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MODELLING OF SH-WAVES IN A FIBER-REINFORCED ANISOTROPIC LAYER OVER A PRE-STRESSED HETEROGENEOUS HALF-SPACE

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Modelling of SH-waves in an anisotropic fiber-reinforced layer provides a great deal of support in the understanding of seismic wave propagation. This paper deals with the propagation of SH-waves in a fiber-reinforced anisotropic layer over a pre-stressed heterogeneous half-space. The heterogeneity of the elastic half-space is caused by linear variations of density and rigidity. As a special case when both media are homogeneous and stress free, the derived equation is in agreement with the general equation of the Love wave. Numerically, it is observed that the velocity of SH-waves decreases with an increase in heterogeneity-reinforced parameters and decrease in initial stress.

Keywords: heterogeneity, fiber reinforced medium, SH-waves, initial stress, anisotropy

1. Introduction

The propagation of seismic waves in anisotropic elastic media is unlike in comparison to their propagation in isotropic media. So, the study of seismic wave propagation in anisotropic elastic layered media becomes important to understand the nature of these waves in some complex media. Other types of layers which may be present in the interior of the Earth are reinforced concrete media. The reinforced layers are comprised due to excessive initial stresses present in the Earth. Fiber-reinforced composites are widely used in engineering structures, geophysical prospecting, civil engineering and mining engineering. So the investigation of shear waves in such media become obligatory with a vision to its application to geomechanics. The normal feature of a reinforced concrete medium is that its constituents, namely steel and concrete together, act as a single anisotropic unit as long as they persist in the elastic condition, i.e., the components are bound side by side so that there is no relative motion between them. There are large numbers of fiber-reinforced composite materials which exhibit anisotropic behavior, for example alumina, reinforced light alloys, fibreglasses and concrete. Spencer (1972) was the first who represented fiber-reinforced anisotropic materials with constitutive equations. Later, Belfield *et al.* (1983) presented the method of introducing a continuous self-reinforcement in an elastic solid. Chattopadhyay and Choudhury (1995) discussed some important results of propagation of seismic waves in fiber-reinforced materials. Chattopadhyay *et al.* (2012) studied propagation of SH-waves in an irregular inhomogeneous self reinforced layer lying over a self-reinforced half-space.

For seismologists, the propagation of seismic wave in elastic and reinforced layered media is useful to understand earthquake disaster prevention, oil exploration and groundwater prospecting. The geotechnical study reveals that the material properties such as heterogeneity and anisotropy of the Earth change rapidly beneath its surface, and these properties affect the propagation of seismic waves. Also, the effect of initial stresses on shear waves, which is largely present in the Earth due to a slow process of creep, temperature, pressure, and gravitation cannot be ignored. In order to understand the underground response of seismic wave propagation towards the material properties and initial stresses of the Earth, researchers and seismologists generally prefer heterogeneous elastic models in semi-infinite domains. Due to large applications, pre-stressed Love/SH-waves in different media attract researchers' interests even nowadays. Li *et al.* (2004) investigated the influence of initial stresses on the Love wave propagation in piezoelectric layered structure. Du *et al.* (2008) presented an emphasis on the effect of initial stress on the Love wave propagation in a piezoelectric layer in the presence of a viscous liquid. Zakharenko (2005) studied the propagation of Love waves in a cubic piezoelectric crystal. Qian *et al.* (2004) developed a mathematical model to study the effect of Love wave propagation in a piezoelectric layered structure with initial stresses. Wang and Quek (2001) discussed propagation of Love waves in a piezoelectric coupled solid medium. Zaitsev *et al.* (2001) discussed propagation of acoustic waves in piezoelectric conductive and viscous plates. The supplement of surface wave analysis and other wave propagation problems to anisotropic elastic materials has been a subject of many studies; see for example Musgrave (1959), Crampin and Taylor (1971), Chadwick and Smith (1977), Dowaikh and Ogden (1990), Mozhaev (1995), Nair and Sotiropoulos (1999), Destrade (2001, 2003), Ting (2002), Ogden and Singh (2011, 2014).

SH-waves cause more destruction to the structure than the body waves due to slower attenuation of the energy. Many authors have studied the propagation of an SH-wave by considering dissimilar forms of asymmetry at the interface. Watanabe and Payton (2002) discussed SH-waves in a cylindrically monoclinic material with Green's function. Gupta and Gupta (2013) studied the effect of initial stress on wave motion in an anisotropic fiber reinforced thermoelastic medium. Sahu *et al.* (2014) showed the effect of gravity on shear waves in a heterogeneous fiber-reinforced layer placed over a half-space. Recently, Kundu *et al.* (2014) analyzed an SH-wave in an initially stressed orthotropic homogeneous and a heterogeneous half space. Chattopadhyay *et al.* (2014) studied the effect of heterogeneity and reinforcement on propagation of a crack due to shear waves.

The coupled effects of initial stress, heterogeneity and reinforcement on the propagation of an SH-wave in a fiber-reinforced anisotropic layer overlying a pre-stressed heterogeneous half-space are studied in this paper. The closed form of the dispersion equation for the shear wave by using the method of separation of variables and Whittaker's function is obtained. The effects of all parameters under considered geometry are discussed graphically.

