

ANALYTICAL SOLUTION OF EXCITED TORSIONAL VIBRATIONS OF PRISMATIC THIN-WALLED BEAMS

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In the paper, the analytical solution of excited torsional vibrations of prismatic thin-walled beams for different types of boundary conditions and different types of external excitation of torsional moment are formulated. The presented solution can be applied, among others, to preliminary analysis of the optimal position of the actuators and the value of the applied voltage to the elements for minimization of the vibrations of the beams.

Keywords: thin-walled beams, torsional vibrations, excited vibrations

1. Introduction

Knowledge of the analytical solution of externally excited vibrations for a considered model of realistic structures is very useful for preliminary analysis in the design process or choosing parameters and algorithms for vibration control of the systems. The model of an excited prismatic beam response for torsional or coupled bending-torsional vibrations can be applied for analysis of vibration of some machine elements, e.g. turbine-blades (Gryboś, 1996; Łączkowski, 1974; Pust and Pesek, 2014; Rao, 1991). Sometimes, the beam cross-section takes form of an open thin-walled structure. In this paper, thin-walled beams of closed cross-sections as general models of blades are analyzed (Librescu and Na, 1998; Song and Librescu, 1993; Song *et al.*, 2002).

Recently, application of piezoelectric elements for vibration reduction of thin structures are commonly discussed in the literature (Elliott and Nelson, 1997; Hansen and Snyder, 1997; Moheimani and Fleming, 2006; Preumont, 2006). See also the papers by Ferdek and Kozień (2013), Kozień and Kołtowski (2011). These types of vibration cancellation were considered by the authors in preliminary analysis of the possibility of reduction of torsional vibrations of prismatic beams with solid cross-sections (Augustyn and Kozień, 2014).

There are well known analytical solutions for problems of the dynamics of thin walled beams with open cross-section, starting with Gere (1954), Gere with Lin (1958), Aggrawal with Cranch (1967), and Carr (1969) and later e.g. by Bishop *et al.* (1989), Dokumaci (1987), Kaliski and Solarz (1992), Tao (1964), Timoshenko *et al.* (1974) and Yaman (1997). Different cases of vibrations in such a type of structures are discussed in many of articles. A review of these different approaches, theories and models for static and dynamic cases is given by Sapountzakis (2013). The effect of variable cross-section on natural frequencies is discussed in (Eisenberger, 1997). Non-linear models are discussed by Crespo Da Silva (1988a,b), Rozmarynowski with Szymczak (1984) and Di Egidio *et al.* (2003a,b). The steady-state forced vibrations are discussed by Crespo Da Silva (1988b) and Di Egidio *et al.* (2003b). The problem of optimal design of a thin-walled beam for a given natural frequency is analyzed by Szymczak (1984). Torsional vibrations of composite thin-walled beams are analyzed by Song and Librescu (1993). The models with the effect of rotary inertia are analysed by Arpaci *et al.* (2003) and with adaptive capabilities by Song *et al.* (2002). A detailed solution for the general case of excited vibrations, especially in the

transient case, are not easily found in the literature. This formulation was given by one of the authors of this paper for bending vibrations of a beam for different types of external excitations (Kozień, 2013).

The main aim of this paper is to formulate the analytical solution for describing the excited torsional-type vibrations of a thin-walled beam for different combinations of boundary conditions (simply supported, free, fixed) and different types of external torsional moment type excitations (harmonic concentrated, harmonic distributed, pulse concentrated, pulse distributed). The presented solution, among others, can be applied to preliminary analysis of the optimal position of the actuators and to calculations of voltage applied to the elements for minimization of vibrations of beams when the application of piezoelectric elements are considered. In this model, the influence of external piezoelectric elements can be modeled by the external concentrated moment of a suitable value, as it was done by Elliott and Nelson (1997), Hansen and Snyder (1997) for pairs of elements in simulations of bending vibrations of beams, as was proposed by Augustyn and Kozień (2014) for double pairs of elements for torsional vibrations of beams.

2. Formulation of the problem

2.1. Equation of motion

Let us consider a prismatic beam with a thin-walled cross section of an open type (Murray, 1986; Piechnik, 2007; Vlasov, 1959) in which it is also assumed that the material is isotropic. The geometry of the cross-section is shown in Fig. 1, where G is the gravity center (centroid), S – shear center. The origin of the co-ordinate system $Sxyz$ lies in the shear center, and the axes Sy and Sz are parallel to the principal axes of the cross-section ($G\eta, G\zeta$).

