11} + 2c_{13}e_{15}e_{31}\mu_{11} - 2c_{11}q_{15}q_{33}\varepsilon_{11} \\
&\quad - 2c_{11}e_{15}e_{33}\mu_{11} + 2c_{13}q_{15}^2\varepsilon_{11} + 2c_{13}e_{15}^2\mu_{11} + 2e_{31}e_{15}q_{31}q_{15} + 2c_{11}e_{15}q_{15}d_{33} \\
&\quad + d_{11}[-2c_{13}e_{15}(q_{15} + q_{31}) - 2c_{13}q_{15}(e_{15} + e_{31})] + d_{11}[2c_{11}(e_{15}q_{33} + q_{15}e_{33}) + 2c_{44}e_{31}q_{31}] \\
e &= c_{11}[\mu_{11}e_{15}^2 + \varepsilon_{11}q_{15}^2 + c_{44}\varepsilon_{11}\mu_{11} - d_{11}(c_{44}d_{11} + 2e_{15}q_{15})] \\
\widetilde{c}^2 &= c_{11}c_{33} - c_{13}(c_{13} + 2c_{44})
\end{aligned}
$$

A2. The parameters a_1, b_1, c_1, d_1, and a_2, b_2, c_2, d_2 in Eq. (2.8) are

$$
\begin{aligned}
a_1 &= \beta_1[c_{33}(\varepsilon_{33}\mu_{33} - d_{33}^2) + \mu_{33}e_{33}^2 + \varepsilon_{33}q_{33}^2 - 2e_{33}d_{33}q_{33}] + \beta_3[-(c_{13} + c_{44})(\varepsilon_{33}\mu_{33} - d_{33}^2) \\
&\quad - (e_{31} + e_{15})(\mu_{33}e_{33} - d_{33}q_{33}) - (q_{31} + q_{15})(q_{33}\varepsilon_{33} - d_{33}e_{33})] \\
&\quad + \gamma_3[-(c_{13} + c_{44})(d_{33}e_{33} - q_{33}\varepsilon_{33}) + (e_{31} + e_{15})(d_{33}c_{33} + q_{33}e_{33}) \\
&\quad - (q_{31} + q_{15})(c_{33}\varepsilon_{33} + e_{33}^2)] + p_3[-(c_{13} + c_{44})(d_{33}q_{33} - e_{33}\mu_{33}) \\
&\quad + (q_{31} + q_{15})(d_{33}c_{33} + q_{33}e_{33}) - (e_{31} + e_{15})(c_{33}\mu_{33} + q_{33}^2)]
\end{aligned}
$$