2. Formulation of the problem

Let H be thickness of a steel fiber reinforced (silica fume concrete) layer placed over a prestressed heterogeneous half-space. We consider x-axis along the direction of wave propagation and z-axis vertically downwards (Fig. 1). Let the rigidity, density in the lower half-space are $\mu_2 = \mu'(1+\varepsilon_1 z)$ and $\rho_2 = \rho'(1+\varepsilon_2 z)$, respectively. Here ε_1 and ε_2 are heterogeneous parameters of the lower half-space and having dimensions that are inverse of length.

3. Solution of the problem

3.1. Solution for the upper layer

The constitutive equations for a fiber reinforced linearly elastic anisotropic medium with respect to a preferred direction (Belfield *et al.*, 1983) are

$$\begin{aligned}\tau_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu_T e_{ij} + \alpha(a_k a_m e_{km}\delta_{ij} + e_{kk}a_i a_j) + 2(\mu_L - \mu_T)(a_i a_k e_{kj} + a_j a_k e_{ki}) \\ + \beta(a_k a_m e_{km} a_i a_j)\end{aligned} \tag{3.1}$$

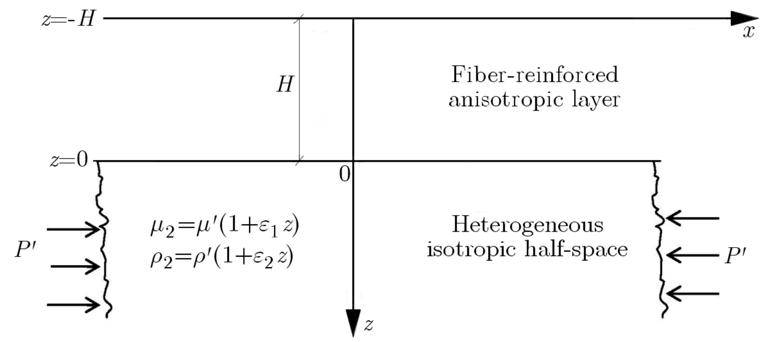

Fig. 1. Geometry of the problem

where $e_{ij} = (\mu_{i,j} + \mu_{j,i})/2$ are components of strain; are reinforced anisotropic elastic parameters with the dimension of stress; λ is an elastic parameter. The preferred direction of fibers is given by $\mathbf{a} = [a_1, a_2, a_3]$, $a_1^2 + a_2^2 + a_3^2 = 1$. If $\mathbf{a}$ has components that are $[1, 0, 0]$ then the preferred direction is the z-axis normal to the direction of propagation. The coefficients μ_L and μ_T are the longitudinal shear and transverse shear moduli of elasticity in the tender direction, respectively.

Equation (3.1) in the presence of initial compression simplifies as given below

$$\begin{aligned} &\tau_{11} = (\lambda + 2\alpha + 4\mu_L + \beta - 2\mu_T)e_{11} + (\lambda + \alpha)e_{22} + (\lambda + \alpha)e_{33} \\ &\tau_{22} = (\lambda + \alpha)e_{11} + (\lambda + 2\mu_T)e_{22} + \lambda e_{33} \\ &\tau_{33} = (\lambda + \alpha)e_{11} + \lambda e_{22} + (\lambda + 2\mu_T)e_{33} \\ &\tau_{12} = 2\mu_T e_{12} \qquad \tau_{13} = 2\mu_T e_{13} \qquad \tau_{23} = 2\mu_T e_{23} \end{aligned} \tag{3.2}$$

The equations of motion in the upper half-space are

$$\begin{aligned} &\frac{\partial \tau_{11}}{\partial x} + \frac{\partial \tau_{12}}{\partial y} + \frac{\partial \tau_{13}}{\partial z} = \rho_1 \frac{\partial^2 u_1}{\partial t^2} \qquad \frac{\partial \tau_{21}}{\partial x} + \frac{\partial \tau_{22}}{\partial y} + \frac{\partial \tau_{23}}{\partial z} = \rho_1 \frac{\partial^2 v_1}{\partial t^2} \\ &\frac{\partial \tau_{31}}{\partial x} + \frac{\partial \tau_{32}}{\partial y} + \frac{\partial \tau_{33}}{\partial z} = \rho_1 \frac{\partial^2 w_1}{\partial t^2} \end{aligned} \tag{3.3}$$

For the SH-wave propagation along the x-axis, we have

$$u_1 = 0 \qquad v_1 = v_1(x, z, t) \qquad w_1 = 0 \tag{3.4}$$

Taking transverse isotropy and setting $a_2 = 0$, we get from Eqs. (3.3)

$$\begin{aligned} &\tau_{12} = \mu_T\Big(P\frac{\partial u_2}{\partial x} + R\frac{\partial u_2}{\partial z}\Big) \qquad \tau_{23} = \mu_T\Big(R\frac{\partial u_2}{\partial z} + Q\frac{\partial u_2}{\partial x}\Big) \\ &\tau_{11} = \tau_{22} = \tau_{33} = \tau_{23} = \tau_{13} = 0 \end{aligned} \tag{3.5}$$

where

$$P = 1 + (\mu^* - 1)a_1^2 \qquad Q = 1 + (\mu^* - 1)a_3^2 \qquad R = (\mu^* - 1)a_1 a_3 \qquad \mu^* = \frac{\mu_L}{\mu_T} \tag{3.6}$$

In the absence of body forces, Eq. (3.3) becomes

$$\frac{\partial \tau_{21}}{\partial x} + \frac{\partial \tau_{22}}{\partial y} + \frac{\partial \tau_{23}}{\partial z} = \rho_1 \frac{\partial^2 v_1}{\partial t^2} \tag{3.7}$$

where ρ_1 is density of the layer.