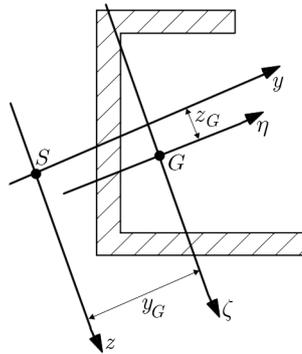


Fig. 1. Geometry of the cross-section

The equations of coupled vibrations of the beam take the following form

$$\begin{aligned}
 EJ_{\zeta} \frac{\partial^4 v(x, t)}{\partial x^4} + \rho A \left[\frac{\partial^2 v(x, t)}{\partial t^2} - z_G \frac{\partial^2 \varphi(x, t)}{\partial t^2} \right] &= q_y(x, t) \\
 EJ_{\eta} \frac{\partial^4 w(x, t)}{\partial x^4} + \rho A \left[\frac{\partial^2 w(x, t)}{\partial t^2} + y_G \frac{\partial^2 \varphi(x, t)}{\partial t^2} \right] &= q_z(x, t) \\
 EJ_{\omega} \frac{\partial^4 \varphi(x, t)}{\partial x^4} - GJ_s \frac{\partial^2 \varphi(x, t)}{\partial x^2} + \rho J_0 \frac{\partial^2 \varphi(x, t)}{\partial t^2} \\
 + \rho A \left[-z_G \frac{\partial^2 v(x, t)}{\partial t^2} + y_G \frac{\partial^2 w(x, t)}{\partial t^2} \right] &= m_s(x, t)
 \end{aligned} \tag{2.1}$$

where $v(x, t)$ is the displacement of the beam in the y direction, $w(x, t)$ is the displacement of the beam in the z direction, $\varphi(x, t)$ is the rotation of the axis of beam around the x direction, J_{η} is the principal moment of inertia of the cross-section with respect to the η axis, J_{ζ} is the principal

moment of inertia of the cross-section with respect to the ζ axis, J_0 is the polar moment of inertia with respect to the shear center, J_s is the equivalent moment of inertia of the cross-section due to torsion, E is Young's modulus, G is the shear modulus, ρ is the material density, y_G , z_G are the positions of the gravity center G (co-ordinates), $q_y(x, t)$ is the distributed force acting in the y direction (external excitation), $q_z(x, t)$ is the distributed force acting in the z direction (external excitation), $m_s(x, t)$ is the distributed twisting moment (external excitation).

When the considered cross-section is such that the position of the shear center and gravity center is the same (e.g. for the cross-section with two axes of symmetry), the following relationship is valid $y_G = z_G = 0$. It means, that the equation of motion for twisting existing in system (2.1) is separated from the equations of bending vibrations, and takes simplified and independent form from bending vibrations form

$$\frac{EJ_\omega}{GJ_s} \frac{\partial^4 \varphi(x, t)}{\partial x^4} - \frac{\partial^2 \varphi(x, t)}{\partial x^2} + \frac{\rho J_0}{GJ_s} \frac{\partial^2 \varphi(x, t)}{\partial t^2} = \frac{1}{GJ_s} m_s(x, t) \quad (2.2)$$

Hence, for the considered cross-section, it is possible to analyze torsional vibrations separately with the bending ones.

If the cross-section is a monolithic type, equation of motion (2.3) takes the following form

$$-GJ_s \frac{\partial^2 \varphi(x, t)}{\partial x^2} + \rho J_0 \frac{\partial^2 \varphi(x, t)}{\partial t^2} = m_s(x, t) \quad (2.3)$$

In general, vibrations of the analyzed structure can be described in form of a three-dimensional model of a solid structure. But for some structures, for which one characteristic dimension (the so-called length) is high enough in comparison to the characteristic dimensions of the cross-section, then so-called one-dimensional models of structures are built for simplicity of the analysis. The results of their application must give good enough results of analysis in comparison with three-dimensional models. The same case takes place during detailed consideration of the cross-section shape. If one characteristic dimension of the cross-section (the so-called thickness) is small enough in comparison with the second one (the so-called width), the whole structure is called a thin-walled one. A detailed rule states that for a thin-walled cross-section, the thickness is more than eight times smaller than the highest way measured along the middle-line of the cross-section between its two end-points. Moreover, the length of this way should be more than eight times smaller than the length of the beam (Piechnik, 2007). For such a beam the assumptions of the Bernoulli-Euler beam theory or the de Saint-Venant rule of loadings, for example, are invalid. Their motion must be analyzed as a three-dimensional body or by the application of an especially formulated theory for thin-walled beams. Due to the form of the middle line of the cross-section, three types of thin-walled beams are defined: with an open cross-section, with a closed cross-section and with a mixed one (Piechnik, 2007). Depending on the type of cross-section, the suitable theory of thin-walled beams should be applied. For a thin-walled beam with an open cross-section, as considered in this article, the Vlasov theory can be applied (Piechnik, 2007; Vlasov, 1959).

2.2. Boundary conditions

The equation of motion of the thin-walled beam is the fourth order due to the spatial variable x . Therefore, a set of the four boundary conditions must be formulated for each beam element. Their formulation is connected with the Vlasov theory of thin-walled beams and must take into account the warping effect.

