

ELASTIC BUCKLING OF A TRIANGULAR FRAME SUBJECT TO IN-PLANE TENSION

KRZYSZTOF MAGNUCKI

*Poznan University of Technology, Institute of Applied Mechanics, Poznań, Poland
e-mail: krzysztof.magnucki@put.poznan.pl*

SZYMON MILECKI

Institute of Rail Vehicles, TABOR, Poznań, Poland

The paper is devoted to elastic buckling of a symmetrical triangular frame under tensile in-plane load. Three mathematical models of the triangular frame are formulated. The first model deals with the pre-buckling state, the second one with the in-plane buckling state, and the third one with the lateral buckling state of the frame. The FEM-numerical model of the frame is formulated and the critical loads are calculated. The comparison of the results obtained in the analytical and numerical-FEM analysis is presented in tables and graphs in figures.

Keywords: triangular frame, elastic buckling, critical load, mathematical model

1. Introduction

The theoretical basis of buckling problems of structures is elaborated in many papers and monographs. Horne and Merchant (1965) described the problem of stability of frames. Thompson and Hunt (1973) presented a general theory of elastic stability of structures. Budiansky (1974) presented the theory of buckling and post-buckling behaviour of elastic structures. Chen and Liu (1987) described the theory of stability and implementation for structures. Bažant and Cederlin (1991) presented an extensive review of stability problems of structures: columns, frames, thin-walled beams, plates and shells. Simitses and Hodges (2006) presented the fundamentals of structural stability for columns, beams, rings and arches. Van der Heijden (2008) described the elastic stability of solids and structures formulated and studied by professor W.T. Koiter. Kaveh and Salimbahram (2007) presented a methodology for efficient calculation of buckling loads for symmetric rectangular frame structures. Şakar *et al.* (2012) presented FEM study of dynamic stability of a multi-span rectangular frame subjected to periodic loading.

The problem of triangular frame stability has been presented in literature only in several publications. For example, Magnucki and Milecki (2012) presented flat elastic buckling of the brake triangle in freight wagons, while Sobaś (2010) described the strength problems, especially the fatigue strength of the brake triangles.

The subject of theoretical study presented in this paper is a symmetrical triangular frame under tensile in-plane load (Fig. 1). The vertex C of the frame is fixed, whereas the vertexes A and B are simply supported in the plane of the triangular frame.

The arms of length L_1 are connected with a cross-beam of length L_2 . The cross-section of the arms is rectangular, while the cross-section of the cross-beam is a circular ring (Fig. 2).

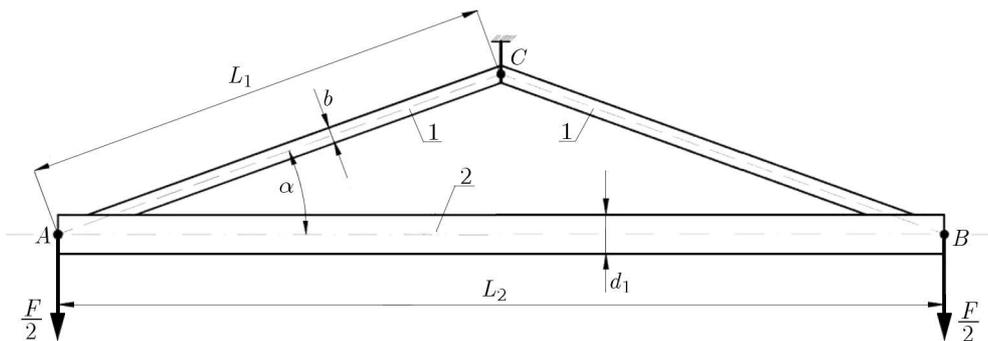


Fig. 1. Scheme of the symmetrical triangular frame with the load

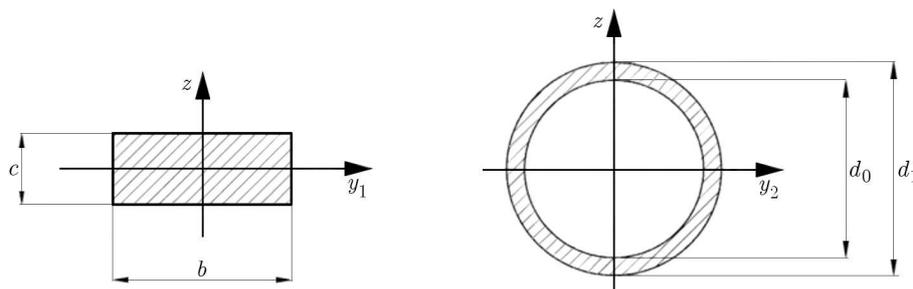


Fig. 2. Cross-section of the arms – 1 and cross-beam – 2

2. Mathematical models

2.1. Pre-buckling state

A half of the symmetrical triangular frame with the load F , internal normal force N_2 and bending moment M_2 for the pre-buckling state is shown in Fig. 3.