Using Eqs. (3.5)-(3.7), we get

$$P\frac{\partial^2 v_1}{\partial x^2}+2R\frac{\partial^2 v_1}{\partial x\partial z}+Q\frac{\partial^2 v_1}{\partial z^2}=\frac{\rho_1}{\mu_T}\frac{\partial^2 v_1}{\partial t^2} \tag{3.8}$$

In order to solve Eq. (3.8), we take

$$v_1(x,z,t)=\xi(z)\mathrm{e}^{\mathrm{i}k(x-ct)} \tag{3.9}$$

Here, k is the wave number; c is the phase velocity of simple harmonic waves with a wave length $2\pi/k$.

From Eq. (3.8) and Eq. (3.9), we get

$$Q\frac{\partial^2\xi(z)}{\partial z^2}+2R\mathrm{i}k\frac{\partial\xi(z)}{\partial z}+\Big(\frac{\rho_1}{\mu_T}\omega^2-Pk^2\Big)\xi(z)=0 \tag{3.10}$$

Let the solution to Eq. (3.10) be

$$\xi(z)=A\mathrm{e}^{-\mathrm{i}ks_1z}+B\mathrm{e}^{-\mathrm{i}ks_2z} \tag{3.11}$$

where

$$s_j=\frac{1}{Q}\Big[R+(-1)^{j+1}\sqrt{R^2+Q\Big(\frac{c^2}{c_1^2}-P\Big)}\Big] \qquad j=1,2 \tag{3.12}$$

where A and B are arbitrary constants and $c_1=\sqrt{\mu_T/\rho_1}$ is the shear velocity.

From Eq. (3.9) and Eq. (3.11), the equation of displacement of the upper reinforced medium is given by

$$u_2(x,z,t)=\Big(A\mathrm{e}^{-\mathrm{i}ks_1z}+B\mathrm{e}^{-\mathrm{i}ks_2z}\Big)\mathrm{e}^{\mathrm{i}k(x-ct)} \tag{3.13}$$

3.2. Solution for the lower half-space

The equation of motion for the lower half-space under initial stress P' acting along the x-axis can be written as (Love, 1911)

$$\begin{aligned}&\frac{\partial\sigma_{11}}{\partial x}+\frac{\partial\sigma_{12}}{\partial y}+\frac{\partial\sigma_{13}}{\partial z}-P'\Big(\frac{\partial\varpi_3}{\partial y}-\frac{\partial\varpi_3}{\partial z}\Big)=\rho_2\frac{\partial^2u_2}{\partial t^2}\\&\frac{\partial\sigma_{21}}{\partial x}+\frac{\partial\sigma_{22}}{\partial y}+\frac{\partial\sigma_{23}}{\partial z}-P'\Big(\frac{\partial\varpi_3}{\partial x}\Big)=\rho_2\frac{\partial^2v_2}{\partial t^2}\\&\frac{\partial\sigma_{31}}{\partial x}+\frac{\partial\sigma_{32}}{\partial y}+\frac{\partial\sigma_{33}}{\partial z}-P'\Big(\frac{\partial\varpi_3}{\partial x}\Big)=\rho_2\frac{\partial^2w_2}{\partial t^2}\end{aligned} \tag{3.14}$$

where σ_{11}, σ_{12}, σ_{13}, σ_{21}, σ_{22}, σ_{23}, σ_{31}, σ_{32} and σ_{33} are incremental stress components, u_2, v_2 and w_2 are components of the displacement vector, P' is initial pressure in the lower half-space and ρ_2 is density of the lower half-space. Here, ϖ_1, ϖ_2 and ϖ_3 are rotational components in the lower half-space, which are defined by

$$\varpi_1=\frac{1}{2}\Big(\frac{\partial w_2}{\partial y}-\frac{\partial v_2}{\partial z}\Big) \qquad \varpi_2=\frac{1}{2}\Big(\frac{\partial u_2}{\partial z}-\frac{\partial w_2}{\partial x}\Big) \qquad \varpi_3=\frac{1}{2}\Big(\frac{\partial v_2}{\partial x}-\frac{\partial u_2}{\partial y}\Big) \tag{3.15}$$

Using the SH-wave conditions $u_2=w_2=0$, $v_2=v_2(x,z,t)$, Eq. (3.14) can be reduced to

$$\frac{\partial\sigma_{21}}{\partial x}+\frac{\partial\sigma_{23}}{\partial z}-\frac{P'}{2}\Big(\frac{\partial^2v_2}{\partial x^2}\Big)=\rho_2\frac{\partial^2v_2}{\partial t^2} \tag{3.16}$$

The stress-strain relations are

$$\begin{aligned} &\sigma_{11} = \sigma_{22} = \sigma_{13} = \sigma_{33} = 0 \qquad \sigma_{21} = 2\mu_2 e_{xy} = 2\mu_2 \frac{1}{2}\Big(\frac{\partial v_2}{\partial x} + \frac{\partial u_2}{\partial y}\Big) \\ &\sigma_{23} = 2\mu_2 e_{yz} = 2\mu_2 \frac{1}{2}\Big(\frac{\partial w_2}{\partial y} + \frac{\partial v_2}{\partial z}\Big) \end{aligned} \tag{3.17}$$