The following types of boundary conditions can be formulated in a natural way (Gere, 1954; Piechnik, 2007):

- Simply supported:

— no rotation of the cross-section around the axis of the beam

$$\varphi = 0 \quad (2.4)$$

— zero normal stress (free of warping of the cross-section)

$$\frac{\partial^2 \varphi}{\partial x^2} = 0 \quad (2.5)$$

- Fixed:

— no rotation of the cross-section around the axis of the beam

$$\varphi = 0 \quad (2.6)$$

— plain cross-section (blocked warping of the cross-section)

$$\frac{\partial \varphi}{\partial x} = 0 \quad (2.7)$$

- Free:

— zeroes for the total torsional moment (free for rotation)

$$GJ_s \frac{\partial \varphi}{\partial x} - EJ_\omega \frac{\partial \varphi^3}{\partial x^3} = 0 \quad (2.8)$$

— zero normal stress (free for warping of the cross-section)

$$\frac{\partial^2 \varphi}{\partial x^2} = 0 \quad (2.9)$$

3. Eigen-problem, natural vibrations

Now, the eigen-problem analysis of the thin-walled prismatic beam with length l is considered. The equation of motion has the form

$$\frac{EJ_\omega}{GJ_s} \frac{\partial^4 \varphi(x, t)}{\partial x^4} - \frac{\partial^2 \varphi(x, t)}{\partial x^2} + \frac{\rho J_0}{GJ_s} \frac{\partial^2 \varphi(x, t)}{\partial t^2} = 0 \quad (3.1)$$

The solution to the problem is proposed in form (3.2)₁ – the Fourier method of solution. After suitable manipulations, it leads to a solution to the problem of separated variables (3.2)₂. It can be written as two independent ordinary differential equations (3.2)_{3,4}. The first of the fourth order for the independent spatial variable x , and the second of the second order for the independent variable t (time). The form of unknown functions in the spatial domain can be written in form (3.2)₅). For the above given homogeneous boundary conditions, these series of functions satisfy orthogonality conditions (3.4) – see Appendix A for detailed analysis. Finally, the solution can be written in form (3.7), where the series of constants a_n and b_n are determined based on initial conditions (3.6)

$$\begin{aligned} \varphi(x, t) &= \sum_{n=1}^{+\infty} X_n(x) T_n(t) \\ \frac{EJ_\omega}{GJ_s} \frac{d^4 X_n(x)}{dx^4} - \frac{d^2 X_n(x)}{dx^2} &= -\frac{\rho J_0}{GJ_s} \frac{d^2 T_n(t)}{dt^2} = \lambda_n^4 \\ \frac{EJ_\omega}{GJ_s} \frac{d^4 X_n(x)}{dx^4} - \frac{d^2 X_n(x)}{dx^2} - \lambda_n^4 X_n(x) &= 0 \quad \frac{d^2 T_n(t)}{dt^2} + \underbrace{\lambda_n^4 \frac{GJ_s}{\rho J_0}}_{\omega_n^2} T_n(t) = 0 \end{aligned} \quad (3.2)$$

$$X_n(x) = A_n \sin(\alpha_n x) + B_n \cos(\alpha_n x) + C_n \sinh(\beta_n x) + D_n \cosh(\beta_n x)$$

where

$$\alpha_n = \sqrt{\frac{-GJ_s + \sqrt{(GJ_s)^2 + 4EJ_\omega GJ_s \lambda_n^4}}{2EJ_\omega}} \quad \beta_n = \sqrt{\frac{GJ_s + \sqrt{(GJ_s)^2 + 4EJ_\omega GJ_s \lambda_n^4}}{2EJ_\omega}} \quad \omega_n = \lambda_n^2 \sqrt{\frac{GJ_s}{\rho J_0}} \quad (3.3)$$

and

$$\int_0^l X_n(x) X_m(x) dx = \begin{cases} \gamma_n^2 & n = m \\ 0 & n \neq m \end{cases} \quad (3.4)$$

Finding the solution to the system of differential equation (3.2)₂ or (3.2)_{3,4} for a given boundary conditions is the well-known eigen-mode problem which gives a set of eigen-values (powered natural frequencies of a system ω_n^2 – solution of equation (3.2)₄ and a set of eigen-functions (eigen-modes, waveforms of eigen-functions $X_n(x)$ – solution of equation (3.2)₃). The equations, which make possible the finding of natural frequencies ω_n (3.3)_{1,2,3} or exact formula for ω_n for a different combination of typical types of boundary conditions (fixed, simply supported, free) are given in Table 1. Moreover, the analytical form of eigen-functions for the same combination of boundary conditions (simply supported, free, fixed) are also given in Table 1. The other name of the problem is the modal problem and the detailed solution of this problem was done by Gere (1954).

If the solution to the equation of motion fulfills initial conditions (3.6), where $\varphi_0(x)$ is the initial angle of torsion of the beam for $t = 0$, and $\Omega_0(x)$ is the initial angular velocity of the beam for $t = 0$, the problem of natural vibrations is completely defined. These initial conditions should be applied for the detailed solution of the second equation of system (3.2)_{3,4}

$$\varphi(x, 0) = \varphi_0(x) \quad \frac{\partial}{\partial t} \varphi(x, 0) = \Omega_0(x) \quad (3.5)$$

Solution (3.6) represents the so-called natural vibrations, which are the response of the system to initial conditions written in the time and spatial domains $\varphi(x, t)$

$$\varphi(x, t) = \sum_{n=1}^{+\infty} X_n(x) [a_n \sin(\omega_n t) + b_n \cos(\omega_n t)] \quad (3.6)$$

$$a_n = \frac{1}{\omega_n \gamma_n^2} \int_0^l \Omega_0(x) X_n(x) dx \quad b_n = \frac{1}{\gamma_n^2} \int_0^l \varphi_0(x) X_n(x) dx$$

4. Excited vibrations

When analyzing the excited vibrations of a realistic beam (with internal damping), the general solution of the homogeneous differential equation is a function relatively fast tending to zero with respect to time due to the internal and external damping.