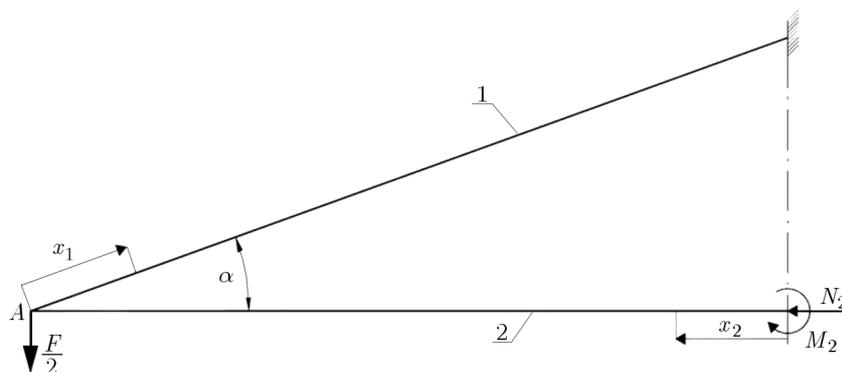


Fig. 3. Scheme of a half of the frame for the pre-buckling state

The normal forces $N(x_i)$ and bending moments $M_b(x_i)$ in the elements of the frame:

- arm – 1

$$N(x_1) = N_1 = N_2 \cos \alpha + \frac{1}{2} F \sin \alpha \quad M_b(x_1) = \left(N_2 \sin \alpha - \frac{1}{2} F \cos \alpha \right) x_1 + M_2 \quad (2.1)$$

- cross-beam – 2

$$N(x_2) = N_2 \quad M_b(x_2) = M_2 \quad (2.2)$$

The elastic strain energy of the frame is

$$U_\varepsilon = \frac{1}{2EA_1} \int_0^{L_1} [N(x_1)]^2 dx_1 + \frac{1}{2EJ_z^{(1)}} \int_0^{L_1} [M_b(x_1)]^2 dx_1 + \frac{1}{2EA_2} \int_0^{L_2/2} [N(x_2)]^2 dx_2 + \frac{1}{2EJ_z^{(2)}} \int_0^{L_2/2} [M_b(x_2)]^2 dx_2 \quad (2.3)$$

where $A_1 = bc$, $A_2 = \pi(d_1^2 - d_0^2)/4$ are the areas of the cross-sections of the arm and the cross-beam, $J_z^{(1)} = bc^3/12$, $J_z^{(2)} = \pi(d_1^4 - d_0^4)/64$ – inertia moments of the cross-sections of the arm and the cross-beam, E – Young's modulus.

Taking into account the theorem of Menabrea, two conditions are formulated

$$\frac{\partial U_\varepsilon}{\partial N_2} = \frac{1}{A_1} \int_0^{L_1} N(x_1) \cos \alpha dx_1 + \frac{1}{J_z^{(1)}} \int_0^{L_1} M_b(x_1) x_1 \sin \alpha dx_1 + \frac{1}{A_2} \int_0^{L_2/2} N_2 dx_2 = 0 \quad (2.4)$$

$$\frac{\partial U_\varepsilon}{\partial M_2} = \frac{1}{J_z^{(1)}} \int_0^{L_1} M_b(x_1) dx_1 + \frac{1}{J_z^{(2)}} \int_0^{L_2/2} M_2 dx_2 = 0$$