The heterogeneity of rigidity and density of the lower half-space are

$$\mu_2 = \mu'(1 + \varepsilon_1 z) \qquad \rho_2 = \rho'(1 + \varepsilon_2 z) \tag{3.18}$$

Now, substituting the heterogeneity of rigidity from Eq. (3.18) into Eq. (3.17), we have

$$\sigma_{21} = \mu'(1 + \varepsilon_1 z)\frac{\partial v_2}{\partial x} \qquad \sigma_{23} = \mu'(1 + \varepsilon_2 z)\frac{\partial v_2}{\partial z} \tag{3.19}$$

Equation of motion (3.16) with the help of equations (3.18) and (3.19) can be written as

$$\Big(1 - \frac{P'}{2\mu'(1 + \varepsilon_1 z)}\Big)\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial x^2} - \frac{\varepsilon_1}{1 + \varepsilon_1 z}\frac{\partial v_2}{\partial z} = \frac{\rho'}{\mu'}\Big(\frac{1 + \varepsilon_2 z}{1 + \varepsilon_1 z}\Big)\frac{\partial^2 v_2}{\partial t^2} \tag{3.20}$$

To solve Eq. (3.20), we take the following substitution

$$v_2 = V(z)\mathrm{e}^{\mathrm{i}k(x - ct)} \tag{3.21}$$

Using Eq. (3.21) in Eq. (3.20), we get

$$\frac{d^2 V(z)}{dz^2} + \frac{\varepsilon_1}{1 + \varepsilon_1 z}\frac{dV(z)}{dz} + \Big[\frac{\rho'}{\mu'}\Big(\frac{1 + \varepsilon_2 z}{1 + \varepsilon_1 z}\Big)c^2 - \Big(1 - \frac{P'}{2\mu'(1 + \varepsilon_1 z)}\Big)\Big]k^2 V(z) = 0 \tag{3.22}$$

After introducing $V(z) = \varPhi(z)/\sqrt{(1 + \varepsilon_1 z)}$ into Eq. (3.22) in order to cancel the term $dV(z)/dz$, we have

$$\frac{d^2\varPhi(z)}{dz^2} + \Big\{\frac{\varepsilon_1^2}{4(1 + \varepsilon_1 z)^2} - k^2\Big[\Big(1 - \frac{P'}{2\mu'(1 + \varepsilon_1 z)}\Big) - \frac{c^2}{c_3^2}\Big(\frac{1 + \varepsilon_2 z}{1 + \varepsilon_1 z}\Big)\Big]\Big\}\varPhi(z) = 0 \tag{3.23}$$

where c is the phase velocity and $c_2 = \sqrt{\mu'/\rho'}$.

Introducing non-dimensional quantities

$$r = \sqrt{1 - \frac{P'}{2\mu'(1 + \varepsilon_1 z)} - \frac{c^2}{c_2^2}\Big(\frac{\varepsilon_2}{\varepsilon_1}\Big)} \qquad s = \frac{2rk(1 + \varepsilon_1 z)}{\varepsilon_1} \qquad \omega = kc$$

in Eq. (3.23), we get

$$\frac{d^2\varPhi}{ds^2} + \Big(\frac{1}{4s^2} + \frac{R}{2s^2} - \frac{1}{4}\Big)\varPhi(s) = 0 \tag{3.24}$$

where $R = \omega^2(\varepsilon_1 - \varepsilon_2)/(c_2^2 rk\varepsilon_1^2)$.

Equation (3.24) becomes the well known Whittaker's equation (Whittaker and Watson, 1990).

The solution to Eq. (3.24) is given by

$$\varPhi(s) = DW_{\frac{r}{2},0}(s) + EW_{-\frac{r}{2},0}(-s) \tag{3.25}$$

where D and E are arbitrary constants and $W_{\frac{r}{2},0}(s)$, $W_{-\frac{r}{2},0}(s)$ are the Whittaker functions. Now considering the condition $V(z) \to 0$ as $z \to \infty$ i.e. $\varPhi(s) \to 0$ as $s \to \infty$ in Eq. (3.21), the exact solution becomes

$$\varPhi(s) = DW_{\frac{r}{2},0}(s) \tag{3.26}$$

The solution to Eq. (3.26) is given by

$$v_2 = V(z)\mathrm{e}^{\mathrm{i}k(x-ct)} = \frac{DW_{\frac{r}{2},0}(s)}{\sqrt{1+\varepsilon_1 z}}\mathrm{e}^{\mathrm{i}k(x-ct)} \tag{3.27}$$

Equation (3.27) is the displacement for the SH-wave in the half space.

Now, expanding Eq. (3.27) up to the linear term, we have

$$v_2 = D\mathrm{e}^{\frac{-rk(1+\varepsilon_1 z)}{\varepsilon}}\sqrt{\frac{2rk}{\varepsilon_1}}\Big[1+(1-R)\frac{2rk}{\varepsilon_1}(1+\varepsilon_1 z)\Big]\mathrm{e}^{\mathrm{i}k(x-ct)} \tag{3.28}$$

4. Boundary conditions

The displacement components and stress components are continuous at $z=-H$ and $z=0$, therefore geometry of the problem leads to the following conditions:

(1) At $z=-H$, the stress component $\tau_{23}=0$.

(2) At $z=0$, the stress component of the layer and the half space is continuous, i.e. $\tau_{23}=\sigma_{23}$.