The complete solution to the problem of excited vibrations, understood as the particular solution to the non-homogeneous equation (2.2), for the excitation function of harmonic type, has the form of a sum of two components: connected with a set of natural frequencies ω_n and connected with the external loading frequency ν . Note, that the set of constants A_n , B_n , C_n and D_n existing in particular solution (2.2)₅, must be found taking into account the complete solution.

Table 1. Formulas for eigenfrequencies and eigenmodes (Gere, 1954)

Boundary conditions	Equation/formula for natural frequency ω_n or parameters α_n and β_n	Eigenfunction $X_n(x)$
Simply supported-simply supported	$\omega_n = \frac{n\pi}{l^2} \sqrt{\frac{n^2 \pi^2 E J_\omega + l^2 G J_s}{\rho J_0}}$	$\sin\left(\frac{n\pi}{l}x\right)$
Fixed-fixed	$\alpha_n \beta_n = \omega_n \sqrt{\frac{\rho J_0}{E J_\omega}}$	$\frac{\beta_n [\cosh(\beta_n l) - \cos(\alpha_n l)]}{\alpha_n \sinh(\beta_n l) - \beta_n \sin(\alpha_n l)} \sin(\alpha_n x) - \cos(\alpha_n x)$ $- \frac{\alpha_n [\cosh(\beta_n l) - \cos(\alpha_n l)]}{\alpha_n \sinh(\beta_n l) - \beta_n \sin(\alpha_n l)} \sinh(\beta_n x) + \cosh(\beta_n x)$
Simply supported-fixed	$\frac{\tanh(\beta_n l)}{\beta_n} = \frac{\tan(\alpha_n l)}{\alpha_n}$	$\sin(\alpha_n x) - \frac{\sin(\alpha_n l)}{\sinh(\beta_n l)} \sinh(\beta_n x)$
Fixed-free	$\frac{\alpha_n^4 + \beta_n^4}{\alpha_n^2 \beta_n^4} \cos(\alpha_n l) \cosh(\beta_n l)$ $- \frac{\alpha_n^2 - \beta_n^2}{\alpha_n \beta_n} \sin(\alpha_n l) \sinh(\beta_n l) + 2 = 0$	$\frac{\beta_n \alpha_n^2 \sinh(\beta_n l) - \alpha_n \beta_n \sin(\alpha_n l)}{\alpha_n \alpha_n^2 \cosh(\beta_n l) + \beta_n^2 \cos(\alpha_n l)} \sin(\alpha_n x) - \cos(\alpha_n x)$ $- \frac{\alpha_n^2 \sinh(\beta_n l) - \alpha_n \beta_n \sin(\alpha_n l)}{\alpha_n^2 \cosh(\beta_n l) + \beta_n^2 \cos(\alpha_n l)} \sinh(\beta_n x) + \cosh(\beta_n x)$
Simply supported-free	$\beta_n^2 \tanh(\beta_n l) = \alpha_n^2 \tan(\alpha_n l)$	$\sin(\alpha_n x) + \frac{\beta_n \cos(\alpha_n l)}{\alpha_n \cosh(\beta_n l)} \sinh(\beta_n x)$
Free-free	$(\beta_n^4 - \alpha_n^4) \sin(\alpha_n l) \sinh(\beta_n l)$ $+ 2\alpha_n^2 \beta_n^2 [\cos(\alpha_n l) \cosh(\beta_n l) - 1] = 0$	$- \frac{\alpha_n \beta_n^3 [\cosh(\beta_n l) - \cos(\alpha_n l)]}{\beta_n \beta_n^3 \sinh(\beta_n l) - \alpha_n^3 \sin(\alpha_n l)} \sin(\alpha_n x) + \frac{\beta_n}{\alpha_n} \cos(\alpha_n x)$ $- \frac{\beta_n^3 [\cosh(\beta_n l) - \cos(\alpha_n l)]}{\beta_n^3 \sinh(\beta_n l) - \alpha_n^3 \sin(\alpha_n l)} \sinh(\beta_n x) + \cosh(\beta_n x)$

Due to the same result of action of the internal and external damping (as for free vibrations), the components connected with the set of natural frequencies ω_n are functions relatively fast tending to zero with respect to time. Therefore, the solution of excited vibrations is usually connected with the only external loading frequency ν (Łączkowski, 1974). Such a case of vibrations is called the steady-state case. The complete solution of the problem which includes free vibrations and excited vibrations in general formulation is known as transient vibrations.