Thus, after integration, two equations are obtained

$$\alpha_{11} \frac{M_2}{L_2} + \alpha_{12} N_2 = \frac{1}{2} \beta_1 F \quad \alpha_{21} \frac{M_2}{L_2} + \alpha_{22} N_2 = \frac{1}{8} F \quad (2.5)$$

from which

$$N_2 = \frac{1}{2} \frac{\alpha_{11} - 4\beta_1 \alpha_{21} \alpha_{11}}{\alpha_{22} - \alpha_{12} \alpha_{21}} F \quad M_2 = \frac{1}{8} \frac{4\beta_1 \alpha_{22} - \alpha_{12}}{\alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21}} F L_2 \quad (2.6)$$

where

$$\alpha_{11} = \frac{A_2 L_2^2 \sin \alpha}{4J_z^{(1)} \cos^2 \alpha} \quad \alpha_{12} = 1 + \frac{A_2}{A_1} \cos \alpha + \frac{A_2 L_2^2 \sin^2 \alpha}{12J_z^{(1)} \cos^3 \alpha}$$

$$\beta_1 = \left(\frac{A_1 L_2^2}{12J_z^{(1)} \cos^2 \alpha} - 1 \right) \frac{A_2}{A_1} \sin \alpha \quad \alpha_{21} = 1 + \frac{J_z^{(1)}}{J_z^{(2)}} \cos \alpha \quad \alpha_{22} = \frac{1}{4} \tan \alpha$$

Thus, the normal tension force of arm (2.1) is as follows

$$N_1 = \frac{1}{2} \left(\frac{\alpha_{11} - 4\beta_1 \alpha_{21}}{\alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21}} \cos \alpha + \sin \alpha \right) F \quad (2.7)$$

The normal force N_2 (2.6)₁ is a compressive force acting on the cross-beam which causes buckling of the frame.

2.2. Flat buckling state – critical load

The critical state for the flat buckling of the symmetrical triangular frame is demonstrated by symmetrical flexure of the cross-beam and the angles of rotation of the vertexes A and B in the xy plane. A scheme of the flat buckling mode for one half of the frame is shown in Fig. 4. Short discussion of this buckling problem was presented by Bažant and Cedolin (1991).

The detailed scheme of the load and displacements for the arm and the cross-beam of the frame is presented in Fig. 5.