(3) At $z=0$, the velocity component of both layers is continuous, i.e. $v_1=v_2$.

5. Dispersion relation

The dispersion relation for SH-waves can be obtained by using the above boundary conditions. Therefore, the displacement for the SH-waves in the in-homogeneous half-space using boundary conditions (3.1), (3.2) and (3.3) in Eq. (3.13) and Eq. (3.28) becomes (taking Whittaker's function $W_{\frac{r}{2},0}(s)$ up to linear terms in s)

$$\begin{aligned}
&A(R-Qs_1)\mathrm{e}^{\mathrm{i}s_1kH}+B(R-Qs_2)\mathrm{e}^{\mathrm{i}s_2kH}=0\\
&\mathrm{i}k[A(R-Qs_1)+B(R-Qs_2)]\\
&\qquad -D\frac{\mu'}{\mu_T\zeta}\mathrm{e}^{-\frac{kr}{\varepsilon_1}}\sqrt{\frac{2kr}{\varepsilon_1}}\Big[\frac{kr}{\varepsilon_1}(1-R)+1\Big]\left[\frac{(1-R)kr}{1+(1-R)\frac{kr}{\varepsilon_1}}-kr\right]=0\\
&A+B-D\mathrm{e}^{-\frac{kr}{\varepsilon_1}}\sqrt{\frac{2kr}{\varepsilon_1}}\Big[1+(1-R)\frac{kr}{\varepsilon}\Big]=0
\end{aligned} \tag{5.1}$$

Now eliminating A, B and D from Eqs. (5.1), we obtain

$$\begin{bmatrix}
(R-Qs_1)\mathrm{e}^{\mathrm{i}s_1kH} & (R-Qs_2)\mathrm{e}^{\mathrm{i}s_2kH} & 0\\
\mathrm{i}k(R-Qs_1) & \mathrm{i}k(R-Qs_2) & -\frac{\mu'}{\mu_T\zeta}\mathrm{e}^{-\frac{kr}{\varepsilon_1}}\sqrt{\frac{2kr}{\varepsilon_1}}\mathcal{A}\Big[\frac{(1-R)kr}{\mathcal{A}}-kr\Big]\\
1 & 1 & -\mathrm{e}^{-\frac{kr}{\varepsilon_1}}\sqrt{\frac{2kr}{\varepsilon_1}}\mathcal{A}
\end{bmatrix}=0 \tag{5.2}$$

where

$$\mathcal{A}=\frac{kr}{\varepsilon_1}(1-R)+1$$

On simplifying Eq. (5.2), we get

$$\tan\left[\frac{kH}{Q}\sqrt{R^2+Q\Big(\frac{c^2}{c_1^2}-P\Big)}\right]=\frac{\mu'}{\mu_T}\frac{1}{\sqrt{R^2+Q\Big(\frac{c^2}{c_1^2}-P\Big)}}\left[r-\frac{1-R}{1+(1-R)\frac{kr}{\varepsilon_1}}\right] \tag{5.3}$$

Equation (5.3) is the dispersion equation of the SH-wave propagation in a fiber-reinforced anisotropic layer over a pre-stressed heterogeneous isotropic elastic half-space.

- *Case 1*

If we take $a_1 = 1$, $a_2 = a_3 = 0$ then $\rho_1 \to \mu_L/\mu_T$ and $\mu_L \to \mu_T \to \mu_1$, then $P \to 1$, $Q \to 1$ and $R \to 1$, therefore Eq. (5.3) reduces to

$$\tan\Big(kH\sqrt{\frac{c^2}{c_1^2}-1}\Big) = \frac{\mu'}{\mu_1}\frac{1}{\sqrt{\frac{c^2}{c_1^2}-1}}\Big[r - \frac{1-R}{1+(1-R)\frac{kr}{\varepsilon_1}}\Big] \tag{5.4}$$

This is the dispersion equation of a homogenous reinforced medium over a pre-stressed heterogeneous half space.

- *Case 2*

When the lower half-space is homogeneous, that is $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$, which implies that $r = \sqrt{1 - \frac{P'}{2\mu'} - \frac{c^2}{c_2^2}}$, therefore Eq. (5.3) reduces to

$$\tan\Big[\frac{kH}{Q}\sqrt{R^2 + Q\Big(\frac{c^2}{c_1^2} - P\Big)}\Big] = \frac{\mu'}{\mu_T}\frac{1}{\sqrt{R^2 + Q\Big(\frac{c^2}{c_1^2} - P\Big)}}\sqrt{1 - \frac{P'}{2\mu'} - \frac{c^2}{c_2^2}} \tag{5.5}$$

This is the dispersion equation of an anisotropic reinforced medium over a pre-stressed homogenous half space.

- *Case 3*

When the lower half-space is stress free and homogeneous, that is $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$, $P' \to 0$, which implies that $r = \sqrt{1 - \frac{c^2}{c_2^2}}$, therefore Eq. (5.3) reduces to

$$\tan\Big[\frac{kH}{Q}\sqrt{R^2 + Q\Big(\frac{c^2}{c_1^2} - P\Big)}\Big] = \frac{\mu'}{\mu_T}\frac{1}{\sqrt{R^2 + Q\Big(\frac{c^2}{c_1^2} - P\Big)}}\sqrt{1 - \frac{c^2}{c_2^2}} \tag{5.6}$$

This is the dispersion equation of an anisotropic reinforced medium over a pre-stressed homogenous half space.