As discussed in the Introduction, let us consider the general form of the solution of excited vibrations. In such a case, the solution can be proposed in form (4.1) in terms of the functions $X_n(x)$ and a series of unknown functions of time $H_n(t)$

$$\varphi(x, t) = \sum_{n=1}^{+\infty} X_n(x) H_n(t) \quad (4.1)$$

Putting proposed solution (4.1) into equation (2.2) and taking into account the first of equation of system (3.2)_{3,4} one can obtain

$$\sum_{n=1}^{+\infty} \left[\underbrace{\left(\frac{EJ_\omega}{GJ_s} \frac{d^4 X_n(x)}{dx^4} - \frac{d^2 X_n(x)}{dx^2} \right)}_{\lambda_n^4 X_n(x)} H_n(t) + \frac{\rho J_0}{GJ_s} \frac{d^2 H_n(t)}{dt^2} X_n(x) \right] = \frac{1}{GJ_s} m_s(x, t) \quad (4.2)$$

and then after suitable manipulations

$$\sum_{n=1}^{+\infty} \left[\frac{d^2 H_n(t)}{dt^2} + \frac{GJ_s}{\rho J_0} \underbrace{\lambda_n^4}_{\omega_n^2} H_n(t) \right] X_n(x) = \frac{GJ_s}{\rho J_0} \frac{1}{GJ_s} m_s(x, t) = \frac{1}{\rho J_0} m_s(x, t) \quad (4.3)$$

The external load function existing in equation (2.2) can be represented in a series form (4.4), in terms of functions $X_n(x)$ and a series of the known functions of time $Q_n(t)$

$$\begin{aligned} \frac{1}{\rho J_0} m_s(x, t) &= \frac{1}{\rho J_0} \sum_{n=1}^{+\infty} X_n(x) Q_n(t) \\ Q_n(t) &= \frac{1}{\gamma_n^2} \int_0^l m_s(x, t) X_n(x) dx \quad Q_n^*(t) = \frac{1}{\rho J_0} Q_n(t) \end{aligned} \quad (4.4)$$

Substituting formulas (4.1) and (4.4) into equation (2.2), the differential equation for the determination of the unknown functions $H_n(t)$ takes the form

$$\frac{d^2 H_n(t)}{dt^2} + \omega_n^2 H_n(t) = Q_n^*(t) \quad (4.5)$$

The solution to this equation has the form

$$H_n(t) = \frac{1}{\omega_n} \int_0^t Q_n^*(\tau) \sin[\omega_n(t - \tau)] d\tau \quad (4.6)$$

known as the Duhamel integral.

The given formulas make possible the formulation of the analytical solution of excited vibrations of a beam with defined boundary conditions.

Let us formulate the general form of the Duhamel integral for the following types of external excitations and different boundary conditions given by the eigen-functions $X_n(x)$:

— Harmonically distributed moment with a constant amplitude m_0

$$\begin{aligned} m_s(x, t) &= m_0 \sin(\nu t) \quad [m_0] = N \\ H_n(t) &= m_0 \frac{1}{\rho J_0} \frac{1}{\gamma_n^2} \frac{1}{\omega_n^2 - \nu^2} \left[\sin(\nu t) - \frac{\nu}{\omega_n} \sin(\omega_n t) \right] \int_0^l X_n(x) dx \\ \varphi(x, t) &= M_0 \frac{1}{\rho J_0} \sum_{n=1}^{\infty} \frac{1}{\gamma_n^2} \frac{1}{\omega_n^2 - \nu^2} \left[\sin(\nu t) - \frac{\nu}{\omega_n} \sin(\omega_n t) \right] \int_0^l X_n(x) X_n(x) dx \end{aligned} \quad (4.7)$$

— Pulsed distributed moment with a constant amplitude Q_m acting at the time $t = t_0$

$$\begin{aligned}
 m_s(x, t) &= Q_m \delta(t - t_0) & [Q_m] &= Ns \\
 H_n(t) &= \begin{cases} 0 & t < t_0 \\ Q_M \frac{1}{\rho J_0} \frac{1}{\gamma_n^2} \frac{1}{\omega_n} \sin[\omega_n(t - t_0)] \int_0^l X_n(x) dx & t \geq t_0 \end{cases} \\
 \varphi(x, t) &= \begin{cases} 0 & t < t_0 \\ Q_M \frac{1}{\rho J_0} \sum_{n=1}^{\infty} \frac{1}{\gamma_n^2} \frac{1}{\omega_n} \sin[\omega_n(t - t_0)] \int_0^l X_n(x) X_n(x) dx & t \geq t_0 \end{cases}
 \end{aligned} \tag{4.8}$$

— Harmonic concentrated moment with a constant amplitude M_0 applied to the point $x = x_0$

$$\begin{aligned}
 m_s(x, t) &= M_0 \delta(x - c) \sin(\nu t) & [M_0] &= Nm \\
 H_n(t) &= M_0 \frac{1}{\rho J_0} \frac{1}{\gamma_n^2} \frac{1}{\omega_n^2 - \nu^2} \left[\sin(\nu t) - \frac{\nu}{\omega_n} \sin(\omega_n t) \right] X_n(x_0) \\
 \varphi(x, t) &= M_0 \frac{1}{\rho J_0} \sum_{n=1}^{\infty} \frac{1}{\gamma_n^2} \frac{1}{\omega_n^2 - \nu^2} \left[\sin(\nu t) - \frac{\nu}{\omega_n} \sin(\omega_n t) \right] X_n(x_0) X_n(x)
 \end{aligned} \tag{4.9}$$