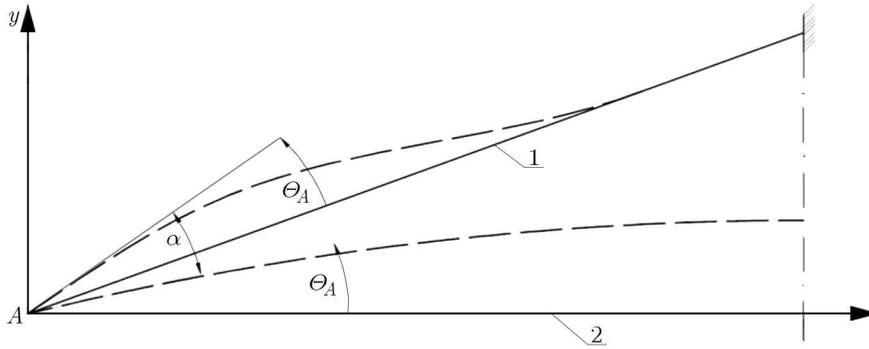


Fig. 4. Scheme of the flat buckling mode

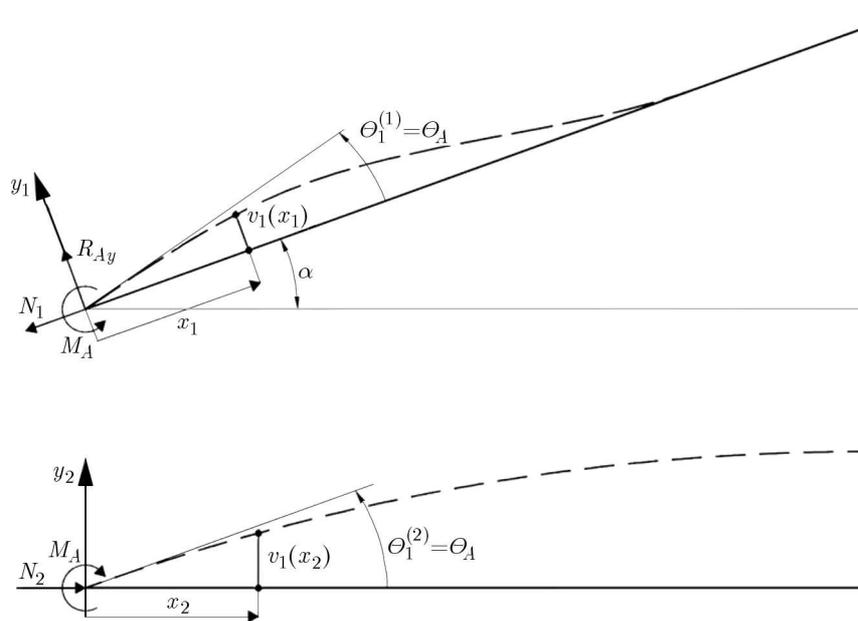


Fig. 5. Scheme of the load and displacements for the flat buckling state

The bending moment in the arm

$$M_b(x_1) = M_A - N_1 v_1(x_1) - R_{Ay} x_1 \tag{2.8}$$

The differential equation of the bending line of the arm

$$EJ_z^{(1)} \frac{d^2 v_1}{dx_1^2} - N_1 v_1(x_1) = -M_A + R_{Ay} x_1 \tag{2.9}$$

or

$$\frac{d^2 v_1}{dx_1^2} - k_{1y}^2 v_1(x_1) = -\frac{M_A}{EJ_z^{(1)}} + \frac{R_{Ay} x_1}{EJ_z^{(1)}} \tag{2.10}$$

where $k_{1y} = \sqrt{N_1 / (EJ_z^{(1)})}$ is a coefficient.

The solution of this equation is in the form

$$v_1(x_1) = C_1^{(1)} \sinh(k_{1y} x_1) + C_2^{(1)} \cosh(k_{1y} x_1) + \frac{M_A}{N_1} - \frac{R_{Ay}}{N_1} x_1 \tag{2.11}$$

where $C_1^{(1)}, C_2^{(1)}$ are the integration constants.

The boundary conditions are as follows

$$v_1(0) = 0 \quad v_1(L_1) = 0 \quad \left. \frac{dv_1}{dx_1} \right|_{L_1} = 0 \quad (2.12)$$

from which

$$C_1^{(1)} = \frac{M_A}{N_1} \frac{1 + k_{1y}L_1 \sinh(k_{1y}L_1) - \cosh(k_{1y}L_1)}{k_{1y}L_1 \cosh(k_{1y}L_1) - \sinh(k_{1y}L_1)} \quad C_2^{(1)} = -\frac{M_A}{N_1} \quad (2.13)$$

and the reaction

$$R_{Ay} = k_{1y}M_A \left[\frac{1 + k_{1y}L_1 \sinh(k_{1y}L_1) - \cosh(k_{1y}L_1)}{k_{1y}L_1 \cosh(k_{1y}L_1) - \sinh(k_{1y}L_1)} \cosh(k_{1y}L_1) - \sinh(k_{1y}L_1) \right] \quad (2.14)$$

Thus, the angle of the A vertex rotation in the xy plane is

$$\theta_A^{(1)} = \left. \frac{dv_1}{dx_1} \right|_0 = \frac{M_A L_1}{E J_z^{(1)}} f_{1y}(k_{1y}L_1) \quad (2.15)$$

where

$$f_{1y}(k_{1y}L_1) = \frac{1}{k_{1y}L_1} \left\{ \sinh(k_{1y}L_1) - \frac{1 + k_{1y}L_1 \sinh(k_{1y}L_1) - \cosh(k_{1y}L_1)}{k_{1y}L_1 \cosh(k_{1y}L_1) - \sinh(k_{1y}L_1)} [\cosh(k_{1y}L_1) - 1] \right\} \quad (2.16)$$

Similarly, the bending moment in the cross-beam is

$$M_b(x_2) = -M_A + N_2 v_2(x_2) \quad (2.17)$$

The differential equation of the bending line

$$\frac{d^2 v_2}{dx_2^2} + k_2^2 v_2(x_2) = \frac{M_A}{E J_z^{(2)}} \quad (2.18)$$

where $k_2 = \sqrt{N_2 / (E J_z^{(2)})}$ is a coefficient.