- *Case 4*

For a homogeneous reinforced medium over an homogeneous half space, we take $\varepsilon_1 = 0$, $\varepsilon_2 = 0$, $P' \to 0$, $a_1 = 1$, $a_2 = a_3 = 0$ then $\rho_1 \to \mu_L/\mu_T$ and $\mu_L \to \mu_T \to \mu_1$, then $P \to 1$, $Q \to 1$ and $R \to 1$ therefore Eq. (5.3) reduces to

$$\tan\Big(kH\sqrt{\frac{c^2}{c_1^2}-1}\Big) = \frac{\mu'}{\mu_1}\frac{\sqrt{1-\frac{c^2}{c_2^2}}}{\sqrt{\frac{c^2}{c_1^2}-1}} \tag{5.7}$$

Equation (5.7) is the classical dispersion equation of SH-waves given by Love (1911) and Ewing *et al.* (1957).

6. Numerical analysis and discussion

To show the effect of heterogeneity parameters, the initial stress parameter and steel reinforced parameters on SH-wave propagation in a fiber-reinforced anisotropic layer over a heterogeneous isotropic elastic half-space, we take the data assumed by Gupta (2014) and Gubbins (1990) as shown in Table 1 and the values of parameters for figures in Table 2. We have plotted the non-dimensional phase velocity c/c_1 against the dimensionless wave number kH on the propagation of SH-wave in the fiber-reinforced anisotropic layer by using MATLAB software. The effects of reinforced parameters a_1^2, a_3^2, initial stress parameter $\zeta = P'/(2\mu')$ and heterogeneity parameters ε_1/k, ε_2/k are shown in Figs. 2-5. Figure 2a illustrates the effect of heterogeneity parameters in the presence of reinforced parameters and stress parameter on the propagation of SH-waves.

Table 1. Data for the fiber-reinforced anisotropic layer and the elastic medium

Symbol	Numerical value	Units
μ_T	$5.65 \cdot 10^9$	N/m^2
μ_L	$2.46 \cdot 10^9$	N/m^2
λ	$5.65 \cdot 10^9$	N/m^2
α	$-1.28 \cdot 10^{10}$	N/m^2
β	$220.09 \cdot 10^9$	N/m^2
ρ_1	7800	kg/m^3
a_3^2	0.75	–
a_1^2	0.25	–
μ'	$6.34 \cdot 10^{10}$	N/m^2
ρ'	3364	kg/m^3

Table 2. Values of parameters for the figures

Figure	a_1^2	a_3^2	ζ	ε_1/k	ε_2/k
2a	0.35	0.65	0.5	–	–
2b	0	0	0.5	–	–
3a	0.35	0.65	0	–	–
3b	0	0	0	–	–
4a	–	–	0.5	0.4	0.4
4b	–	–	0.5	0.4	0.4
5a	–	–	0	0.4	0.4
5b	–	–	0	0.4	0.4
6a	0.35	0.65	–	0.4	0.4
6b	0.35	0.65	–	0.4	0.4

It is clear from this figure that the phase velocity decreases with an increase in the heterogeneity parameters. Figure 2b represents the variation of dimensionless phase velocity with the dimensionless wave number on the propagation of SH-waves for different values of heterogeneity parameters in the absence of the reinforced parameter for the initially stressed half-space. It is observed from these curves that as the heterogeneity parameters in the half-space increase, the velocity of SH-wave decreases. From Figs. 2a and 2b, it is clear that the SH-wave propagation is more influenced by the heterogeneity parameters in comparison to reinforcement in the upper layer. It is also seen that for a large value of heterogeneity parameters, the curves of phase velocities are significantly distanced from each other.

Figure 3a shows the effect of heterogeneity parameters in the presence of reinforced parameters on the propagation of SH-waves when the lower half is stress free. It is clear from this

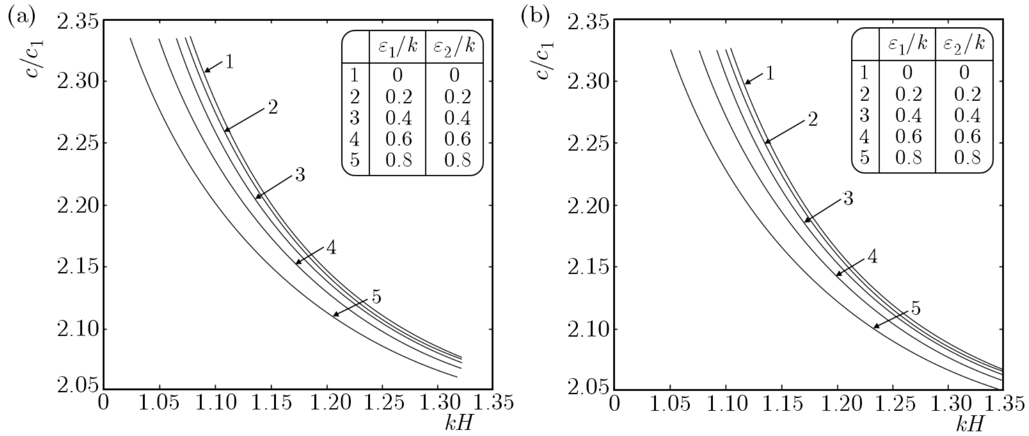

Fig. 2. Variation of the phase velocity against the wave number for different values of heterogeneity parameters in the (a) presence of the reinforced parameters and (b) absence of the reinforced parameter for the initially stressed half-space