— Pulsed concentrated moment with a constant amplitude Q_m acting at the time $t = t_0$ at the point $x = x_0$

$$\begin{aligned}
 m_s(x, t) &= Q_M \delta(x - x_0) \delta(t - t_0) & [Q_M] &= Nms \\
 H_n(t) &= \begin{cases} 0 & t < t_0 \\ Q_M \frac{1}{\rho J_0} \frac{1}{\gamma_n^2} \frac{1}{\omega_n} \sin[\omega_n(t - t_0)] X_n(x_0) & t \geq t_0 \end{cases} \\
 \varphi(x, t) &= \begin{cases} 0 & t < t_0 \\ Q_M \frac{1}{\rho J_0} \sum_{n=1}^{\infty} \frac{1}{\gamma_n^2} \frac{1}{\omega_n} \sin[\omega_n(t - t_0)] X_n(x_0) X_n(x) & t \geq t_0 \end{cases}
 \end{aligned} \tag{4.10}$$

5. Example – excited torsional vibrations of a simply supported thin-walled beam

5.1. Formulation of the problem

As an example, let us consider excited vibrations of a thin-walled beam simply supported on both ends. The beam is made of steel ($E = 2.1 \cdot 10^{11}$ Pa, $G = 8.1 \cdot 10^{10}$ Pa, $\rho = 7800$ kg/m³) and has length $l = 6$ m. The shape of cross-section with its detailed dimensions is shown in Fig. 2.

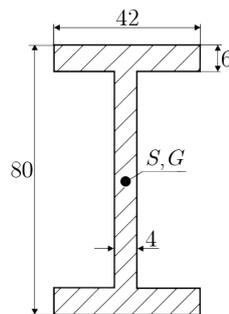


Fig. 2. Cross-section of the analyzed beam (dimensions given in mm)

The boundary conditions take form

$$\begin{aligned} \varphi(0, t) = 0 & \quad \frac{\partial^2}{\partial x^2} \varphi(0, t) = 0 \\ \varphi(l, t) = 0 & \quad \frac{\partial^2}{\partial x^2} \varphi(l, t) = 0 \end{aligned} \quad (5.1)$$

5.2. Natural vibrations – a comparison of the results for different models

The following models of beam are applied for comparison of the lowest natural frequencies for torsional vibrations:

- three-dimensional solid model solved by application of the finite element method,
- thin-walled Vlasov analytical model,
- model with a monolithic cross-section.

The finite element solution has been obtained by application of the *Ansys* finite element package. The model is built of 76800 three dimensional 8-nodes solid elements of the type *solid45*. The generated mesh is shown in Fig. 3. Due to the degrees of freedom of the applied elements, the boundary conditions on the two ends of the beam are modeled as free sliding along the axis of the beam and as blocked displacements for all nodes in the plane of its ends. Therefore, deplanation of the ends is possible.

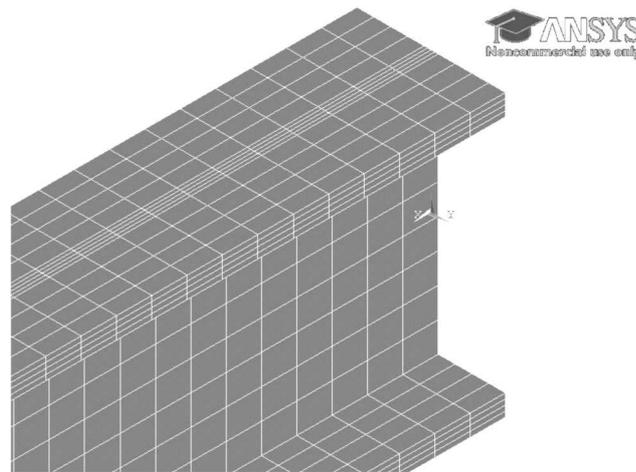


Fig. 3. Part of the FEM mesh of the beam

The second group of values of the natural frequencies is obtained based on the thin-walled model described in the text, especially based on the formula given in the first row in Table 1. The values of J_s , J_w and J_0 are calculated based on the formulas given by Gere (1954).

The third group of results is obtained for the beam modeled with the assumption of a monolithic type cross-section (shaft) with free-free ($\varphi(0, t) = \varphi(l, t) = 0$) or fixed-fixed ($\partial\varphi(0, t)/\partial x = \partial\varphi(l, t)/\partial x = 0$) boundary conditions. The values of natural frequencies are the same for these cases of boundary conditions (Woroszył, 1984).

In Table 2, a comparison of the lowest natural frequencies for the analyzed beam for the three considered models and the relative percentage error is given to one model obtained from three dimensional solid models, as the reference model. It should be noted that for the considered beam, the values of natural frequencies are correctly estimated theoretically with a monolithic cross-section only for the first few torsional modes. For higher modes, only the theory for thin-walled beam gives good enough results.