The solution of this equation is

$$v_2(x_2) = C_1^{(2)} \sinh(k_2 x_2) + C_2^{(2)} \cosh(k_2 x_2) + \frac{M_A}{N_2} \quad (2.19)$$

The boundary conditions

$$v_2(0) = 0 \quad v_2(L_2) = 0 \quad (2.20)$$

from which the integration constants are

$$C_1^{(2)} = -\frac{M_A}{N_2} \frac{1 - \cos(k_2 L_2)}{\sin(k_2 L_2)} \quad C_2^{(2)} = -\frac{M_A}{N_2} \quad (2.21)$$

Thus, the angle of the A vertex rotation in the xy plane is

$$\theta_A^{(2)} = \left. \frac{dv_2}{dx_2} \right|_0 = -\frac{M_A L_2}{E J_z^{(2)}} \frac{1 - \cos(k_2 L_2)}{k_2 L_2 \sin(k_2 L_2)} \quad (2.22)$$

The consistency condition for the angles of in-plane rotation of the vertex A for the arm and cross-beam

$$\theta_A^{(1)} = \theta_A^{(2)} \tag{2.23}$$

from which the nonlinear algebraic equation is in the following form

$$k_{J_{zz}} f_{1y}(k_{1y} L_1) + 2 \frac{1 - \cos(k_2 L_2)}{k_2 L_2 \sin(k_2 L_2)} = 0 \tag{2.24}$$

where $k_{J_{zz}} = J_z^{(2)} / (J_z^{(1)} \cos \alpha)$ is a dimensionless parameter.

Taking into account the normal forces N_2 (2.6)₁ and N_1 (2.7), the critical load from this equation is determined.

2.3. Lateral buckling state – critical load

The critical state for the lateral buckling of the triangular frame is demonstrated by symmetrical flexure of the cross-beam in the xz plane and the corresponding angles of rotation of the vertexes A and B . A scheme of the lateral buckling mode for the half of the frame is shown in (Fig. 6).

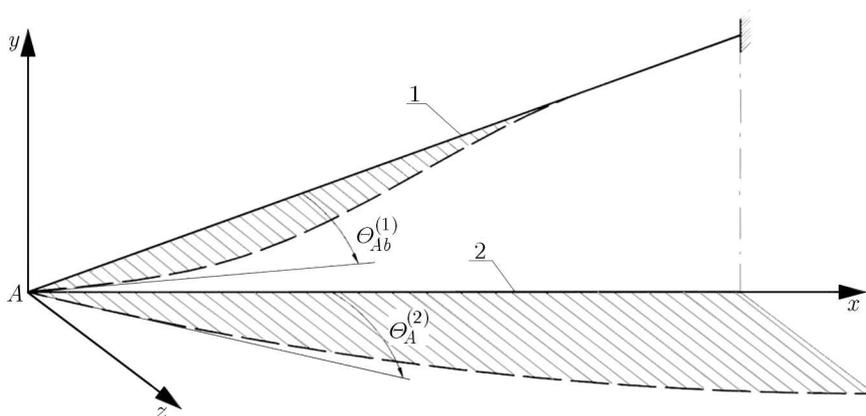


Fig. 6. Scheme of the lateral buckling mode

The flexure of the cross-beam in the xz plane causes bending and torsion of the arms. A scheme of the moments in the vertex A is shown in Fig. 7.

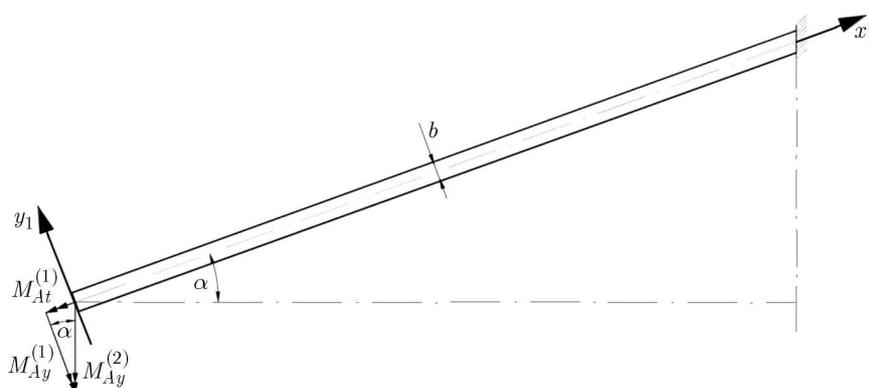


Fig. 7. Scheme of the moments in the vertex A

The bending moment $M_{Ay}^{(1)}$ and torsional moment $M_{At}^{(1)}$ in the vertex A for the arm are

$$M_{Ay}^{(1)} = M_{Ay}^{(2)} \cos \alpha \quad M_{At}^{(1)} = M_{Ay}^{(2)} \sin \alpha \quad (2.25)$$

where $M_{Ay}^{(2)}$ is the bending moment in the vertex A for the cross-beam.