$$
\begin{aligned}
b_1 &= \beta_1[c_{33}(2d_{11}d_{33} - \varepsilon_{33}\mu_{11} - \mu_{33}\varepsilon_{11}) + c_{44}(d_{33}^2 - \varepsilon_{33}\mu_{33}) - \varepsilon_{11}q_{33}^2 - \mu_{11}e_{33}^2 \\
&\quad + 2d_{33}(e_{33}q_{15} + q_{33}e_{15}) + 2d_{11}e_{33}q_{33} - 2q_{15}q_{33}\varepsilon_{33} - 2e_{15}e_{33}\mu_{33}] \\
&\quad + \beta_3[-(c_{13} + c_{44})(2d_{11}d_{33} - \varepsilon_{33}\mu_{11} - \mu_{33}\varepsilon_{11}) \\
&\quad + (q_{13} + q_{15})(q_{15}\varepsilon_{33} + q_{33}\varepsilon_{15} - d_{11}e_{33} - d_{33}e_{15}) \\
&\quad + (e_{31} + e_{15})(e_{15}\mu_{33} + e_{33}\mu_{11} - d_{11}q_{33} - d_{11}q_{15})] \\
&\quad + \gamma_3[(c_{13} + c_{44})(d_{11}e_{33} + d_{33}e_{15} - q_{15}\varepsilon_{33} - q_{33}\varepsilon_{11}) \\
&\quad - (e_{31} + e_{15})(c_{44}d_{33} + c_{33}d_{11} + q_{15}e_{33} + e_{15}q_{33}) + (q_{31} + q_{15})(c_{44}\varepsilon_{33} + c_{33}\varepsilon_{11} + 2e_{15}e_{33})] \\
&\quad + p_3[(c_{13} + c_{44})(d_{11}q_{33} + d_{33}q_{15} - e_{15}\mu_{33} - e_{33}\mu_{11}) \\
&\quad - (q_{31} + q_{15})(c_{44}d_{33} + c_{33}d_{11} + q_{15}e_{33} + e_{15}q_{33}) + (e_{31} + e_{15})(c_{44}\mu_{33} + c_{33}\mu_{11} + 2q_{15}q_{33})] \\
c_1 &= \beta_1[c_{44}(\varepsilon_{11}\mu_{33} + \varepsilon_{33}\mu_{11} - 2d_{11}d_{33}) + c_{33}(\varepsilon_{11}\mu_{11} - d_{11}^2) + \varepsilon_{33}q_{15}^2 + \mu_{33}e_{15}^2 \\
&\quad - 2d_{11}(e_{15}q_{33} + q_{15}e_{33}) + 2q_{15}q_{33}\varepsilon_{11} + 2\mu_{11}e_{15}e_{33}] \\
&\quad + \beta_3[(c_{13} + c_{44})(d_{11}^2 - \varepsilon_{11}\mu_{11}) - (e_{31} + e_{15})(\mu_{11}e_{15} - d_{11}q_{15}) \\
&\quad - (q_{31} + q_{15})(\varepsilon_{11}q_{15} - d_{11}e_{15})] + \gamma_3[(c_{13} + c_{44})(q_{15}\varepsilon_{11} - e_{15}d_{11}) \\
&\quad + (e_{31} + e_{15})(d_{11}c_{44} + q_{15}e_{15}) - (q_{31} + q_{15})(c_{44}\varepsilon_{11} + e_{15}^2)] \\
&\quad + p_3[(c_{13} + c_{44})(e_{15}\mu_{11} - q_{15}d_{11}) \\
&\quad + (q_{31} + q_{15})(d_{11}c_{44} + q_{15}e_{15}) - (e_{31} + e_{15})(c_{44}\mu_{11} + q_{15}^2)] \\
d_1 &= -\beta_1[c_{44}(\varepsilon_{11}\mu_{11} - d_{11}^2) + \mu_{11}e_{15}^2 + \varepsilon_{11}q_{15}^2 - 2e_{15}q_{15}d_{11}] \\
a_2 &= c_{44}[\beta_3(\varepsilon_{33}\mu_{33} - d_{33}^2) + \gamma_3(d_{33}e_{33} + q_{33}\varepsilon_{33}) + p_3(d_{33}q_{33} - e_{33}\mu_{33})] \\
b_2 &= \beta_1[(c_{13} + c_{44})(\varepsilon_{33}\mu_{33} - d_{33}^2) - (e_{31} + e_{15})(d_{33}q_{33} - \mu_{33}e_{33}) \\
&\quad - (q_{31} + q_{15})(d_{33}e_{33} - \varepsilon_{33}q_{33})] - \beta_3[c_{11}(\varepsilon_{33}\mu_{33} + d_{33}^2) \\
&\quad + c_{44}(\mu_{11}\varepsilon_{33} + \mu_{33}\varepsilon_{11}) - 2(q_{31} + q_{15})(e_{31} + e_{15})d_{33} + (q_{31} + q_{15})^2\varepsilon_{33} + (e_{31} + e_{15})^2\mu_{33}] \\
&\quad - \gamma_3[c_{11}(d_{33}e_{33} - q_{33}\varepsilon_{33}) + c_{44}(d_{11}e_{33} + d_{33}e_{15} - q_{15}\varepsilon_{33} - q_{33}\varepsilon_{11}) \\
&\quad - (c_{13} + c_{44})d_{33}(e_{31} + e_{15}) - q_{33}(e_{31} + e_{15})^2 + \varepsilon_{33}(q_{31} + q_{15})(c_{13} + c_{44}) \\
&\quad + e_{33}(q_{31} + q_{15})(e_{31} + e_{15})] - p_3[c_{11}(d_{33}q_{33} - e_{33}\mu_{33}) \\
&\quad + c_{44}(d_{11}q_{33} + d_{33}q_{15} - e_{15}\mu_{33} - e_{33}\mu_{11}) - (c_{13} + c_{44})d_{33}(q_{31} + q_{15}) \\
&\quad - e_{33}(q_{31} + q_{15})^2 + \mu_{33}(e_{31} + e_{15})(c_{13} + c_{44}) + q_{33}(q_{31} + q_{15})(e_{31} + e_{15})]
\end{aligned}
$$