Table 2. Comparison of the lowest natural frequencies for the analyzed beam [Hz]

Mode	3-D Solid	Thin-walled		Monolytic	
No.	value [Hz]	value [Hz]	error [%]	value [Hz]	error [%]
1	25.15	25.54	1.6	25.40	1.0
2	50.97	51.89	1.8	50.81	-0.3
3	78.11	79.84	2.2	76.21	-2.4
4	107.16	110.05	2.7	101.62	-5.2
5	138.64	143.16	3.3	127.02	-8.4
6	172.98	179.66	3.9	152.42	-11.9
7	210.56	219.94	4.5	177.83	-15.5
8	251.65	264.36	5.1	203.23	-19.2
9	296.50	313.14	5.6	228.64	-22.9
10	345.26	366.49	6.1	254.04	-26.4

A comparison of the natural frequencies for the two beam models – the thin-walled one and the monolithic one are shown in Fig. 4 in form of the relative percentage error in comparison to the values obtained for the thin-walled model as the reference model. On the horizontal axis, there are identification numbers of the first hundred torsional modes ($n = 1, 2, \dots, 100$). It can be shown that for the considered beam, the error of the estimated values of natural frequencies found by application of the theory with monolithic cross-section grow rapidly with an increase in the number of torsional modes.

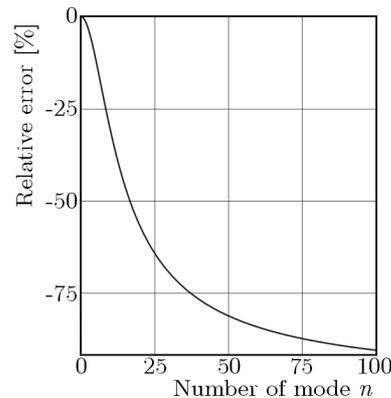


Fig. 4. Relative error of the natural frequencies for torsional modes for the beam models

5.3. Excited vibrations

The vibrations are excited by a concentrated moment with an amplitude $M_0 = 5$ Nm acting at the point distanced $0.4l$ (2.4 m) from the end of beam.

The initial conditions are zeroes (i.e. $\varphi_0(x) = 0$ and $\Omega_0 = 0$ in formulas (3.6))

$$m_s(x, t) = M_0 \delta(x - 0.4l) \sin(\nu t) \quad (5.2)$$

The angular frequency of the excitation is $\nu = 200$ rad/s. Material damping is neglected. The analytical solution has the form

$$\varphi(x, t) = \frac{2}{l} \frac{1}{\rho J_0} M_0 \sum_{n=1}^{\infty} \frac{1}{\omega_n^2 - \nu^2} \left[\sin(\nu t) - \frac{\nu}{\omega_n} \sin(\omega_n t) \right] \sin(\lambda_n 0.4l) \sin(\lambda_n x) \quad (5.3)$$

In the analysis, the first five modes are considered.

The displacement of the middle of the beam just after the beginning of vibrations and for the steady-state are shown in Fig. 5. The control point is the place of action of the concentrated excitation moment. It is distanced by $0.4l$ (2.4 m) from the end of the beam.

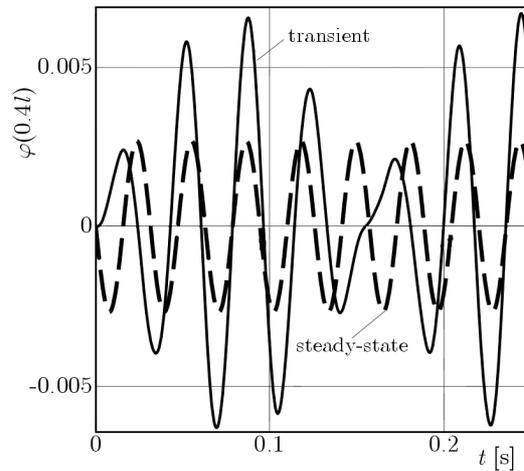


Fig. 5. Displacement at the control point – complete analytical solution, transient and steady-state

The plots show the necessity of taking into account the full solution instead of the steady-state case if the results are important just after the start of action of external loadings. Due to internal material damping (not taken into account in the considered model), the transient component existing in solution is a function that relatively fastly tends to zero. Therefore, enough time after the start of action of the harmonic excitation, the solution takes the steady-state form.

6. Conclusions

- The solution of the problem for a monolithic cross-section is not an asymptotic case of the solution of the thin-walled case if the value of J_ω tends to zero.
- The given formulas can be applied to the analysis of reduction of torsional vibrations by coupled sets of piezoelectric elements. The action of piezoelectric elements can be approximately modeled by a concentrated moment put in the suitable cross-section, see Elliott and Nelson (1997), Hansen and Snyder (1997) for beams.
- The given formulas make it possible to find an analytical solution for the transient case of the response (just after the excitation is applied) and for the steady-state case.
- The solution can be generalized by assuming the Voigt-Kelvin model of the viscous type of damping for description of the internal material damping.