Analogically, the bending angle $\theta_{Ab}^{(1)}$ and torsion angle $\theta_{At}^{(1)}$ in the vertex A for the arm are

$$\theta_{Ab}^{(1)} = \theta_A^{(2)} \cos \alpha \quad \theta_{At}^{(1)} = \theta_A^{(2)} \sin \alpha \quad (2.26)$$

where $\theta_A^{(2)}$ is the bending angle in the vertex A for the cross-beam.

The detailed scheme of the load and displacements for the arm and the cross-beam of the frame is presented in Fig. 8.

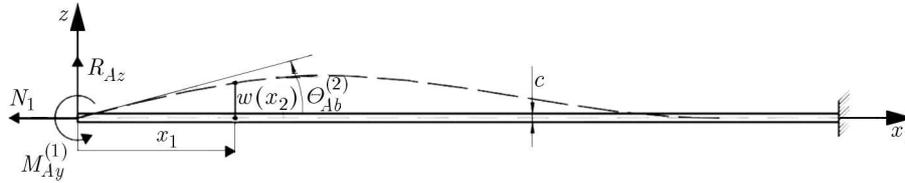


Fig. 8. Scheme of the load and displacements for lateral buckling state

The bending moment in the arm is

$$M_b(x_1) = M_{Ay}^{(1)} - N_1 w_1(x_1) - R_{Az} x_1 \quad (2.27)$$

This bending problem is analogous to the bending of the arm for flat buckling state (10), thus the bending angle of the vertex A is

$$\theta_{Ab}^{(1)} = \left. \frac{dw_1}{dx_1} \right|_0 = \frac{M_{Ay}^{(1)} L_1}{E J_y^{(1)}} f_{1z}(k_{1z} L_1) \quad (2.28)$$

where

$$f_{1z}(k_{1z} L_1) = \frac{1}{k_{1z} L_1} \left\{ \sinh(k_{1z} L_1) - \frac{1 + k_{1z} L_1 \sinh(k_{1z} L_1) - \cosh(k_{1z} L_1)}{k_{1z} L_1 \cosh(k_{1z} L_1) - \sinh(k_{1z} L_1)} [\cosh(k_{1z} L_1) - 1] \right\} \quad (2.29)$$

and $k_{1z} = \sqrt{N_1 / (E J_y^{(1)})}$ is a coefficient, $J_y^{(1)} = b^3 c / 12$ – inertia moment of the cross-sections of the arm.

The torsion angle $\theta_{At}^{(1)}$ in the vertex A

$$\theta_{At}^{(1)} = \frac{M_{At}^{(1)} L_1}{G J_t^{(1)}} = 2(1 + \nu) \frac{M_{At}^{(1)} L_1}{E J_t^{(1)}} \quad (2.30)$$

where: $G = E / [2(1 + \nu)]$ is the shear modulus of elasticity, ν – Poisson's ratio, and the torsional constant $J_t^{(1)}$ of the rectangular cross-section

- for $b \leq c$

$$J_t^{(1)} = \mu b^3 c \quad \mu = \frac{1}{3} - \left[0.178 + 0.153 \frac{b}{c} - 0.138 \left(\frac{b}{c} \right)^2 \right] \quad (2.31)$$

- for $c \leq b$

$$J_t^{(1)} = \mu b c^3 \quad \mu = \frac{1}{3} - \left[0.178 + 0.153 \frac{c}{b} - 0.138 \left(\frac{c}{b} \right)^2 \right] \quad (2.32)$$

The bending moment in the cross-beam is

$$M_b(x_2) = -M_{Ay}^2 + N_2 w_2(x_2) \tag{2.33}$$

This bending problem is analogous to the bending of the cross-beam for flat buckling state (2.17), thus the bending angle of the vertex A is

$$\theta_A^{(2)} = \left. \frac{dw_2}{dx_2} \right|_0 = -\frac{M_{Ay}^{(2)} L_2}{E J_z^{(2)}} \frac{1 - \cos(k_2 L_2)}{k_2 L_2 \sin(k_2 L_2)} \tag{2.34}$$

Taking into account expressions (2.26) for the bending angle $\theta_{Ab}^{(1)}$ and torsion angle $\theta_{At}^{(1)}$, the consistency condition for the angles of the vertex A is as follows

$$\theta_{Ab}^{(1)} \cos \alpha + \theta_{At}^{(1)} \sin \alpha = \theta_A^{(2)} \tag{2.35}$$