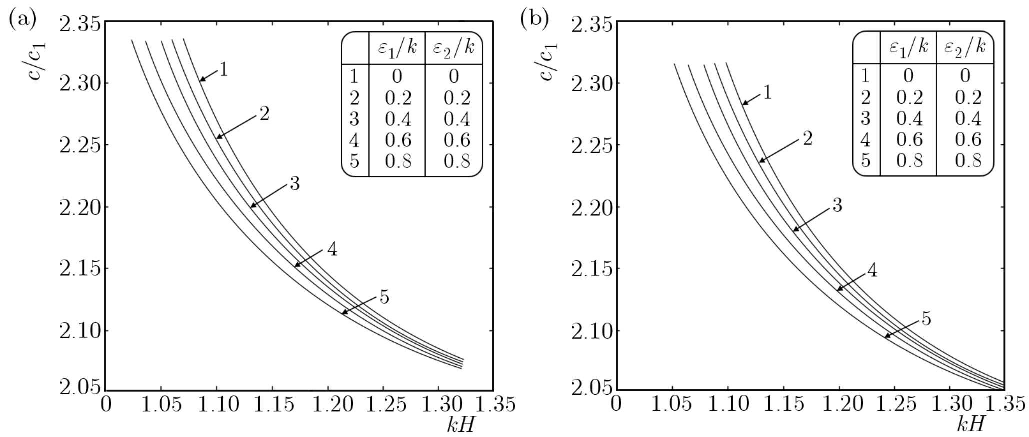

Fig. 3. Variation of the phase velocity against the wave number for different values of heterogeneity parameters in the (a) presence of the reinforced parameters and (b) absence of the reinforced parameter for the stress free half-space

figure that the phase velocity decreases with an increase in the heterogeneity parameters. Figure 3b represents the variation of dimensionless phase velocity with the dimensionless wave number on the propagation of SH-waves for different values of heterogeneity parameters in the absence of the initial stress and reinforced parameter. It is observed from these curves that as the heterogeneity parameters in the half-space increase, the velocity of SH-wave decreases.

Figure 4a shows the effect of reinforced parameters a_1^2 and a_3^2on the propagation of SH-waves at constant stress and heterogeneity parameters. It is seen from the diagram that as a_1^2 increases as well as a_3^2 decreases, the velocity of SH-wave decreases. Figure 4b shows the effect of reinforced parameters a_1^2 and a_3^2on the propagation of SH-waves at constant stress and heterogeneity parameters. It is seen from the diagram that as a_1^2 decreases as well as a_3^2 increases, the velocity of SH-wave decreases.

Figure 5a shows the effect of reinforced parameters a_1^2 and a_3^2on the propagation of SH-waves at constant heterogeneity parameters in the absence of the initial stress. It is seen from the diagram that as a_1^2 increases as well as a_3^2 decreases, the velocity of SH-wave decreases. Figure 5b shows the effect of reinforced parameters a_1^2 and a_3^2on the propagation of SH-waves at constant heterogeneity parameters in the absence of the initial stress. It is seen from the diagram that as a_1^2 decreases as well as a_3^2 increases, the velocity of SH-wave decreases.

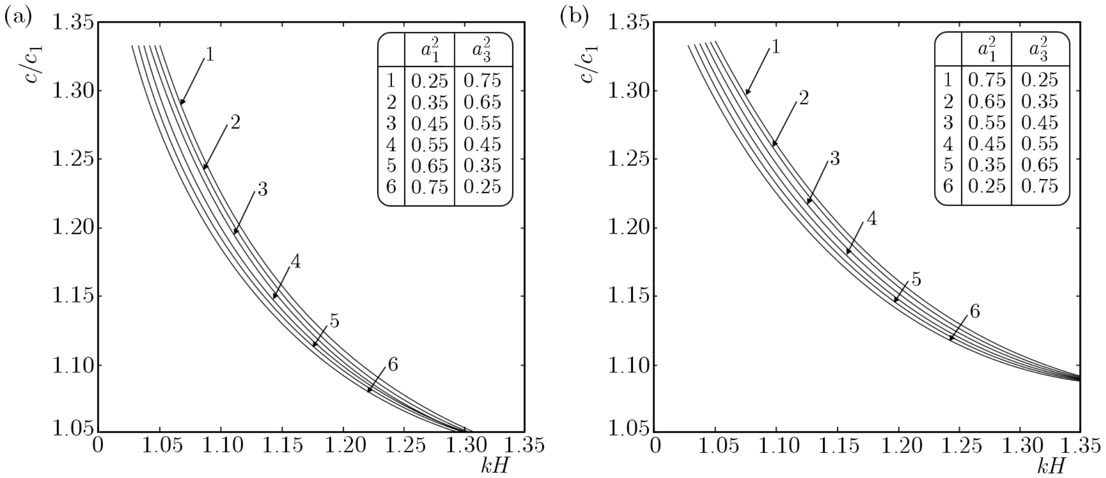

Fig. 4. Variation of the phase velocity against the wave number for different values reinforced parameters at constant stress and heterogeneity parameters