Appendix A. Orthogonality of the eigen-functions

Formula (3.2)₂ can be written in the following form

$$\begin{aligned} GJ_s \lambda_n^4 X_n(x) &= EJ_\omega \frac{d^4 X_n(x)}{dx^4} - GJ_s \frac{d^2 X_n(x)}{dx^2} \\ GJ_s \lambda_m^4 X_m(x) &= EJ_\omega \frac{d^4 X_m(x)}{dx^4} - GJ_s \frac{d^2 X_m(x)}{dx^2} \end{aligned} \quad (\text{A.1})$$

Let us multiply the first equation of system (A.1) by the eigen-function $X_m(x)$ and the second equation suitably by $X_n(x)$, and then subtract the equations by sides and integrate.

Finally, formula (A.1) is

$$\begin{aligned}
 GJ_s(\lambda_n^4 - \lambda_m^4) \int_0^l X_n(x)X_m(x) dx &= EJ_\omega \int_0^l \frac{d^4 X_n(x)}{dx^4} X_m(x) dx - EJ_\omega \int_0^l \frac{d^4 X_m(x)}{dx^4} X_n(x) dx \\
 &- GJ_s \int_0^l \frac{d^2 X_n(x)}{dx^2} X_m(x) dx + GJ_s \int_0^l \frac{d^2 X_m(x)}{dx^2} X_n(x) dx
 \end{aligned} \tag{A.2}$$

By integrating the right side of equation (4.2) by parts, one obtains the following formulas for each component

$$\begin{aligned}
 EJ_\omega \int_0^l \frac{d^4 X_n(x)}{dx^4} X_m(x) dx &= EJ_\omega \left[\frac{d^3 X_n(x)}{dx^3} X_m(x) \Big|_0^l - \frac{d^2 X_n(x)}{dx^2} \frac{dX_m(x)}{dx} \Big|_0^l + \int_0^l \frac{d^2 X_n(x)}{dx^2} \frac{d^2 X_m(x)}{dx^2} dx \right] \\
 EJ_\omega \int_0^l \frac{d^4 X_m(x)}{dx^4} X_n(x) dx &= EJ_\omega \left[\frac{d^3 X_m(x)}{dx^3} X_n(x) \Big|_0^l - \frac{d^2 X_m(x)}{dx^2} \frac{dX_n(x)}{dx} \Big|_0^l + \int_0^l \frac{d^2 X_m(x)}{dx^2} \frac{d^2 X_n(x)}{dx^2} dx \right] \\
 GJ_s \int_0^l \frac{d^2 X_n(x)}{dx^2} X_m(x) dx &= GJ_s \frac{dX_n(x)}{dx} X_m(x) \Big|_0^l - \int_0^l \frac{dX_n(x)}{dx} \frac{dX_m(x)}{dx} dx \\
 GJ_s \int_0^l \frac{d^2 X_m(x)}{dx^2} X_n(x) dx &= GJ_s \frac{dX_m(x)}{dx} X_n(x) \Big|_0^l - \int_0^l \frac{dX_m(x)}{dx} \frac{dX_n(x)}{dx} dx
 \end{aligned} \tag{A.3}$$

Finally, the relationship (A.2) can be written in the form

$$\begin{aligned}
 GJ_s(\lambda_n^4 - \lambda_m^4) \int_0^l X_n(x)X_m(x) dx &= \left(EJ_\omega \frac{d^3 X_n(l)}{dx^3} - GJ_s \frac{d^2 X_n(l)}{dx^2} \right) X_m(l) \\
 &- \left(EJ_\omega \frac{d^3 X_n(0)}{dx^3} - GJ_s \frac{d^2 X_n(0)}{dx^2} \right) X_m(0) - \left(EJ_\omega \frac{d^3 X_m(l)}{dx^3} - GJ_s \frac{d^2 X_m(l)}{dx^2} \right) X_n(l) \\
 &+ \left(EJ_\omega \frac{d^3 X_m(0)}{dx^3} - GJ_s \frac{d^2 X_m(0)}{dx^2} \right) X_n(0) - EJ_\omega \frac{d^2 X_n(l)}{dx^2} \frac{dX_m(l)}{dx} \\
 &+ EJ_\omega \frac{d^2 X_n(0)}{dx^2} \frac{dX_m(0)}{dx} - EJ_\omega \frac{d^2 X_m(l)}{dx^2} \frac{dX_n(l)}{dx} + EJ_\omega \frac{d^2 X_m(0)}{dx^2} \frac{dX_n(0)}{dx}
 \end{aligned} \tag{A.4}$$

For any physically possible combination of the discussed boundary conditions, all components existing on the right-hand side are equal to zero. It means that for $n \neq m$

$$\int_0^l X_n(x)X_m(x) dx = 0 \tag{A.5}$$

For $n = m$

$$\int_0^l X_n(x)X_n(x) dx = \gamma_n^2 \tag{A.6}$$

The value of γ_n^2 depends on the form of the eigen-functions (see Table 1).

These relationships are known as the orthogonality conditions of the eigen-function $X_n(x)$.

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