$$\begin{aligned}
c_2 = {} & \beta_1[-(c_{13}+c_{44})(\varepsilon_{11}\mu_{33}+\varepsilon_{33}\mu_{11})+(e_{31}+e_{15})(d_{11}q_{33}+d_{33}q_{15}) \\
& +(q_{31}+q_{15})(d_{11}e_{33}+d_{33}e_{15})-(q_{31}+q_{15})(q_{15}\varepsilon_{33}+q_{33}\varepsilon_{11}) \\
& -(e_{31}+e_{15})(e_{15}\mu_{33}+e_{33}\mu_{11})+2(c_{13}+c_{44})d_{11}d_{33}]+\beta_3[c_{44}(\varepsilon_{11}\mu_{11}+d_{11}^2) \\
& +c_{11}(\mu_{11}\varepsilon_{33}+\mu_{33}\varepsilon_{11})+\mu_{11}(e_{31}+e_{15})^2+\varepsilon_{11}(q_{31}+q_{15})^2-2(e_{31}+e_{15})(q_{31}+q_{15})d_{11}] \\
& -\gamma_3[c_{44}(q_{15}\varepsilon_{11}-e_{15}d_{11})+(e_{31}+e_{15})((c_{13}+c_{44})d_{11}+(e_{31}+e_{15})q_{15}) \\
& -(q_{31}+q_{15})((c_{13}+c_{44})\varepsilon_{11}+(e_{31}+e_{15})e_{15})-c_{11}(d_{11}e_{33}+e_{15}d_{33}-\varepsilon_{11}q_{33}-\varepsilon_{33}q_{15})] \\
& -p_3[c_{44}(e_{15}\mu_{11}-q_{15}d_{11})+(q_{31}+q_{15})((c_{13}+c_{44})d_{11}+(q_{31}+q_{15})e_{15}) \\
& -(e_{31}+e_{15})((c_{13}+c_{44})\mu_{11}+(q_{31}+q_{15})q_{15})-c_{11}(d_{11}q_{33}+q_{15}d_{33}-\mu_{11}e_{33}-\mu_{33}e_{15})]
\end{aligned}$$

$$\begin{aligned}
d_2 = {} & \beta_1[(c_{13}+c_{44})\varepsilon_{11}\mu_{11}-(e_{31}+e_{15})d_{11}q_{15}-(q_{31}+q_{15})d_{11}e_{15}+(q_{31}+q_{15})q_{15}\varepsilon_{11} \\
& +(e_{31}+e_{15})\mu_{11}e_{15}-(c_{13}+c_{44})d_{11}^2]-\beta_3[c_{11}(\varepsilon_{11}\mu_{11}-d_{11}^2)]-\gamma_3[c_{11}(e_{15}d_{11}-q_{15}\varepsilon_{11})] \\
& -p_3[c_{11}(q_{15}d_{11}-e_{15}\mu_{11})]
\end{aligned}$$

A3. The parameters a_{3i} and a_{4i} in Eq. (2.8) are defined by the parameter a_{1i} as follows

$$\begin{aligned}
a_{3i} = {} & \Big\{\{\beta_1[(e_{31}+e_{15})(q_{33}\lambda_i^2-q_{15})-(c_{13}+c_{44})(d_{11}-d_{33}\lambda_i^2)] \\
& +\beta_3[(e_{31}+e_{15})(q_{31}+q_{15})\lambda_i+(c_{44}\lambda_i^2-c_{11})(d_{11}-d_{33}\lambda_i^2)] \\
& +p_3[\lambda_i(c_{13}+c_{44})(q_{31}+q_{15})+(c_{44}\lambda_i^2-c_{11})(q_{33}\lambda_i^2-q_{15})]\}a_{1i}\lambda_i \\
& +\beta_1[(c_{33}\lambda_i^2-c_{44})(d_{11}-d_{33}\lambda_i^2)-(q_{33}\lambda_i^2-q_{15})(e_{33}\lambda_i^2-e_{15})] \\
& +\beta_3\lambda_i[(c_{13}+c_{44})(d_{11}-d_{33}\lambda_i^2)-(q_{31}+q_{15})(e_{33}\lambda_i^2-e_{15})] \\
& +p_3\lambda_i[(c_{13}+c_{44})(q_{33}\lambda_i^2-q_{15})-(q_{31}+q_{15})(c_{33}\lambda_i^2-c_{44})]\Big\} \\
& \cdot\{p_3\lambda_i[(e_{31}+e_{15})(q_{33}\lambda_i^2-q_{15})-(q_{31}+q_{15})(e_{33}\lambda_i^2-e_{15})] \\
& +\beta_1[(e_{33}\lambda_i^2-e_{15})(d_{11}-d_{33}\lambda_i^2)-(q_{33}\lambda_i^2-q_{15})(\varepsilon_{11}-\varepsilon_{33}\lambda_i^2)] \\
& +\beta_3\lambda_i[(e_{31}+e_{15})(d_{11}-d_{33}\lambda_i^2)-(q_{31}+q_{15})(\varepsilon_{11}-\varepsilon_{33}\lambda_i^2)]\}^{-1}
\end{aligned}$$