Substituting expressions (2.28) and (2.30) into equation (2.35) and making simple transformation, the nonlinear algebraic equation is obtained in the following form

$$k_{J_{zy}} f_{1z}(k_{1z} L_1) \cos^2 \alpha + 2(1 + \nu) k_{J_{zt}} \sin^2 \alpha + 2 \frac{1 - \cos(k_2 L_2)}{k_2 L_2 \sin(k_2 L_2)} = 0 \tag{2.36}$$

where: $k_{J_{zy}} = J_z^{(2)} / (J_y^{(1)} \cos \alpha)$, $k_{J_{zt}} = J_z^{(2)} / (J_t^{(1)} \cos \alpha)$ are dimensionless parameters.

Taking into account the normal forces N_2 (2.6)₁ and N_1 (2.7), the critical load from this equation is determined.

2.4. Example of the triangular frame – values of the critical load

An exemplary calculation of the critical load is carried out for a symmetrical triangular frame: length of the cross-beam $L_2 = 1352$ mm, angle between the arms and the cross-beam $\alpha = \pi/9$, diameters of the cross-section of the cross-beam $d_0 = 50$ mm, $d_1 = 60$ mm, area of the cross section of the arms $A_1 = 1000$ mm² and material constants $E = 2 \cdot 10^5$ MPa, $\nu = 0.3$. Models with different b parameter have been investigated. Values of other parameters are the same as the parameters of the construction which is used in railway industry. The values of critical loads $F_{CR, Flat}^{(Anal)}$ and $F_{CR, Lat}^{(Anal)}$ have been calculated on the basis of non-linear algebraic equations (2.24) and (2.36), respectively. The results of calculations are specified in Table 1.

Table 1. Values of critical loads – analytical solution

b [mm]	15	20	25	31.623	35	40	50	60
$F_{CR, Flat}^{(Anal)}$ [kN]	360.6	403.0	448.2	510.0	541.5	587.1	672.5	747.8
$F_{CR, Lat}^{(Anal)}$ [kN]	363.9	394.6	403.4	399.2	388.7	377.1	357.1	342.8

Flat buckling occurs when the width b of the arm is small ($b \leq 18$ mm), and the lateral buckling occurs for greater width (18 mm $\leq b$).

3. Numerical FEM model

3.1. FEM model of the triangular frame

The symmetrical triangular frame is a beam structure (Fig. 9), hence it is modeled with beam elements – system SolidWorks Simulation 2013 (Fig. 10). The model of the frame is defined in the rectangular coordinate system. The loading force is applied to the vertexes of the frame (Fig. 11).

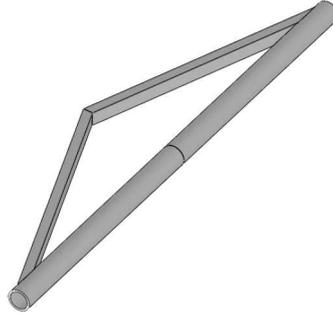


Fig. 9. Numerical model of triangular frame

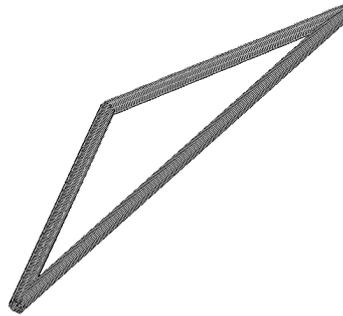


Fig. 10. Discretization of numerical model with the beam elements



Fig. 11. Boundary conditions and loads of the numerical model

3.2. Example of the triangular frame – values of the critical load

An exemplary calculation of the critical load is carried out for the symmetrical triangular frame: length of the cross-beam $L_2 = 1352$ mm, angle between arms and the cross-beam $\alpha = \pi/9$, diameters of the cross-section of the cross-beam $d_0 = 50$ mm, $d_1 = 60$ mm, the area of the cross section of the arms $A_1 = 1000$ mm² and material constants $E = 2 \cdot 10^5$ MPa, $\nu = 0.3$. The values of critical loads $F_{CR, Flat}^{(FEM)}$ and $F_{CR, Lat}^{(FEM)}$ are calculated on the basis of the finite element method. The results of calculations are specified in Table 2. An exemplary FEM calculation of the triangular frame with dimension $b = 25$ mm is shown in Fig. 12.