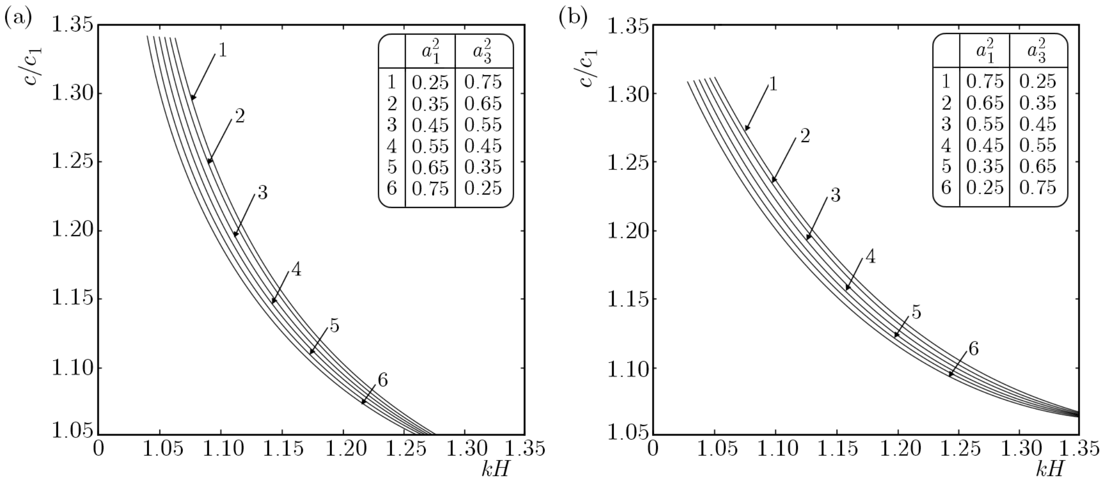

Fig. 5. Variation of the phase velocity against the wave number for different values reinforced parameters at constant heterogeneity parameters in the absence of the initial stress

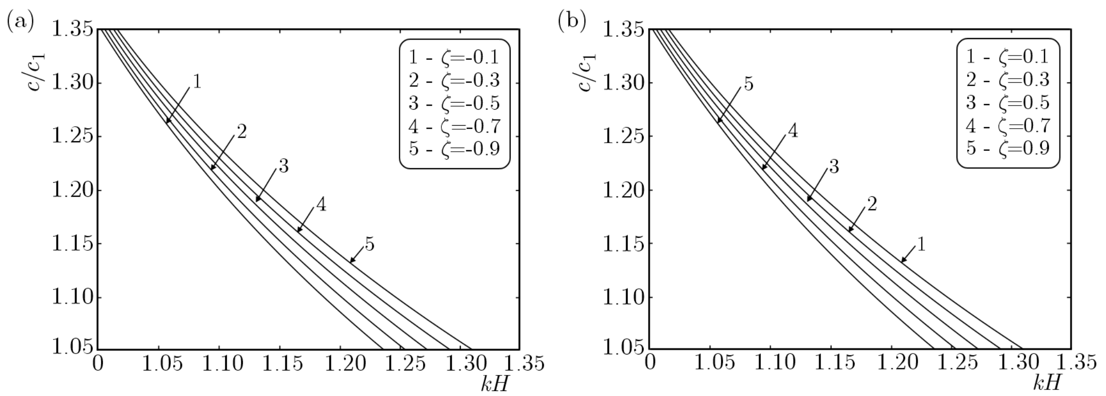

Fig. 6. Variation of the phase velocity against the wave number for different values (a) tensile stress (b) compressive stress in the presence of the heterogeneity parameter and reinforced parameters

In Fig. 6a, the curves show the effect of tensile stress $\zeta = P'/(2\mu') < 0$ on the propagation of the SH-wave in a fiber-reinforced anisotropic layer in the presence of the heterogeneity parameter in the lower half-space and reinforced parameters in the layer. It is clear from this figure that the phase velocity increases with an increase in the tensile stress. In Fig. 6b, the curves show the effect of compressive stress $\zeta = P'/(2\mu') > 0$ on the propagation of SH-waves in a fiber-reinforced anisotropic layer in the presence of the heterogeneity parameter in the lower half-space and reinforced parameters in the layer. It is clear from this figure that the phase velocity decreases with an increase in the compressive stress.

7. Conclusions

Two layers are considered in the problem analysed in this paper: a fiber-reinforced anisotropic upper layer and a pre-stressed heterogeneous lower layer with exponential variation in rigidity and density. The Whittaker function and the method of separation of variables are employed in order to find the dispersion of SH-waves in the fiber-reinforced layer placed over a pre-stressed heterogeneous elastic half-space. Displacement of the upper fiber-reinforced layer is derived in a closed form and the dispersion curves are drawn for various values of heterogeneity, stress and reinforced parameters. In a particular case, the dispersion equation coincides with the well--known classical equation of the Love wave when the upper and lower layer are homogeneous and stress free. The above results may be used to study the surface wave propagation in a fiber reinforced medium. This validates the solution.

From above numerical analysis, it may be concluded that:

- In all the figures, the dimensionless phase velocity of SH-waves decreases with an increase in the dimensionless wave number.
- The dimensionless phase velocity of SH-wave shows a remarkable change with heterogeneity and reinforced parameters.
- It is observed that the depth increases, the velocity of SH-waves decreases.
- The velocity of SH-waves decreases with an increase in the reinforced parameter of the upper layer and the inhomogeneous parameter of the lower half-space. This is the property of seismic wave propagation in the crustal layer.
- The phase velocity increases with an increase in the tensile stress but decreases with an increase in the compressive stress of the lower half-space.

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