$$\begin{aligned}
a_{4i} = {} & -\Big\{\{\beta_1[(e_{31}+e_{15})(e_{33}\lambda_i^2-e_{15})-(c_{13}+c_{44})(\varepsilon_{11}-\varepsilon_{33}\lambda_i^2)] \\
& +\beta_3[(e_{31}+e_{15})^2\lambda_i+(c_{44}\lambda_i^2-c_{11})(\varepsilon_{11}-\varepsilon_{33}\lambda_i^2)] \\
& +p_3[\lambda_i(c_{13}+c_{44})(e_{31}+e_{15})+(c_{44}\lambda_i^2-c_{11})(e_{33}\lambda_i^2-e_{15})]\}a_{1i}\lambda_i \\
& +\beta_1[(c_{33}\lambda_i^2-c_{44})(\varepsilon_{11}-\varepsilon_{33}\lambda_i^2)-(e_{33}\lambda_i^2-e_{15})^2] \\
& +\beta_3\lambda_i[(c_{13}+c_{44})(\varepsilon_{11}-\varepsilon_{33}\lambda_i^2)-(e_{31}+e_{15})(e_{33}\lambda_i^2-e_{15})] \\
& +p_3\lambda_i[(c_{13}+c_{44})(e_{33}\lambda_i^2-e_{15})-(e_{31}+e_{15})(c_{33}\lambda_i^2-c_{44})]\Big\} \\
& \cdot\{p_3\lambda_i[(e_{31}+e_{15})(q_{33}\lambda_i^2-q_{15})-(q_{31}+q_{15})(e_{33}\lambda_i^2-e_{15})] \\
& +\beta_1[(e_{33}\lambda_i^2-e_{15})(d_{11}-d_{33}\lambda_i^2)-(q_{33}\lambda_i^2-q_{15})(\varepsilon_{11}-\varepsilon_{33}\lambda_i^2)] \\
& +\beta_3\lambda_i[(e_{31}+e_{15})(d_{11}-d_{33}\lambda_i^2)-(q_{31}+q_{15})(\varepsilon_{11}-\varepsilon_{33}\lambda_i^2)]\}^{-1}
\end{aligned}$$

A4. The roots of characteristic equation (2.9) are presented by the formulae (eigenvalues of multifield materials)

$$\begin{aligned}
\lambda_1^2 &= -\frac{b}{4a}-\frac{1}{2}\sqrt{R_5+R_6}-\frac{1}{2}\sqrt{2R_5-R_6+\frac{1}{4}\frac{R_7}{\sqrt{R_5+R_6}}} \\
\lambda_2^2 &= -\frac{b}{4a}-\frac{1}{2}\sqrt{R_5+R_6}+\frac{1}{2}\sqrt{2R_5-R_6+\frac{1}{4}\frac{R_7}{\sqrt{R_5+R_6}}}
\end{aligned}$$

$$\lambda_3^2 = -\frac{b}{4a} + \frac{1}{2}\sqrt{R_5 + R_6} - \frac{1}{2}\sqrt{2R_5 - R_6 - \frac{1}{4}\frac{R_7}{\sqrt{R_5 + R_6}}}$$
$$\lambda_4^2 = -\frac{b}{4a} + \frac{1}{2}\sqrt{R_5 + R_6} + \frac{1}{2}\sqrt{2R_5 - R_6 - \frac{1}{4}\frac{R_7}{\sqrt{R_5 + R_6}}}$$

where

$$R_1 = 2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace \qquad R_2 = c^2 - 3bd + 12ae$$
$$R_3 = \sqrt{R_1^2 - 4R_2^3} \qquad R_4 = \sqrt[3]{\frac{1}{2}(R_1 + R_3)}$$
$$R_5 = \frac{b^2}{4a^2} - \frac{2c}{3a} \qquad R_6 = \frac{R_2}{3aR_4} + \frac{R_4}{3a} \qquad R_7 = \frac{b^3}{a^3} - \frac{4bc}{a^2} + \frac{8d}{a}$$

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Manuscript received May 12, 2015; accepted for print November 8, 2015