Flat buckling, similarly to the analytical model, occurs when the width b of the arm is small ($b \leq 18$ mm), and the lateral buckling occurs for greater width ($18 \text{ mm} \leq b$).

Table 2. The values of critical loads – FEM solution

b [mm]	15	20	25	31.623	35	40	50	60
$F_{CR, Flat}^{(FEM)}$ [kN]	353.7	390.7	435.6	492.4	520.9	562.1	638.2	704.7
$F_{CR, Lat}^{(FEM)}$ [kN]	361.6	393.5	400.5	393.2	385.5	373.4	352.7	338.6

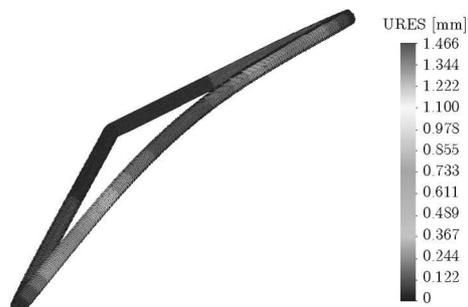


Fig. 12. Scheme of the FEM model of the triangular frame

4. Comparison analysis

The values of critical loads calculated analytically and numerically (FEM) are similar. The difference between them is less than 3%. Comparison of these values is graphically shown for flat buckling (Fig. 13) and for lateral buckling (Fig. 14).

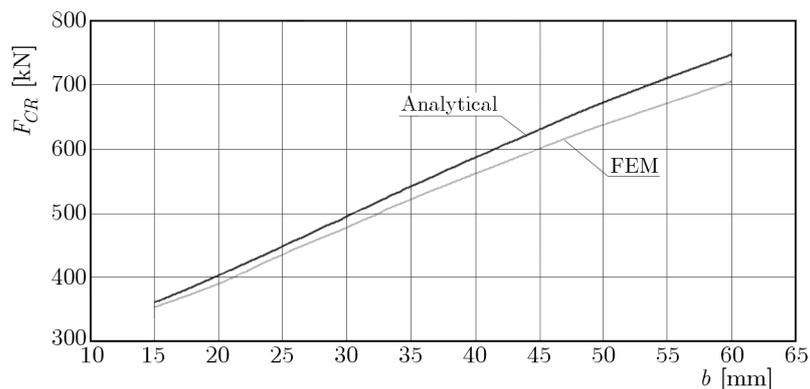


Fig. 13. Comparison of the critical values obtained from the analytical and numerical (FEM) methods for flat buckling

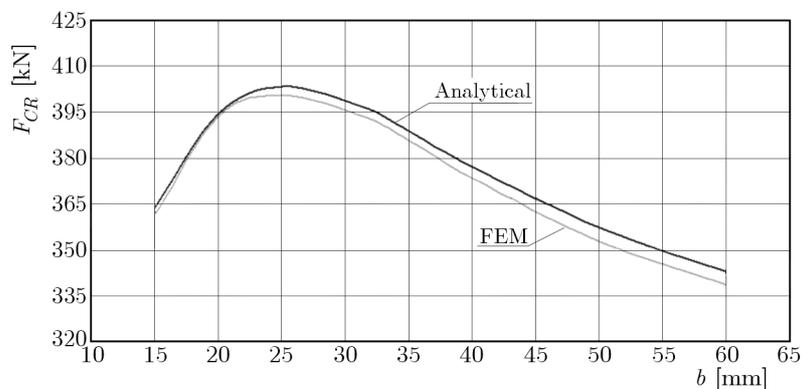


Fig. 14. Comparison of the critical values obtained from the analytical and numerical (FEM) methods for lateral buckling

5. Conclusions

Theoretical studies of the buckling problem of the symmetrical triangular frame carried out on the basis of the analytical and numerical FEM models allow one to conclude that:

- flat buckling of the frame occurs when the width b of the arm is small ($b \leq 18$ mm),
- lateral buckling of the frame occurs when the width b of the arm is greater (18 mm $\leq b$),
- maximum of the critical load exists for the width $b = 25$ mm,
- the analytical model of the frame accurately describes the flat and lateral buckling phenomena. The analytical and FEM results comply with each other.

The presented study, considering the lateral buckling, has not been undertaken before.

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